

THE SECOND CONJUGATE ALGEBRAS OF BANACH ALGEBRAS

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ABSTRACT. In this paper, we study Arens regularity of a Banach algebra A . In particular, we give characterizations for A to be Arens regular.

KEY WORDS AND PHRASES. Banach algebra, Arens products, Arens regularity, weakly compact operators.

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1. INTRODUCTION.

Let A be a Banach algebra. It is an interesting and difficult problem to determine whether A is Arens regular. Many papers have been written on this subject. For example, see [2], [5], [9], [10], [11] and [12]. In particular let A be a B^* -algebra. It is well known that A is Arens regular. However, it is not easy to prove this result. There are many different proofs of this result. For example, see [4], [5], and [8].

In this paper, we give characterizations for A to be Arens regular. It follows from this result and a result of C.A. Akemann that a B^* -algebra is Arens regular. We also show that if A is a Banach algebra which is Arens regular, then any closed subalgebra of A is also Arens regular.

2. NOTATION AND PRELIMINARIES.

Definitions not explicitly given are taken from Rickart [7].

Let A be a Banach algebra and let A^* and A^{**} be the conjugate and second conjugate spaces of A . We will denote by π the canonical embedding of A into A^{**} . The two Arens products on A^{**} are defined in stages according to the following rules (see [3]). Let $x, y \in A$, $f \in A^*$ and $F, G \in A^{**}$.

Define fox by $(fox)(y) = f(xy)$. Then $fox \in A^*$.

Define Gof by $(Gof)(x) = G(fox)$. Then $Gof \in A^*$.

Define FoG by $(FoG)(f) = F(Gof)$. Then $FoG \in A^{**}$.

A^{**} is a Banach algebra under the Arens product \circ , and we denote this algebra by (A^{**}, \circ) .

Define $xo'f$ by $(xo'f)(y) = f(yx)$. Then $xo'f \in A^*$.

Define $fo'F$ by $(fo'F)(x) = F(xo'f)$. Then $fo'F \in A^*$.

Define $Fo'G$ by $(Fo'G)(f) = G(fo'F)$. Then $Fo'G \in A^{**}$.

A^{**} is a Banach algebra under the Arens product \circ' and we denote this algebra by (A^{**}, \circ') .

Both of the Arens products extend the given multiplication on A when A is canonically embedded in A^{**} . In general, \circ and \circ' are distinct on A^{**} . If they agree on A^{**} , then A is called Arens regular.

In this paper, all algebras and linear spaces under consideration are over the complex field C .

3. ARENS REGULARITY FOR BANACH ALGEBRAS.

Let A be a Banach algebra and $f \in A^*$. Define $L_f: A \rightarrow A^*$ by

$$L_f(x) = fox \quad (x \in A).$$

Then L_f is clearly a continuous linear operator from A to A^* .

For each $F \in A^{**}$, define $F.L_f$ by

$$F.L_f(x) = F(L_f(x)) = F(fox) = (Fof)(x).$$

Then $F.L_f \in A^*$. Define $L_f^*: A^{**} \rightarrow A^*$ by

$$L_f^*(F) = F.L_f = Fof \quad (F \in A^{**}).$$

Then L_f^* is clearly a continuous linear operator from A^{**} to A^* .

For each $F \in A^{**}$, define $F.L_f^*$ by

$$F.L_f^*(G) = F(L_f^*(G)) = F(Gof) \quad (G \in A^{***}).$$

Then $F.L_f^* \in A^{***}$. Finally, we define $L_f^{**}: A^{**} \rightarrow A^{***}$ by

$$L_f^{**}(F) = F.L_f^* \quad (F \in A^{**}).$$

Then clearly L_f^{**} is a continuous linear operator from A^{**} to A^{***} .

THEOREM 1. Let A be a Banach algebra. Then the following statements are equivalent:

- (1) A is Arens regular.
- (2) For each $f \in A^*$, $L_f^{**}(A^{**})$ is contained in $\pi(A^*)$, where $\pi(A^*)$ is a subspace of A^{***} .
- (3) For each $f \in A^*$, L_f is weakly compact.
- (4) Let $F \in A^{**}$ and $\{x_\alpha\}$ a bounded net in A . If $\pi(x_\alpha) \rightarrow F$ weakly, then $f \circ F$ is a weakly limit point of $\{fox_\alpha\}$.

PROOF. (1) \Rightarrow (2). Assume (1). Let $F, G \in A^{**}$. Then $L_f^{**}(f) = F.L_f^*$ and by (1)

$$F.L_f^*(G) = F(L_f^*(G)) = F(Gof) = (F \circ G)(f) = (F \circ G)(f) = G(f \circ F) = \pi(f \circ F)(G).$$

Therefore $F.L_f^* = \pi(f \circ F) \in \pi(A^*)$ and so $L_f^{**}(F) = F.L_f^* \in \pi(A^*)$. This proves (2).

(2) \Rightarrow (3). This follows immediately from [6; p. 482, Theorem 2].

(3) \Rightarrow (4). Assume that L_f is weakly compact. Let F and $G \in A^{**}$. Then by Goldstine's theorem [6; p. 424, Theorem 5] there exists a bounded net $\{x_\alpha\}$ in A such that $\pi(x_\alpha) \rightarrow F$ weakly. Similarly, there exists a bounded net $\{y_\beta\}$ such that $\pi(y_\beta) \rightarrow G$ weakly. Since L_f is weakly compact, we can assume that $L_f(x_\alpha) \rightarrow g$ weakly for some $g \in A^*$. Hence $fox_\alpha \rightarrow g$ weakly. Therefore

$$\begin{aligned} \lim_\alpha G(fox_\alpha) &= G(g) = \lim_\beta \pi(y_\beta)(g) = \lim_\beta \lim_\alpha \pi(y_\beta)(fox_\alpha) \\ &= \lim_\beta \lim_\alpha f(x_\alpha y_\beta) = \lim_\beta \lim_\alpha (y_\beta \circ f)(x_\alpha) \\ &= \lim_\beta \lim_\alpha \pi(x_\alpha)(y_\beta \circ f) = \lim_\beta F(y_\beta \circ f) \\ &= \lim_\beta (f \circ F)(y_\beta) = \lim_\beta \pi(y_\beta)(f \circ F) = G(f \circ F). \end{aligned}$$

Therefore $f \circ F$ is a weak limit point of $\{fox_\alpha\}$. This proves (4).

(4) \Rightarrow (1). Assume (4). Let $F, G \in A^{**}$. Then by Goldstine's theorem, there exists a bounded net $\{x_\alpha\}$ in A such that $\pi(x_\alpha) \rightarrow F$ weakly. Since $f \circ F$ is a weakly limit point of $\{fox_\alpha\}$, we can assume that

$$G(f \circ F) = \lim_\alpha G(fox_\alpha) = \lim_\alpha (Gof)(x_\alpha) = \lim_\alpha \pi(x_\alpha)(Gof) = F(Gof) = F \circ G(f).$$

Therefore $(F \circ G)(f) = G(f \circ F) = F \circ G(f)$ and so A is Arens regular. This completes the proof of the theorem.

COROLLARY 2. Let A be a Banach algebra such that each continuous linear map T of A into A^* is weakly compact, then A is Arens regular.

PROOF. Since each $L_f(f \in A^*)$ is weakly compact, A is Arens regular by Theorem 1.

Let A be a B^* -algebra and B a Banach space such that B^* is a W^* -algebra. Then by [1; p.293, Corollary II.9], any continuous linear map T of A into B is weakly compact. Therefore it follows from Corollary 1 that A is Arens regular. The property that "any continuous linear map T of A into B is weakly compact" is a very strong one. In order for A to be Arens regular, we need only to show that L_f is weakly compact for all f in A^* . Therefore, a simple proof for a B^* -algebra to be Arens regular may exist.

4. SUBALGEBRAS OF A BANACH ALGEBRA WHICH IS ARENS REGULAR.

Let A be a Banach algebra which is Arens regular. It is well known that a subalgebra of A may not be Arens regular. In fact, let M be the group algebra of an infinite abelian locally compact group. Then M is an A^* -algebra. Let A be the completion of M in an auxiliary norm. By [5; p.857, Theorem 3.14] M is not Arens regular. Since A is a B^* -algebra, A is Arens regular.

Let A be a Banach algebra and M a closed subalgebra of A . For each $f \in A^*$, we define f_M by $f_M(x) = f(x)$ for all $x \in M$. Then $f_M \in M^*$.

THEOREM 3. Let M be a closed subalgebra of A . If A is Arens regular, then so is M .

PROOF. Let $f \in M^*$. Then there exists some $\tilde{f} \in A^*$ such that $\tilde{f}_M = f$. Let $F \in M^{**}$. Define \tilde{F} by

$$\tilde{F}(g) = F(g_M) \quad (g \in A^*).$$

Then it is clear that $\tilde{F} \in A^{**}$. Since A is Arens regular, by Theorem 1, $L_{\tilde{f}}$ is weakly compact on A . Let $\{x_\alpha\}$ be a bounded net in M , then $L_{\tilde{f}}(x_\alpha) = \tilde{f} \circ x_\alpha \rightarrow g$ weakly for some $g \in A^*$. Since $(\tilde{f} \circ x_\alpha)_M = f \circ x_\alpha \in M^*$, it follows that

$$F(g_M) = \tilde{F}(g) = \lim_{\alpha} \tilde{F}(\tilde{f} \circ x_\alpha) = \lim_{\alpha} F((\tilde{f} \circ x_\alpha)_M) = \lim_{\alpha} F(f \circ x_\alpha).$$

Therefore $L_f(x_\alpha) \rightarrow g_M$ weakly and so by Theorem 1, M is Arens regular. This completes the proof.

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