

TOTALLY REAL SURFACES IN CP^2 WITH PARALLEL MEAN CURVATURE VECTOR

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Abstract. It has been shown that a totally real surface in CP^2 with parallel mean curvature vector and constant Gaussian curvature is either flat or totally geodesic.

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1. INTRODUCTION.

Let J be the almost complex structure on CP^2 and g be the Hermitian metric on CP^2 of constant holomorphic sectional curvature 4. If $\bar{\nabla}$ is the Riemannian connection with respect to g and \bar{R} is the curvature tensor of $\bar{\nabla}$, then

$$(\bar{\nabla}_X J)(Y) = 0, \quad (1.1)$$

$$\bar{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ, \quad (1.2)$$

where X, Y, Z are vector fields on CP^2 .

Let M be a 2-dimensional totally real submanifold of CP^2 and v be the normal bundle of M . If $\chi(M)$ is the lie-algebra of vector fields on M , then for each $X \in \chi(M)$, $JX \in v$. The Riemannian connection $\bar{\nabla}$ induces the Riemannian connection ∇ on M and the connection ∇^{\perp} in the normal bundle v . We then have the following Gauss and Weingarten formulae

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X N = -A_N X + \nabla_X^{\perp} N, \quad X, Y \in \chi(M), \quad N \in v, \quad (1.3)$$

where $h(X, Y)$ and $A_N X$ are the second fundamental forms and are related by $g(h(X, Y), N) = g(A_N X, Y)$. The mean curvature vector H of M is given by

$$H = (1/2) \sum h(e_i, e_i),$$

where $\{e_1, e_2\}$ is a local orthonormal frame on M . If $H = 0$, then M is said to be a minimal submanifold of CP^2 . It is known that if M is a minimal totally real surface of constant Gaussian curvature in CP^2 , then either M is flat or totally geodesic (cf. [2]). The mean curvature vector H is said to be parallel if $\nabla_X^{\perp} H = 0$, $X \in \chi(M)$. In this paper we consider the totally real surfaces of constant Gaussian curvature with parallel mean curvature vector in CP^2 .

The Gaussian curvature K of M is given by

$$K = 1 + g(h(X, X), 2h(Y, Y)) - g(h(X, Y), h(X, Y)), \quad (1.4)$$

where $\{X, Y\}$ is an orthonormal frame on M . The Codazzi equation gives

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \quad X, Y, Z \in \chi(M). \quad (1.5)$$

For a totally real surface M , using (1.1) and (1.3), we get

$$h(X, Y) = JA_{JY}X, \quad \nabla_X^\perp JY = J\nabla_X Y, \quad X, Y \in \chi(M). \quad (1.6)$$

Using (1.6) and the symmetry of $h(X, Y)$, we have

$$g(h(Y, Z), JX) = g(h(X, Y), JZ) = g(h(X, Z), JY), \quad X, Y, Z \in \chi(M). \quad (1.7)$$

2. MAIN RESULTS

THEOREM 2.1. Let M be a connected totally real surface in CP^2 of constant Gaussian curvature c with parallel mean curvature vector. Then either M is flat or totally geodesic.

PROOF. Let $UM = \{X \in TM : \|X\| = 1\}$ be the unit tangent bundle of M . Define the function $f: UM \rightarrow \mathbb{R}$ by $F(X) = g(h(X, X), JX)$, which is clearly a smooth function. First suppose that f is constant. Then $f(-X) = -f(X)$ gives $f(X) = 0$ and therefore $g(h(X, X), JX) = 0$, $X \in UM$. Now consider a local orthonormal frame $\{X, Y\}$ on M . Then we have $g(h(X, X), JX) = 0$, $g(h(Y, Y), JY) = 0$,

$$g\left(h\left(\frac{X+Y}{\sqrt{2}}, \frac{X+Y}{\sqrt{2}}\right), J\left(\frac{X+Y}{\sqrt{2}}\right)\right) = 0, \quad g\left(h\left(\frac{X-Y}{\sqrt{2}}, \frac{X-Y}{\sqrt{2}}\right), J\left(\frac{X-Y}{\sqrt{2}}\right)\right) = 0$$

These equations, in view of (1.7), imply that $g(h(X, X), JY) = 0$, $g(h(Y, Y), JX) = 0$, $g(h(X, Y), JX) = 0$, and $g(h(X, Y), JY) = 0$. Since $\{JX, JY\}$ is a local orthonormal frame in the normal bundle ν , we conclude that $h(X, X) = 0$, $h(X, Y) = 0$ and $h(Y, Y) = 0$, which means that M is totally geodesic.

We therefore assume that f is not a constant. Since the unit tangent bundle UM is compact, f attains a maximum at some $e_1 \in UM$. It is known that $g(h(e_1, e_1), JY) = 0$ for any vector in TM which is orthogonal to e_1 (cf. [1]). Choose e_2 such that $\{e_1, e_2\}$ is an orthonormal frame on M . Then we can set

$$h(e_1, e_1) = \alpha Je_1, \quad h(e_2, e_2) = \beta Je_2, \quad \text{and} \quad h(e_1, e_2) = \gamma Je_2, \quad (2.1)$$

where α, β and γ are smooth functions. Using the structure equations of M we have locally

$$\nabla_{e_1} e_1 = ae_2, \quad \nabla_{e_2} e_2 = be_1, \quad \nabla_{e_1} e_2 = -ae_1, \quad \nabla_{e_2} e_1 = -be_2, \quad (2.2)$$

where a, b are smooth functions. Inserting different combinations of the frame vectors e_1, e_2 in (1.5) and using (2.1) and (2.2) we get, upon equating components,

$$e_1 \cdot \beta = a\gamma + 2b\beta - b\alpha, \quad e_2 \cdot \alpha = a(\alpha - 2\beta), \quad e_2 \cdot \beta - e_1 \cdot \gamma = 3a\beta - b\gamma. \quad (2.3)$$

Since the mean curvature vector $H = (1/2)(h(e_1, e_1) + h(e_2, e_2))$ is parallel, we have

$$\nabla_{e_1}^\perp (h(e_1, e_1) + h(e_2, e_2)) = 0 \quad \text{and} \quad \nabla_{e_2}^\perp (h(e_1, e_1) + h(e_2, e_2)) = 0.$$

Using (1.6), (2.1) and (2.2) in the above equations we conclude, upon equating components, that

$$e_1 \cdot (\alpha + \beta) = a\gamma, \quad e_1 \cdot \gamma = -a(\alpha + \beta) \quad (2.4)$$

$$e_2 \cdot (\alpha + \beta) = -b\gamma, \quad e_2 \cdot \gamma = b(\alpha + \beta). \quad (2.5)$$

From (2.3), (2.4) and (2.5), we have

$$\begin{aligned} e_1 \cdot \alpha &= b(\alpha - 2\beta), & e_1 \cdot \beta &= a\gamma + 2b\beta - b\alpha, & e_1 \cdot \gamma &= -a(\alpha + \beta), \\ e_2 \cdot \alpha &= a(\alpha - 2\beta), & e_2 \cdot \beta &= -b\gamma + 2a\beta - a\alpha, & e_2 \cdot \gamma &= b(\alpha + \beta). \end{aligned} \quad (2.6)$$

In view of (2.1) and (1.4), the Gaussian curvature c is given by $c = 1 + \alpha\beta - \beta^2$. If we operate on this equation by e_1 and e_2 with c constant, and use (2.6), we obtain

$$(\alpha - 2\beta)(a\gamma + b(3\beta - \alpha)) = 0 \text{ and } (\alpha - 2\beta)(-b\gamma + a(3\beta - \alpha)) = 0. \quad (2.7)$$

We have two cases:

Case (i). Suppose $\alpha \neq 2\beta$, then the two equations in (2.7) give $(a^2 + b^2)\gamma = 0$ and $(a^2 + b^2)(3\beta - \alpha) = 0$. If $a^2 + b^2 = 0$, then from (2.2) it follows that M is flat (as c is constant). If $a^2 + b^2 \neq 0$, then we have $\gamma = 0$ and $3\beta - \alpha = 0$. Since a and b cannot both be zero and $\gamma = 0$ it follows from equations (2.4) and (2.5) that $\alpha + \beta = 0$. Thus we have $\gamma = 0$ and $\alpha + \beta = 0$, which implies that $H = 0$, that is, M is minimal.

Case (ii). Suppose $\alpha = 2\beta$. Then from (2.6) we get that α is constant, and consequently β is also constant. With $\alpha = 2\beta$ and β constant equations (2.6) give $a\gamma = 0$ and $b\gamma = 0$. Thus either $a = b = 0$ or $\gamma = 0$, which results in either M being flat or $\gamma = 0$. If M is not flat, that is, not both a and b are zero, and $\gamma = 0$, then from (2.4) and (2.5) we get $\alpha + \beta = 0$. This shows that $H = 0$. Hence either M is flat or minimal. But since a minimal totally real surface is constant curvature in CP^2 is either flat or totally geodesic [2], the theorem is proved.

In the following we first prove that in any submanifold of a Riemannian manifold if the second fundamental form is parallel, then the mean curvature vector is parallel. Though this is a simple observation, it does not seem to appear in the literature and is worth mentioning. As a corollary then we obtain the same result as in Section 2 for the totally real surfaces of CP^2 with parallel second fundamental form.

THEOREM 2.2. Let M be a submanifold of a Riemannian manifold \bar{M} with parallel second fundamental form. Then the mean curvature vector of M is parallel.

PROOF. Suppose $\dim M = n$. Then for a local orthonormal frame $\{e_1, e_2, \dots, e_n\}$ of M , the mean curvature H is given by

$$H = (1/n) \sum_{i=1}^n h(e_i, e_i).$$

Since the second fundamental form is parallel we have

$$(\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0 \text{ for } X, Y, Z \in \chi(M).$$

Thus for each frame vector e_i we can write

$$\nabla_X^\perp h(e_i, e_i) = 2h(\nabla_X e_i, e_i).$$

Adding these equations we get

$$n \nabla_X^\perp H = 2 \sum_{i=1}^n h(\nabla_X e_i, e_i).$$

Let ω_j^i be the connection forms on M . Then we have

$$\nabla_X e_i = \sum_{j=1}^n \omega_j^i(x) e_j.$$

Substituting this into the above equation we get

$$n \nabla_X^\perp H = 2 \sum_{i,j=1}^n \omega_i^j(X) h(e_i, e_j).$$

Since $\omega_i^j(X) = -\omega_j^i(X)$ and $h(e_i, e_j) = h(e_j, e_i)$, we conclude that $\nabla_X^\perp H = 0$, $X \in \chi(M)$.

As a direct consequence of this theorem and the theorem in the previous section we have

COROLLARY 2.1. Let M be a connected totally real surface in CP^2 with parallel second fundamental form and constant Gaussian curvature. Then M is either flat or totally geodesic.

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