

## AN APPLICATION OF KKM-MAP PRINCIPLE

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**ABSTRACT.** The following theorem is proved and several fixed point theorems and coincidence theorems are derived as corollaries. Let  $C$  be a nonempty convex subset of a normed linear space  $X$ ,  $f: C \rightarrow X$  a continuous function,  $g: C \rightarrow C$  continuous, onto and almost quasi-convex. Assume that  $C$  has a nonempty compact convex subset  $D$  such that the set

$$A = \{y \in C: \|g(x) - f(y)\| \geq \|g(y) - f(y)\| \text{ for all } x \in D\}$$

is compact.

Then there is a point  $y_0 \in C$  such that  $\|g(y_0) - f(y_0)\| = d(f(y_0), C)$ .

**KEY WORDS AND PHRASES.** Almost quasi-convex functions, fixed points, coincidence points.  
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### 1. INTRODUCTION.

There has been given a variety of applications of KKM-map principle by Ky Fan [1] in areas like fixed point theory, approximation theory, minimax theory, potential theory and variational problems. For further applications we refer to [2].

Recently Prolla [4] proved the following result using fixed point theorems for multivalued mappings. In this note we extend his theorem and our proof will follow KKM-map principle.

Let  $C$  be a compact, convex subset of a Banach space  $X$ ,  $f: C \rightarrow X$  a continuous function and  $g: C \rightarrow C$  a continuous, almost affine and onto map. Then there is a  $y_0 \in C$  such that

$$\|g(y_0) - f(y_0)\| = d(f(y_0), C).$$

Recall that a map  $g: C \rightarrow X$  is almost affine if

$$\|g(\lambda x_1 + (1 - \lambda)x_2) - y\| \leq \lambda \|g(x_1) - y\| + (1 - \lambda) \|g(x_2) - y\|$$

for all  $x_1, x_2 \in C$  and  $y \in X$ .

Clearly a linear map is almost affine, but not conversely.

We have taken an almost quasi-convex map  $g$ .

**DEFINITION.** A map  $g: C \rightarrow X$  is said to be almost quasi-convex if, for every  $t \in X$  and  $r > 0$ , the set  $\{u \in C: \|g(u) - t\| < r\}$  is convex.

An almost quasi-convex condition is more general than almost affine condition.

We use the following well-known result (Lin [3]) to derive our theorem given below.

**THEOREM 1.1.** Let  $C$  be a nonempty convex subset of a topological vector space. Let  $B \subset C \times C$  be such that

- i) for each  $x \in C$ , the set  $\{y \in C : (x, y) \in B\}$  is closed in  $C$ ;
- ii) for each  $y \in C$  the set  $\{x \in C : (x, y) \notin B\}$  is empty or convex;
- iii)  $(x, x) \in B$  for each  $x \in C$ ; and
- iv)  $C$  has a nonempty compact convex subset  $D$  such that the set

$$A = \{y \in C : (x, y) \in B \text{ for all } x \in D\}$$

is compact.

Then there exists a point  $y_0 \in C$  such that  $C \times \{y_0\} \subseteq B$ .

## 2. MAIN RESULTS.

Now we prove our results.

**THEOREM 2.1.** Let  $C$  be a nonempty convex subset of a normed linear space  $X$ ,  $f: C \rightarrow X$  a continuous function,  $g: C \rightarrow C$  continuous, onto and almost quasi-convex function. (\*) Assume that  $C$  has a nonempty compact convex subset  $D$  such that the set

$$A = \{y \in C : \|g(x) - f(y)\| \geq \|g(y) - f(y)\| \text{ for all } x \in D\}$$

is compact.

Then there is a point  $y_0 \in C$  such that  $\|g(y_0) - f(y_0)\| = d(f(y_0), C)$ .

**PROOF.** Set

$$B = \{(x, y) \in C \times C : \|g(x) - f(y)\| \geq \|g(y) - f(y)\|\}.$$

Then the set  $\{y \in C : (x, y) \in B\}$  is closed in  $C$  since  $f$  and  $g$  are continuous. It is easy to see that  $(x, x) \in B$  for each  $x \in C$ .

We have to show that the set

$$M = \{x \in C : (x, y) \notin B\} = \{x \in C : \|g(x) - f(y)\| < \|g(y) - f(y)\|\}$$

is convex or empty.

Since  $g$  is an almost quasi-convex function, therefore  $M$  is convex.

By Theorem 1.1 we get that there is a point  $y_0 \in C$  such that

$$\|g(y_0) - f(y_0)\| = d(f(y_0), C).$$

In case the convex set  $C$  is compact we may take  $C = D$ .

**NOTE.** Condition (\*) is equivalent to the following.

Let  $D$  be a nonempty compact convex subset of  $C$ ,  $K$  be a nonempty compact subset of  $C$  such that for each  $y \in C \setminus K$  there exists an  $x_0 \in D$  such that

$$\|g(x_0) - f(y)\| < \|g(y) - f(y)\|.$$

If  $C = K = D$  and  $g$  is almost affine then we get Prolla's result stated below. Let  $C$  be a compact convex subset of a normed linear space  $X$  and  $f: C \rightarrow X$  a continuous function. Let  $g: C \rightarrow X$  be a continuous, onto and almost affine map. Then there exists a  $y_0 \in C$  such that

$$\|g(y_0) - f(y_0)\| = d(f(y_0), C).$$

**NOTE.** (i) If  $f(y_0) \in C$  then we get a coincidence result; and (ii) If  $g = I$ , an identity function, then the above result is a well-known theorem due to by Ky Fan [1]. This theorem has interesting applications in fixed point theory, approximation theory and variational problems. We give a sample application in fixed point theory.

EXAMPLE: Let  $C$  be a compact convex subset of a normed linear space  $X$  and  $f: C \rightarrow X$  a continuous map. If  $f(x) \neq x$ , then assume that the line segment  $[x, f(x)]$  has at least two elements of  $C$ . Then  $f$  has a fixed point.

By taking  $g = I$ , we get there is a  $y_0 \in C$  such that

$$\|y_0 - f(y_0)\| = d(f(y_0), C).$$

Now, if  $y_0 \neq f(y_0)$  then there is a  $z \in C$  such that

$$z = \lambda f(y_0) + (1 - \lambda)y_0, \quad 0 < \lambda < 1$$

and

$$\begin{aligned} \|y_0 - f(y_0)\| &\leq \|z - f(y_0)\| = \|\lambda f(y_0) + (1 - \lambda)y_0 - f(y_0)\| \\ &= (1 - \lambda) \|f(y_0) - y_0\| < \|f(y_0) - y_0\| \end{aligned}$$

a contradiction, so  $y_0 = f(y_0)$ .

We could derive several other interesting results on fixed point theorems as corollaries.

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