

# GENERALIZED GREEN'S FUNCTIONS FOR HIGHER ORDER BOUNDARY VALUE MATRIX DIFFERENTIAL SYSTEMS

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**ABSTRACT.** In this paper, a Green's matrix function for higher order two point boundary value differential matrix problems is constructed. By using the concept of rectangular co-solution of certain algebraic matrix equation associated to the problem, an existence condition as well as an explicit closed form expression for the solution of possibly not well-posed boundary value problems is given avoiding the increase of the problem dimension.

**KEYWORDS AND PHRASES.** Two point boundary value problem, Green's matrix function, co-solution, algebraic matrix equation, Moore-Penrose pseudoinverse.

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## 1.- INTRODUCTION.

Two-point boundary value problems for higher order matrix of differential systems of the type

$$\begin{aligned} X^{(p)} + A_{p-1} X^{(p-1)} + \dots + A_1 X^{(1)} + A_0 X &= f(t); \quad 0 \leq t \leq b \\ \sum_{j=1}^p \left\{ E_{1j} X^{(j-1)}(0) + F_{1j} X^{(j-1)}(b) \right\} &= r_1; \quad 1 \leq i \leq q. \end{aligned} \quad (1.1)$$

where  $f(t)$ ,  $X(t)$ ,  $r_i$  are matrices in  $\mathbb{C}^{n \times m}$  for  $1 \leq i \leq q$  and  $A_k$ ,  $E_{ij}$ ,  $F_{ij}$  are matrices in  $\mathbb{C}^{n \times n}$  for  $1 \leq i \leq q$ ,  $1 \leq j \leq p$ ,  $0 \leq k \leq p-1$ , appear in different physical problems [1, chap 1].

The standard approach to study such problems is based on the consideration of an extended first order problem

$$Y'(t) = C Y(t) + F(t); \quad B_a Y(a) + B_b Y(b) = R. \quad (1.2)$$

where  $Y = (X, X', \dots, X^{(p-1)})^T$ ,  $F = (0, \dots, f)^T$  are matrices in  $\mathbb{C}^{np \times m}$ ,  $B_a$  and  $B_b$  are appropriate matrices in  $\mathbb{C}^{nq \times np}$ ,  $R$  is a matrix in  $\mathbb{C}^{nq \times n}$ , and  $C$  is the companion matrix defined by

$$C = \begin{bmatrix} 0 & I & \dots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & I \\ -A_0 & -A_1 & \dots & -A_{p-1} \end{bmatrix} \quad (1.3)$$

See [1][3][14].

This classical approach has the inconvenience of the lack of explicitness due to the relationship  $X(t) = [I, 0, \dots, 0] Y(t)$ , as well as the computational cost due to the increase of the problem dimension. In particular it needs the computation of the matrix exponential  $\exp(tC)$  and it is well known that it is not an easy task [11].

These inconveniences motivates the study of some alternative approach that avoids the increase of the problem dimension. In [4], a solution for a very particular second order problem of the type (1.1) is proposed avoiding the increase of the problem dimension, however, the method is not applicable to more general problems. In a recent paper [7] a method for solving problems of the type (1.1) for the case  $p=2$ , without considering the extended system (1.2) have been proposed. Results of [7] are based on the existence of an appropriate pair of solutions of the characteristic algebraic matrix equation

$$Z^2 + A_1 Z + A_0 = 0. \quad (1.4)$$

Unfortunately, equation (1.4) may be unsolvable [6] and in such case, the method given in [7] is not available.

The aim of this paper is to study an existence condition for the solution of problem (1.1) as well as an explicit expression of a solution of the problem in terms of a generalized Green's matrix function  $G(t,s)$ , taking advantage of the ideas developed in [7] but without the restriction of the existence of solutions of the associated algebraic matrix equation

$$Z^p + A_{p-1} Z^{p-1} + \dots + A_1 Z + A_0 = 0. \quad (1.5)$$

The paper is organized as follows. In section 2, we introduce the concept of rectangular co-solution for the equation (1.5) and we state some results recently given [8], that will be used in the following sections. In section 3, we construct a generalized Green's matrix function of problem (1.1) by using an appropriate set of co-solutions of equation (1.5) and a procedure analogous to the one developed in [5] for the scalar case. Finally, in section 4 an explicit closed form solution of problem (1.1) in terms of a generalized Green's matrix function is given.

If  $S$  is a matrix in  $\mathbb{C}^{m \times n}$ , we denote by  $S^+$  its Moore-Penrose pseudoinverse. We recall that an account of uses and properties of this concept may be found in [2] and that the computation of  $S^+$  is an easy matter using MATLAB [10].

## 2.- RECTANGULAR CO-SOLUTIONS OF POLYNOMIAL MATRIX EQUATIONS AND APPLICATIONS.

We begin by introducing the concept of rectangular co-solution of equation (1.5), recently given in [8].

**DEFINITION 2.1.** We say that  $(X, T)$  is a  $(n, q)$  co-solution of equation (1.5) if  $X \in \mathbb{C}^{n \times q}$ ,  $T \in \mathbb{C}^{q \times q}$ ,  $X \neq 0$  and

$$XT^p + A_{p-1} XT^{p-1} + \dots + A_0 X = 0. \quad (2.1)$$

**DEFINITION 2.2.** Let  $(X_i, T_i)$  be a  $(n, m_i)$  co-solution for  $1 \leq i \leq k$ . We say that  $\{(X_i, T_i), 1 \leq i \leq k\}$  is a  $k$ -complete set of co-solutions of (1.5) if the block matrix  $W = (W_{ij})$ , with  $W_{ij} = X_j T_i^{-1}$  for  $1 \leq i \leq p$ ,  $1 \leq j \leq k$ , is invertible.

**THEOREM 1.** ([8]) Let  $C$  be the companion matrix. If  $M = (M_{ij})$  with  $M_{ij} \in \mathbb{C}^{n \times m_j}$ , is a nonsingular matrix in  $\mathbb{C}^{np \times np}$ ,  $1 \leq i \leq p$ ,  $1 \leq j \leq k$ , and if the Jordan canonical form  $J$  of  $C$  is  $J = \text{diag}(J_1, \dots, J_k)$ , with  $J_j \in \mathbb{C}^{m_j \times m_j}$ ,  $m_1 + \dots + m_k = np$ , such that

$$M \text{diag}(J_1, \dots, J_k) = C M \quad (2.2)$$

then  $\{(M_{is}, J_s), 1 \leq s \leq k\}$  is a  $k$ -complete set of co-solutions of (1.5).

**COROLLARY 1.** ([8]) Let us suppose the notation of theorem 1, and let  $\{(M_{is}, J_s), 1 \leq s \leq k\}$  be a  $k$ -complete set of co-solutions of equation (1.5). Then, the general solution of the matrix differential equation (1.1) is given by

$$X(t) = \sum_{s=1}^k M_{is} \exp(t J_s) D_s \quad (2.3)$$

where  $D_s$  is an arbitrary matrix in  $\mathbb{C}^{m_s \times m_s}$ . If  $W$  is the block partitioned matrix associated to the set  $\{(M_{is}, J_s), 1 \leq s \leq k\}$  by definition 2.2, the only solution of (1.1) satisfying the Cauchy conditions  $X^{(j)}(0) = C_j$ ,  $0 \leq j \leq p-1$ , is given by (2.3), where the matrices  $D_s$ , for  $1 \leq s \leq k$ , are uniquely determined by the expression

$$\begin{bmatrix} D_1 \\ \vdots \\ D_k \end{bmatrix} = W^{-1} \begin{bmatrix} C_0 \\ \vdots \\ C_{p-1} \end{bmatrix}. \quad (2.4)$$

For the sake of clarity in the presentation, we recall a result about the solutions of rectangular systems of equations, that will be used in the following sections.

**THEOREM 2.** ([13, p.24]) The matrix system  $SP=Q$ , where  $S$ ,  $P$ ,  $Q$  are matrices in  $\mathbb{C}^{m \times n}$ ,  $\mathbb{C}^{n \times r}$  and  $\mathbb{C}^{m \times r}$  respectively, is compatible if and only if  $S S^* Q = Q$  and in this case, the solution of the system is given by

$$P = S^* Q + (I - S^* S)Z,$$

where  $Z$  is an arbitrary matrix in  $\mathbb{C}^{n \times r}$ .

Note that under the conditions of theorem 2, a particular solution of system  $SP=Q$  is given by  $P=S^+Q$ .

### 3.- CONSTRUCTION OF GREEN'S MATRIX FUNCTIONS.

Let us consider the homogeneous problem

$$X^{(p)} + A_{p-1} X^{(p-1)} + \dots + A_1 X^{(1)} + A_0 X = 0; \quad 0 \leq t \leq b \quad (3.1)$$

$$\sum_{j=1}^p \left\{ E_{1j} X^{(j-1)}(0) + F_{1j} X^{(j-1)}(b) \right\} = 0; \quad 1 \leq i \leq q. \quad (3.2)$$

and let  $\{(M_{1i}, J_i), 1 \leq i \leq k\}$  be the  $k$ -complete set of co-solutions of equation (1.5) provided by corollary 1. Then, the general solution of equation (3.1) is given by

$$X(t) = \sum_{i=1}^k U_i(t) D_i \quad (3.3)$$

where  $D_i$  is an arbitrary matrix in  $\mathbb{C}^{m \times m}$  and

$$U_i(t) = M_{1i} \exp(tJ_i), \\ i = 1, \dots, k.$$

Let us consider the matrix function  $G(t,s)$  defined by

$$G(t,s) = \begin{cases} \sum_{i=1}^k U_i(t) P_i(s), & 0 \leq t \leq s \\ \sum_{i=1}^k U_i(t) Q_i(s), & s \leq t \leq b \end{cases} \quad (3.5)$$

where the  $\mathbb{C}^{m \times m}$  valued matrix functions  $P_i(s)$ ,  $Q_i(s)$  have to be determined so that

1.-  $G(t,s)$  is a continuous matrix function in  $[0,b] \times [0,b]$  and moreover,  $\partial^{(j)} G / \partial t^{(j)}$  is a continuous function in  $(t,s)$ , for  $(t,s)$  in the triangles  $0 \leq t < s \leq b$  and  $0 \leq s < t \leq b$  for  $j=1, \dots, p-2$ .

2.- If  $I$  is the identity matrix in  $\mathbb{C}^{n \times n}$ , one gets the jump discontinuity

$$\frac{\partial^{(p-1)} G}{\partial t^{(p-1)}}(s+0, s) - \frac{\partial^{(p-1)} G}{\partial t^{(p-1)}}(s-0, s) = I. \quad (3.6)$$

3.- As a function of  $t$ ,  $G(t,s)$  satisfies (3.1) and (3.2) in  $[0,b]$ , if  $t \neq s$ .

From (3.5) the continuity condition at  $t=s$  of Green's function gives us that

$$\sum_{i=1}^k U_i(s) P_i(s) = \sum_{i=1}^k U_i(s) Q_i(s),$$

or

$$\sum_{i=1}^k U_i(s)(P_i(s) - Q_i(s)) = 0. \quad (3.7)$$

On the other hand, by the continuity condition of the partial derivatives of the Green's function until order  $p-2$  at  $t=s$ , we obtain

$$\sum_{i=1}^k U_i^{(j)}(s)P_i(s) = \sum_{i=1}^k U_i^{(j)}(s)Q_i(s), \quad j = 1, \dots, p-2$$

and then

$$\sum_{i=1}^k U_i^{(j)}(s)(P_i(s) - Q_i(s)) = 0, \quad j = 1, \dots, p-2. \quad (3.8)$$

From (3.5) and (3.6) it follows that

$$\sum_{i=1}^k U_i^{(p-1)}(s)(P_i(s) - Q_i(s)) = I. \quad (3.9)$$

Let us write

$$R_i(s) = P_i(s) - Q_i(s), \quad i = 1, \dots, k \quad (3.10)$$

then, conditions (3.7) - (3.10) may be written in the compact form

$$U(s) \begin{bmatrix} R_1(s) \\ \vdots \\ R_k(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \quad (3.11)$$

where

$$U(s) = \begin{bmatrix} U_1(s) & \dots & U_k(s) \\ \vdots & & \vdots \\ U_1^{(p-1)}(s) & \dots & U_k^{(p-1)}(s) \end{bmatrix} \quad (3.12)$$

Note that the matrix function  $U(s)$  defined by (3.12) is invertible for all  $s$ , because we may decompose  $U(s)$  in the form

$$U(s) = W \operatorname{diag} \{ \exp(sJ_i), 1 \leq i \leq k \} \quad (3.13)$$

where

$$W = \begin{bmatrix} M_{11} & \dots & M_{1k} \\ \vdots & & \vdots \\ M_{11}J_1^{p-1} & \dots & M_{1k}J_k^{p-1} \end{bmatrix} \quad (3.14)$$

is invertible since  $\{(M_{1i}, J_i), 1 \leq i \leq k\}$  is a  $k$ -complete set of co-solution of equation (1.5).

Let us denote  $Y = [Y_{ij}]_{1 \leq i \leq p, 1 \leq j \leq k}$  with  $Y_{ij} \in \mathbb{C}^{n_1 \times n}$  the inverse of the matrix  $W$ . Then, from (3.11) and (3.13), it follows that

$$R_i(s) = \exp(-sJ_i) Y_{ik}; \quad 1 \leq i \leq k \quad (3.15)$$

If we impose that  $G(t,s)$  defined by (3.5) satisfies the initial conditions (3.2), we obtain

$$\sum_{j=1}^p \left\{ E_{ij} \sum_{n=1}^k U_n^{(j-1)}(0) P_n(s) + F_{ij} \sum_{n=1}^k U_n^{(j-1)}(b) Q_n(s) \right\} = 0, \quad (3.16)$$

$$i = 1, \dots, q.$$

From (3.10) we have

$$Q_i(s) = P_i(s) - R_i(s), \quad i = 1, \dots, k. \quad (3.17)$$

Substituting (3.17) into (3.16), it follows that

$$\sum_{j=1}^p \left\{ E_{ij} \sum_{n=1}^k U_n^{(j-1)}(0) P_n(s) + F_{ij} \sum_{n=1}^k U_n^{(j-1)}(b) [P_n(s) - R_n(s)] \right\} = 0$$

$$i = 1, \dots, q,$$

and

$$\begin{aligned} & \sum_{n=1}^k \sum_{j=1}^p \left( E_{ij} U_n^{(j-1)}(0) + F_{ij} U_n^{(j-1)}(b) \right) P_n(s) = \\ & = \sum_{n=1}^k \sum_{j=1}^p F_{ij} U_n^{(j-1)}(b) R_n(s), \quad i = 1, \dots, q. \end{aligned} \quad (3.18)$$

Let  $S$  be the block matrix

$$S = \left[ \sum_{j=1}^p \left( E_{ij} U_n^{(j-1)}(0) + F_{ij} U_n^{(j-1)}(b) \right) \right]_{1 \leq i \leq q, 1 \leq n \leq k} \quad (3.19)$$

and let  $S^+$  be the Moore-Penrose pseudoinverse matrix

$$S^+ = [T_{ni}]_{1 \leq n \leq k, 1 \leq i \leq q}, \quad \text{with } T_{ni} \in \mathbb{C}^{n_1 \times n}. \quad (3.20)$$

Note that the conditions (3.18) may be written in the form

$$S \begin{bmatrix} P_1(s) \\ \vdots \\ P_k(s) \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^k \sum_{j=1}^p F_{1j} U_m^{(j-1)}(b) R_m(s) \\ \vdots \\ \sum_{m=1}^k \sum_{j=1}^p F_{kj} U_m^{(j-1)}(b) R_m(s) \end{bmatrix} \quad (3.21)$$

From theorem 2, the equation (3.21) is solvable if and only if

$$S S^+ \begin{bmatrix} \sum_{m=1}^k \sum_{j=1}^p F_{1j} U_m^{(j-1)}(b) R_m(s) \\ \vdots \\ \sum_{m=1}^k \sum_{j=1}^p F_{kj} U_m^{(j-1)}(b) R_m(s) \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^k \sum_{j=1}^p F_{1j} U_m^{(j-1)}(b) R_m(s) \\ \vdots \\ \sum_{m=1}^k \sum_{j=1}^p F_{kj} U_m^{(j-1)}(b) R_m(s) \end{bmatrix} \quad (3.22)$$

Let us suppose the algebraic equation (3.21) is compatible. Then, from theorem 2 and (3.20) a solution of (3.21) is given by

$$\begin{bmatrix} P_1(s) \\ \vdots \\ P_k(s) \end{bmatrix} = \begin{bmatrix} T_{11} & \dots & T_{1q} \\ \vdots & & \vdots \\ T_{k1} & \dots & T_{kq} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^p \sum_{m=1}^k F_{1j} U_m^{(j-1)}(b) \exp(-sJ_m) Y_{mp} \\ \vdots \\ \sum_{j=1}^p \sum_{m=1}^k F_{qj} U_m^{(j-1)}(b) \exp(-sJ_m) Y_{mp} \end{bmatrix}$$

and then,

$$P_i(s) = \sum_{j=1}^p \sum_{m=1}^k \sum_{l=1}^q T_{il} F_{lj} U_m^{(j-1)}(b) \exp(-sJ_m) Y_{mp} \quad (3.23)$$

$$i = 1, \dots, k.$$

Hence and from (3.15), (3.17) it follows that

$$\begin{aligned} Q_i(s) &= P_i(s) - R_i(s) = \\ &= \left( \sum_{l=1}^p \sum_{j=1}^p \sum_{m=1}^k T_{il} F_{lj} U_m^{(j-1)}(b) \exp(-sJ_m) Y_{mp} \right) - \exp(-sJ_i) Y_{ip} \\ & \quad i = 1, \dots, k. \end{aligned} \quad (3.24)$$

Thus the following result has been established

**THEOREM 3.** Let  $\{(M_{i1}, J_i), 1 \leq i \leq k\}$  be the  $k$ -complete set of co-solutions of equation (1.5) given by theorem 1 and let  $\{U_i(t), 1 \leq i \leq k\}$  be defined by (3.4). If condition (3.22) is given, then the boundary value matrix problem (3.1) - (3.2) has a generalized Green's matrix function defined by (3.5), where  $P_i(s)$  and  $Q_i(s)$  are given by (3.23) and (3.24).

**REMARK.** If the matrix  $S$  has full rank, then, from [2,p.12]  $S^*S=I$ , (3.21) has only one solution and there exists a unique Green's matrix function.

#### 4.- SOLUTION OF THE NON-HOMOGENEOUS BOUNDARY PROBLEM.

Let us consider the intermediate boundary value problem,

$$X^{(p)} + A_{p-1} X^{(p-1)} + \dots + A_1 X^{(1)} + A_0 X = f(t) \quad (4.1)$$

$$\sum_{j=1}^p E_{1,j} X^{(j-1)}(0) + F_{1,j} X^{(j-1)}(b) = 0, \quad i = 1, \dots, q$$

where  $f(t)$  is a  $\mathbb{C}^{n \times m}$  valued continuous matrix function in  $[0, b]$ .

Let  $X(t)$  be defined by

$$\begin{aligned} X(t) &= \int_0^b G(t,s) f(s) ds = \\ &= \int_0^t G(t,s) f(s) ds + \int_t^b G(t,s) f(s) ds. \end{aligned} \quad (4.2)$$

Taking derivatives and using the Leibniz' rule, we have

$$\begin{aligned} X'(t) &= \int_0^t \frac{\partial G(t,s)}{\partial t} f(s) ds + G(t,t) f(t) + \int_t^b \frac{\partial G(t,s)}{\partial t} f(s) ds - \\ &- G(t,t) f(t) = \int_0^b \frac{\partial G(t,s)}{\partial t} f(s) ds. \end{aligned}$$

$$\begin{aligned} X''(t) &= \int_0^t \frac{\partial^2 G(t,s)}{\partial t^2} f(s) ds + \frac{\partial G(t,t)}{\partial t} f(t) + \int_t^b \frac{\partial^2 G(t,s)}{\partial t^2} f(s) ds - \\ &- \frac{\partial G(t,t)}{\partial t} f(t) = \int_0^b \frac{\partial^2 G(t,s)}{\partial t^2} f(s) ds. \end{aligned}$$

$$X^{(p-1)}(t) = \int_0^b \frac{\partial^{(p-1)} G(t,s)}{\partial t^{(p-1)}} f(s) ds$$

$$\begin{aligned} X^{(p)}(t) &= \int_0^t \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s) ds + \frac{\partial^{(p-1)} G(t,t-0)}{\partial t^{(p-1)}} f(t) + \\ &+ \int_t^b \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s) ds - \frac{\partial^{(p-1)} G(t,t+0)}{\partial t^{(p-1)}} f(t) = \\ &= + \int_0^b \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s) ds + \left( \frac{\partial^{(p-1)} G(t,t-0)}{\partial t^{(p-1)}} - \frac{\partial^{(p-1)} G(t,t+0)}{\partial t^{(p-1)}} \right) f(t) = \\ &= \int_0^b \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s) ds + f(t). \end{aligned}$$



Hence and from the properties of  $G(t,s)$  it follows that

$$\begin{aligned} X^{(p)} + A_{p-1} X^{(p-1)} + \dots + A_1 X^{(1)} + A_0 X &= \\ &= \int_0^b \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} f(s) ds + f(t) + \\ &+ A_{p-1} \int_0^b \frac{\partial^{(p-1)} G(t,s)}{\partial t^{(p-1)}} f(s) ds + \dots + A_0 \int_0^b G(t,s) f(s) ds = \\ \int_0^b \left( \frac{\partial^{(p)} G(t,s)}{\partial t^{(p)}} + A_{p-1} \frac{\partial^{(p-1)} G(t,s)}{\partial t^{(p-1)}} + \dots + A_0 G(t,s) \right) f(s) ds &+ f(t) = \\ &= f(t). \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^p \left\{ E_{1j} X^{(j-1)}(0) + F_{1j} X^{(j-1)}(b) \right\} &= \\ \int_0^b \left( \sum_{j=1}^p \left\{ E_{1j} \frac{\partial^{(j-1)} G(0,s)}{\partial t^{(j-1)}} + F_{1j} \frac{\partial^{(j-1)} G(b,s)}{\partial t^{(j-1)}} \right\} \right) f(s) ds &= 0, \\ i = 1, \dots, q. \end{aligned}$$

Now let us consider the auxiliary problem

$$X^{(p)} + A_{p-1} X^{(p-1)} + \dots + A_1 X^{(1)} + A_0 X = 0 \quad (4.3)$$

$$\sum_{j=1}^p E_{1j} X^{(j-1)}(0) + F_{1j} X^{(j-1)}(b) = r_1, \quad (4.4)$$

$$i = 1, \dots, q.$$

Then, from the corollary 1, the form of the solutions of (4.3) is

$$X(t) = \sum_{m=1}^k U_m(t) Q_m.$$

The boundary value conditions of (4.4) give us the next expression

$$\sum_{j=1}^p \left\{ E_{1j} \sum_{m=1}^k U_m^{(j-1)}(0) Q_m + F_{1j} \sum_{m=1}^k U_m^{(j-1)}(b) Q_m \right\} = r_1$$

$$i = 1, \dots, q.$$

or

$$\sum_{j=1}^p \sum_{m=1}^k \left\{ E_{1j} U_m^{(j-1)}(0) + F_{1j} U_m^{(j-1)}(b) \right\} Q_m = r_1.$$

$$i = 1, \dots, q.$$

If we set the last expression in matrix form

$$S \begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_q \end{bmatrix} \quad (4.5)$$

From theorem 2 of section 1, under the compatibility condition

$$S S^+ \begin{bmatrix} r_1 \\ \vdots \\ r_q \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_q \end{bmatrix}, \quad (4.6)$$

and taking into account (3.20), a solution of (4.5) is given by

$$\begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} = \begin{bmatrix} T_{11} & \dots & T_{1q} \\ \vdots & & \vdots \\ T_{k1} & \dots & T_{kq} \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_q \end{bmatrix}.$$

Thus,  $Q_m = \sum_{i=1}^q T_{mi} r_i$ ,  $1 \leq m \leq k$ , and a solution  $G(t)$  of (4.3) - (4.4) is given by the next expression

$$G(t) = \sum_{m=1}^k U_m(t) \left( \sum_{i=1}^q T_{mi} r_i \right). \quad (4.7)$$

Thus, the following result has been proved:

**THEOREM 4.** Let  $\{(M_{1i}, J_i), 1 \leq i \leq k\}$  be a  $k$ -complete set of co-solutions of equation (1.5) and let  $\{U_i(t), 1 \leq i \leq k\}$  be defined by (3.4). If the conditions (3.22) and (4.6) are satisfied, i.e., the algebraic equations (3.21) and (4.5) are compatible,  $S$  is defined by (3.19),  $S^+ = [T_{mi}]_{1 \leq m \leq k, 1 \leq i \leq q}$  is the Moore-Penrose pseudo-inverse and  $f(t)$  is continuous, then the boundary value problem (1.1) has a solution given by

$$X(t) = \int_0^b G(t,s) f(s) ds + G(t),$$

where  $G(t)$ , is given by (4.6) and  $G(t,s)$  is the generalized Green's matrix function constructed by theorem 3.

**REMARK.** It is interesting to recall that the Jordan canonical form of a matrix may be efficiently computed with MACSYMA [9] and the matrix exponential of a Jordan block has a well known expression [12,p.66].

In the next example, we construct a generalized Green's matrix function for a not well-posed boundary value matrix problem.

**EXAMPLE.** Let us consider the second order differential equation,

$$X''(t) + A_1 X'(t) + A_0 X(t) = 0 \quad t \in [0,1] \quad (4.8)$$

$$E_{11} X(0) + F_{11} X(1) = 0 \quad (4.9)$$

$$E_{22} X'(0) + F_{22} X'(1) = 0$$

and

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, E_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, F_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (4.10)$$

$$E_{22} = \begin{bmatrix} 0 & 0 \\ -e & e \end{bmatrix}, F_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, the matrices  $M$ ,  $J$  and  $M^{-1}$  are given by

$$M = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

and

$$\text{diag}[\exp(sJ_1)] = e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, a complete set of co-solutions is

$$\left\{ \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right), \left( \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\},$$

and we can compute the expressions  $U_1(t)$ ,  $U_2(t)$ , their derivatives and  $R_1(t)$ ,  $R_2(t)$ .

$$U_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad U_1'(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$U_2(t) = e^t \begin{bmatrix} -1 & 1-t & -t^2/2+t-1 \\ 0 & 1 & t-1 \end{bmatrix}, \quad U_2'(t) = e^t \begin{bmatrix} -1 & -t & -t^2/2 \\ 0 & 1 & t \end{bmatrix},$$

$$R_1(s) = \exp(-sJ_1)Y_{12} = [-1, 0],$$

$$R_2(s) = \exp(-sJ_2)Y_{22} = e^{-s} \begin{bmatrix} 1 & -s + \frac{s^2}{2} \\ 0 & 1-s \\ 0 & 1 \end{bmatrix}.$$

From the boundary value conditions (4.9) the corresponding matrix  $S$  defined by (3.19) takes the form

$$S = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1+e^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & e & 0 & e \end{bmatrix}$$

that clearly is not invertible. Thus problem (4.8)-(4.10) is not well posed, however, the equality

$$S \begin{bmatrix} a & b \\ 0 & e^{-s} \\ 0 & e^{-s}(1-s) \\ 0 & e^{-s}(1-s) \end{bmatrix} = e^{1-s} \begin{bmatrix} 0 & 0 \\ 0 & 1-s \\ 0 & 0 \\ 0 & 2-s \end{bmatrix}$$

means that the corresponding algebraic equation (3.21) is compatible. Then, the Moore-Penrose pseudo-inverse is given by

$$S^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1+e^{-1} & -e^{-1} & 0 & e^{-1} \\ -e^{-1} & e^{-1} & 0 & 0 \\ -1-e^{-1} & e^{-1} & 0 & 0 \end{bmatrix}$$

Therefore, we can obtain  $P_1(s)$  and  $P_2(s)$ ,

$$P_1(s) = [0, 0],$$

$$P_2(s) = e^{-s} \begin{bmatrix} 0 & 1 \\ 0 & 1-s \\ 0 & 1-s \end{bmatrix}$$

and

$$Q_1(s) = [1, 0].$$

$$Q_2(s) = e^{-s} \begin{bmatrix} 1 & 1+s-\frac{s^2}{2} \\ 0 & 0 \\ 0 & -s \end{bmatrix}.$$

Finally, a generalized Green's matrix function of problem (4.8)-(4.10) is given by

$$G(t, s) = \begin{cases} e^{t-s} \begin{bmatrix} 0 & \frac{st^2}{2} + st - \frac{t^2}{2} - t - 1 \\ 0 & -st - s + t + 1 \end{bmatrix} & 0 \leq t < s \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{t-s} \begin{bmatrix} -1 & \frac{st^2}{2} + \frac{s^2}{2} - s - 1 \\ 0 & -st \end{bmatrix} & s < t \leq 1. \end{cases}$$

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**REFERENCES.**

- 1.- U.M. Ascher, R.M.M. Mattheij and R.D. Russell, "Numerical solution of boundary value problems for ordinary differential equations", Prentice Hall Inc., New Jersey, 1988.
- 2.- S.L. Campbell and C.D. Meyer Jr., "Generalized inverses of linear transformations", Pitman Pubs.Co., 1979.
- 3.- R. Conti, "Recent trends in the theory of boundary value problems for ordinary differential equations", Boll. Un. Mat. Ital., 22(1967), 135-178.
- 4.- K.A. Heimes, "Green's functions for linear second order systems", SIAM J. Math. Anal., 9(1978), 207-214.
- 5.- E.L. Ince, "Ordinary differential equations", Dover 1957.
- 6.- L. Jódar, "Explicit solutions for second order operator differential equations with two boundary value conditions", Linear Algebra Appl., 103(1988), 73-86.
- 7.- L. Jódar, "Boundary value problems and Green's operator functions", Glasnick Matematskiy, 24(44)(1989), 507-518.
- 8.- L. Jódar and E. Navarro, "Rectangular co-solutions of polynomial matrix equations and applications", Applied Maths. Letters, Vol No.2 (1991), 13-16.
- 9.- MACSYMA, MACSYMA Symbolics Inc., 1989.
- 10.- C.B. Moler, "Matlab user's guide", Technical report CS81-1, Department of Computer Science, Univ. of Mexico, Albuquerque, 1980.
- 11.- C. Moler and Van Loan, "Nineteen dubious ways to compute the exponential of a matrix", SIAM Review, 20(1978), 801-836.
- 12.- J.M. Ortega, "Numerical analysis. A second course", Academic Press, 1972.
- 13.- C.R. Rao and S.K. Mitra, "Generalized inverse of matrices and its applications", Wiley, New York 1971.
- 14.- W.T. Reid, "Ordinary differential equations", John-Wiley, New York 1971.

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