

## DYNAMICS OF THE RADIX EXPANSION MAP

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**ABSTRACT** The chaotic dynamics of the map  $\phi(x) = (\beta x + \alpha) \pmod{1}$  are studied using Parry's  $\beta$ -expansion. It is shown that for  $1 < \beta < 2, \alpha \geq 0$ , the number of periodic points of period  $n$  is  $O(\beta^n)$ .

**KEY WORDS AND PHRASES:** Dynamics, Chaos, Radix Expansion Map, Invariant Measure  
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### 1. INTRODUCTION

When  $\beta$  is an integer, the dynamics of the map  $\phi(x) = (\beta x + \alpha) \pmod{1}$  are rather simple.  $\phi$  is topologically conjugate to the shift automorphism  $\sigma$  on the space  $\sum_\beta \{s_1 s_2 \dots s_i \in \{0, 1, \dots, \beta-1\}\}$ , which, in the terminology of Devaney (1989), implies that it is chaotic (see Def. 2 below). The number of periodic points of period  $n$  is  $\beta^n$ , because the period points of  $\sigma$  on  $\sum_\beta$  are precisely the sequences of the form  $\overline{s_1 \dots s_n}$ ,  $s_i \in \{0, 1, \dots, \beta-1\}$ . And there is a unique invariant Borel measure, differentiable on  $[0, 1]$ : the characteristic function  $\chi_{[0,1]}$  (Parry, 1960).

When  $\beta$  is not an integer, things are not quite so tidy. There is still an invariant Borel measure differentiable on  $[0, 1]$  (Sinai, 1981), although it is zero almost everywhere. However, although the map may still be proved chaotic by demonstrating a conjugacy with a shift map, the space  $A_\beta$  on which this shift map acts is somewhat problematic. We will deal only with the case  $1 < \beta < 2$ , although most of the results generalize straightforwardly. In this case, an analysis of  $A_\beta$  reveals that the number of periodic points of period  $n$  is asymptotically proportional to  $\#(n)$ , the number of binary sequences  $s_1 s_2 \dots s_n$  so that:

$$\begin{aligned} \beta^{-1}s_1 + \dots + \beta^{-n}s_n &\leq 1 \\ \beta^{-1}s_2 + \dots + \beta^{-n+1}s_n &\leq 1 \\ &\vdots \\ \beta^{-1}s_n &\leq 1 \end{aligned}$$

If  $X$  is a random variable on  $R$ , with probability density  $f$  so that  $\text{supp}(f) = [0, 1]$ , let  $\psi f$  denote the probability density of  $\phi(X)$ . Then, we show that  $\psi^n \chi_{[0,1]}(0) = \beta^{-n} \#(n)$ . It follows that

$\#(n) = O(\beta^n)$ , since  $\lim_{n \rightarrow \infty} \psi^n \chi_{[0,1]}(0)$  exists by the ergodicity of the mapping  $\phi$  (Sinai, 1981). This result is new and apparently cannot be obtained from standard methods such as kneading theory (Collet and Eckmann, 1980). From here on, we will assume  $1 < \beta < 2$ ,  $\alpha \geq 0$ .

## 2 INVARIANT MEASURES

Where  $\alpha = 0$ , Parry (1960) has studied the invariant measure of  $\phi$ , and shown that it is unique. In general, any invariant measure of  $\phi$  is characterized by a Frobenius-Perron operator  $\psi$ .

**LEMMA 1** If  $0 \leq x \leq 1$ , then  $\psi f(x) = \beta^{-1} [f(\beta^{-1}x - \beta^{-1}\alpha) + f(\beta^{-1}x - \beta^{-1}\alpha + \beta^{-1}) + f(\beta^{-1}x - \beta^{-1}\alpha + 2\beta^{-1})]$

**PROOF.**  $P(\alpha \leq \psi X \leq x) = P(\beta^{-1}\alpha - \beta^{-1}\alpha \leq X \leq \beta^{-1}x - \beta^{-1}\alpha) + P(\beta^{-1}\alpha - \beta^{-1}\alpha + \beta^{-1} \leq X \leq \beta^{-1}x - \beta^{-1}\alpha + \beta^{-1}) + P(\beta^{-1}\alpha - \beta^{-1}\alpha + 2\beta^{-1} \leq X \leq \beta^{-1}x - \beta^{-1}\alpha + 2\beta^{-1})$

In terms of densities, this means

$$\int_a^x \psi f(t) dt = \int_{\beta^{-1}\alpha - \beta^{-1}\alpha}^{\beta^{-1}x - \beta^{-1}\alpha} f(t) dt + \int_{\beta^{-1}\alpha - \beta^{-1}\alpha + \beta^{-1}}^{\beta^{-1}x - \beta^{-1}\alpha + \beta^{-1}} f(t) dt + \int_{\beta^{-1}\alpha - \beta^{-1}\alpha + 2\beta^{-1}}^{\beta^{-1}x - \beta^{-1}\alpha + 2\beta^{-1}} f(t) dt$$

Differentiating, one obtains the lemma.

A measure  $f$  is invariant under  $\phi$  if and only if it is a fixed point of the operator  $\psi$  (see Lasota, 1973). From this equation, one may deduce certain simple properties of  $\tilde{f}$ . For instance, it is easy to see that if  $\alpha = 0$ ,  $f(1) = \beta^{-1}f(\beta^{-1}) = \frac{\beta-1}{\beta}f(0)$ .

Next, let us derive a formula for the  $n$ 'th iterate of  $\psi$ .

**LEMMA 2.**  $\psi^n f(x) = \beta^{-n} \sum_{i=1}^{2^n} f(\beta^{-n}x - \beta^{-n}\alpha + \alpha_i^{(n)})$ , where  $\{\alpha_1^{(n)}, \dots, \alpha_{2^n}^{(n)}\}$  is the set of all expansions  $\beta^{-1}s_1 + \beta^{-2}s_2 + \dots + \beta^{-n}s_n$  so that  $s_i \in \{0, 1\}$  and

$$\begin{aligned} \beta^{-1}x + \beta^{-1}s_1 + \beta^{-2}s_2 + \dots + \beta^{-n}s_n &\leq 1 \\ \beta^{-n+1}x + \beta^{-1}s_2 + \dots + \beta^{-n+1}s_n &\leq 1 \\ &\vdots \\ \beta^{-1}x + \beta^{-1}s_n &\leq 1 \end{aligned}$$

**PROOF.** Lemma 1 takes care of the case  $n = 1$ , so we may proceed by induction. Assume that  $\alpha = 0$  and that the statement is true for  $n = k - 1$ . Then

$$\begin{aligned} \psi^k f(x) &= \beta^{-1} [\psi^{k-1}(\beta^{-1}x) + \psi^{k-1}(\beta^{-1}x + \beta^{-1})] = \\ &= \beta^{-1} \left[ \beta^{-k+1} \sum_{i=1}^{2^{k-1}} f(\beta^{-k+1}\beta^{-1}x + \alpha_i^{(k-1)}) + \right. \\ &\quad \left. \beta^{-k+1} \sum_{i=1}^{2^{k-1}} f(\beta^{-k+1}(\beta^{-1}x + \beta^{-1}) + \alpha_i^{(k-1)}) \right] \\ &= \beta^{-k} \sum_{i=1}^{2^k} [f(\beta^{-k}x + \alpha_i^{(k-1)}) + f(\beta^{-k}x + \beta^{-k} + \alpha_i^{(k-1)})] = \\ &= \beta^{-k} \sum_{i=1}^{2^k} f(\beta^{-k}x + \alpha_i^{(k)}). \end{aligned}$$

This shows that the statement is true for  $n = k$ . If  $\alpha \neq 0$ , the lemma follows from the observation that, where  $\psi_1$  is the operator corresponding to  $\phi(x) = (\beta x + \alpha) \pmod{1}$ , and  $\psi$  is the operator corresponding to  $\phi(x) = \beta x \pmod{1}$ ,  $\psi_1 f(x) = \psi f(x - \alpha)$ .

This lemma permits us to estimate asymptotically the number  $\#(n)$  defined above. The problem of determining  $\#(n)$  for arbitrary  $n$  is apparently unsolved and seems to be very difficult.

**THEOREM 1**  $\#(n) = O(\beta^n)$

**PROOF** Take  $\alpha = 0$ .  $\psi^n f(\alpha) = \beta^{-n} \sum_{i=1}^{2^n} f(\alpha_i^{(n)})$ . Take  $f = \chi_{[0,1]}$ , then  $\psi^n f(\alpha) = \beta^{-n} \#(n)$ .  $\psi^n$  converges to an invariant measure, because  $\psi$  is ergodic (Sinai, 1981). Thus  $\#(n) = O(\beta^n)$ .

### 3 APPROXIMATION WITH INTERVAL MAPS

Computer simulations have played a large role in the development of the theory of chaotic dynamical systems. One way to simulate the probabilistic behavior of a chaotic map like  $\phi$  is to approximate the map by a sequence of interval maps.

Given a measure  $f$  so that  $\text{supp}(f) = [0, 1]$ , let  $P_f^{(n)} = P^{(n)} = (P_1^{(n)}, \dots, P_n^{(n)})^T$ , where  $P_i = \int_{(i-1)/n}^{i/n} f(x) dx$ . Let  $\psi_n$  be the  $n \times n$  matrix defined by  $(\psi_n P^{(n)})_j = \int_{(j-1)/n}^{j/n} \psi f(x) dx$ .

LEMMA 3.  $\exists k > 0$  so that  $(\psi_n^k)_{ij} > 0$  for all  $i, j$ .

PROOF.  $(\psi_n^k)_{ij}$  is the probability that, after  $k$  iterations of  $\phi$ , the image of a point selected from a uniform distribution on  $\left[\frac{i-1}{n}, \frac{i}{n}\right]$  is in  $\left[\frac{j-1}{n}, \frac{j}{n}\right]$ . Because  $\phi$  is ergodic, for any two open intervals  $U$  and  $V$  there is some  $k$  so that the Lebesgue measure of  $\psi(U) \cap V$  is nonzero.

THEOREM 2.  $\psi_n$  has a unique fixed point  $\tilde{P}^{(n)}$ .

PROOF. Since  $\|\psi_n\|_1 = 1$ ,  $P(\psi_n) \leq 1$ . And since  $\sum_{j=1}^n (A_n)_{ij} = 1$ , 1 is an eigenvalue of  $A_n^T$  with corresponding eigenvector  $(1, \dots, 1)$ , and hence 1 is an eigenvalue of  $A_n$  and  $\rho(A_n) = 1$ . According to a standard linear algebra result, Lemma 3 implies that the multiplicity of 1 is 1. Thus there is a unique eigenvector  $\tilde{P}^{(n)}$  corresponding to the eigenvalue 1.

Given this result, it is easy to see that these  $\tilde{P}^{(n)}$  converge to the invariant measure. For any  $x \in [0, 1]$ , let  $I_n(x)$  denote the interval  $\left[\frac{i-1}{n}, \frac{i}{n}\right]$  which contains  $x$ . Let  $S^n$  denote the set of all step functions constant on each interval  $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ . Let  $\tilde{f}^{(n)}$  be the element of  $S^n$  naturally induced by  $\tilde{P}^{(n)}$ .

Then we have

THEOREM 3.  $\lim_{n \rightarrow \infty} \tilde{f}^{(n)} = \tilde{f}$ .

These results show that the interval maps  $\psi_n$  are qualitatively faithful representations of  $\psi$ . The chaotic behavior of  $\phi$  is necessarily absent from any discrete approximation, but the probabilistic implications of this chaos are accurately mirrored.

### 4. DYNAMICS

DEFINITION 1. Where  $s = s_1 \dots s_n \dots$  is a binary sequence, let

$B_\beta(s) = \beta^{-1}s_1 + \beta^{-2}s_2 + \dots + \beta^{-n}s_n + \dots$  Let  $F_\beta$  be the set of all  $s = s_1 \dots s_n$  or  $s = s_1 s_2 \dots$  so that  $B_\beta(s) \geq 1$ . Let  $A_\beta$  be the set of all  $s = s_1 s_2 \dots$  so that  $s_k s_{k+1} \dots s_{k+m} \notin F_\beta$ ,  $s_k s_{k+1} \dots \notin F_\beta$ , for any  $k$  and  $\phi$ .

$F_\beta$  is the set of all "forbidden subsequences", and  $A_\beta$  is the set of all sequences containing no forbidden subsequences.

LEMMA 4. For each  $x \in [0, 1]$  there is a unique binary sequence  $s \in A_\beta$  so that  $B_\beta(s) = x$ .

PROOF. Existence is clear, one forms an expansion  $B_\beta$  exactly as one forms an expansion in an integer base. To show uniqueness, assume  $x = B_\beta(s_1 s_2 \dots s_{n-1} 1 s_{n+1} \dots)$  and  $x = B_\beta(s_1 s_2 \dots s_{n-1} 0 s'_{n+1} s'_{n+2} \dots)$ . Then the second expansion is not in  $A_\beta$ .

Chaos may be defined in many different ways. Here we will adopt the topological approach found in Devaney (1989).

DEFINITION 2. A function  $r: [0, 1] \rightarrow [0, 1]$  is chaotic if i) it is topologically transitive, ii) its periodic points are dense, iii) it is sensitive to initial conditions.

LEMMA 5.  $\phi: [0, 1] \rightarrow [0, 1]$  is topologically conjugate to the shift map  $\sigma: A_\beta \rightarrow A_\beta$ , where  $\sigma(s_1 s_2 s_3 \dots) = s_2 s_3 \dots$

PROOF. Follows from Lemma 4 by standard arguments.

**THEOREM 4**  $\phi$  is chaotic on  $[0,1] \rightarrow [0,1]$

**PROOF.** i)  $\phi$  is topologically transitive because, where  $\{w_1 w_2 w_3, \dots\}$  is an enumeration of  $A_\beta$ ,  $s = w_1 00 w_2 00 w_3 00 \dots$  is in  $A_\beta$  and  $B_\beta(s)$  is a dense orbit. ii) The set of periodic points in  $\phi$  of period  $n$  is  $\{B_\beta(s) : s \in A_\beta, s = s_1 \dots s_n, s_i \in \{0,1\}\}$ . For example, everything of the form  $B_\beta(\overline{s_1 s_2 \dots s_n 00})$  is periodic, and these points are clearly dense. iii) Follows from standard arguments

**THEOREM 5** If  $\alpha$  and  $\beta$  are rational, all periodic points of  $\phi$  are rational.

**PROOF.** if  $\alpha = 0$ , a little algebra shows that all periodic points are of the form  $[s_1 \beta^{-1} + \dots + s_n \beta^{-n}] / [\beta^n - 1]$ ,  $s_i \in \{0,1\}$ . The general case follows similarly

These results can also be obtained by more conventional methods (Collet and Eckmann, 1980).

However the present techniques tell us considerably more about the periodic points of  $\phi$  than Theorem 4 requires. Let  $Per_n(\phi)$  denote the set of periodic points of  $\phi$  of period  $n$ . For  $\alpha = 0$  and  $\beta$  integral, it is obvious that  $Card[Per_n(\phi)] = \beta^n$ . This result cannot be directly generalized to the case of nonintegral  $\beta$ , since  $Card[Per_n(\phi)]$  is integral but  $\beta^n$  is not. However, the following result is an asymptotic generalization.

**THEOREM 6.**  $Per_n(\phi) = O(\beta^n)$

**PROOF.**  $\{B_\beta(s) : s \in A_\beta, s = \overline{s_1 \dots s_{n-2}}, s_i \in \{0,1\}\} \subset Per_n(\phi)$ , but

$Per_n(\phi) \subset \{B_\beta(s) : s \in A_\beta, s = \overline{s_1 \dots s_n}, s_i \in \{0,1\}\}$ . So  $\#(n-2) \leq Card[Per_n(\phi)] \leq \#(n)$ , and the result follows from Theorem 1.

**NOTE:** Since submitting this paper, the authors have become aware of unpublished work by Leo Flatto which contains the result  $Card[Per_n(\phi)] = O(\beta^n)$  for the case  $\alpha = 0$ . Flatto's techniques are combinatorial rather than operator-theoretic, and apparently cannot be extended to the case  $\alpha \neq 0$ . We would like to thank J. Lagarias for informing us of Flatto's work.

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