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THE SPACE OF HENSTOCK INTEGRABLE FUNCTIONS OF TWO VARIABLES

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ABSTRACT. We consider the space of Henstock integrable functions of two variables. Equipped with the Alexiewicz norm the space is proved to be barrelled. We give a partial description of its dual. We show by an example that the dual can't be described in a manner analogous to the one-dimensional case, since in two variables there exist functions whose distributional partials are measures and which are not multipliers for Henstock integrable functions.

KEY WORDS AND PHRASES: Henstock integral, barrel, barrelled space, normed space, continuous linear functionals.
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1. DEFINITION.

We will write $I_0 = [0,1]^2$. A function $f : I_0 \to R$ is Henstock integrable, with

$$\int_{I_0} f(x,y)dxdy \tag{1}$$

written for the value of the integral, if for every $\varepsilon > 0$ there exists a positive $\delta : I_0 \to R$ such that if

$$\pi = \{((x_i,y_i),I_i) : i = 1,2,\dots,n\} \tag{2}$$

is a partition of I_0 (i.e., I_i's are nonoverlapping subintervals of I_0 whose union is I_0) for which

$$(x_i,y_i) \in I_i \subset \Delta((x_i,y_i),\delta(x_i,y_i)), \tag{3}$$

where $\Delta((a,b),r)$ stands for the disk centered at (a,b) of radius r, then

$$\left| \sum_{i=1}^{n} f(x_i,y_i)\, \lambda(I_i) - \int_{I_0} f(x,y)dxdy \right| < \varepsilon, \tag{4}$$

where $\lambda(I_i)$ denotes the area of I_i.

We will write H for the class of Henstock integrable functions on I_0. H is a linear space. If we replace $\lambda(I_i)$, for $I_i = [a_i,b_i] \times [c_i,d_i]$, in (4) by $g(a_i,c_i) - g(a_i,d_i) - g(b_i,c_i) + g(b_i,d_i)$, for a certain $g : I_0 \to \mathbb{R}$, then we obtain the definition of the Henstock integral of f with respect to g, written as $\int_{I_0} f dg$.

Henstock integral in the plane is fully discussed in [7].

2. DEFINITION.

Let $f \in H$, set

$$\tilde{f}(x,y) = \int_{[0,x]\times[0,y]} f(s,t)dsdt. \tag{5}$$

It is shown in [3] (page 549) that $\tilde{f}$ is continuous. Let

$$||f|| = \sup_{(x,y)\in I_0} |\tilde{f}(x,y)|. \tag{6}$$

We will call (6) the Alexiewicz norm on H.

3. PROPOSITION.

$T \in H^*$ if and only if there is a finite signed Borel measure μ on $(0,1]\times(0,1]$ such that

$$T(f) = \int_{I_0} F(x,y)d\mu(x,y) \tag{7}$$

The norm of T is equal to the norm of μ.

PROOF. Let C be the space of continuous real-valued functions on I_0. Define

$$C_0 = \{F \in C: F(x,y) = 0 \text{ if } x = 0 \text{ or } y = 0\}. \tag{8}$$

Then if we assign

$$H \ni f \to \tilde{f} \in C_0 \tag{9}$$

H is mapped isomorphically into a dense subset of C_0 (since every polynomial is the indefinite Henstock integral of its second mixed partial). Thus, we can identify H^* with C_0^*. But C_0 is a closed subspace of C and $C_0^* = C^*/C_0^{\perp}$, which may be seen to be the space of finite signed Borel measures on $(0,1]\times(0,1]$. (7) follows from the general form of a continuous linear functional on C_0 .

4. DEFINITION.

A function $g: I_0 \to \mathbb{R}$ is a multiplier for H if for every $f \in H$ we have also $fg \in H$.

5. REMARK.

In the one-dimensional case the dual of the space of Henstock-integrable functions is given by the class of multipliers (see [6]). The multipliers are functions whose distributional derivatives are measures. The two-dimensional case is different.

In [4] Kurzweil defines $g: I_0 \to \mathbb{R}$ to be of strongly bounded variation if for every $x, g(x,\cdot)$ is of bounded variation, for every $y, g(\cdot,y)$ is of bounded variation, and

$$M(g) = \sup \sum_{i=1}^{n} | g(a_i,c_i) - g(a_i,d_i) - g(b_i,c_i) + g(b_i,d_i) | < +\infty \tag{10}$$

where sup is taken over all partitions (of I_0) $\{I_i\}_{i=1}^{n}$, $I_i = [a_i,b_i] \times [c_i,d_i]$, consisting of non-overlapping, nondegenerate closed intervals. Then he shows that functions of strongly bounded variation are multipliers for H, and for $f \in H$, g of strongly bounded variation.

$$| \int_{I_0} \tilde{f}(x,y) \, dg(x,y) | \le ||f|| \, M(g) , \tag{11}$$

so that every g of strongly bounded variation is a continuous linear functional on H.

The connection between this result and Proposition 3 is not known. It is not known either if functions of strongly bounded variation and those equivalent to them are the only multipliers.

6. EXAMPLE.

There exists a function $g : I_0 \to R$ whose distributional partials are measures and which is not a multiplier. Define

$$g(x,y) = \begin{cases} \sqrt[6]{x-y} & \text{for } x \geq y, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Note that Krickeberg shows in [2] that $g : I_0 \to R$ has its distributional partials being measures if and only if it is of bounded variation in the sense of Tonelli.

For g, var $g(\cdot,y) = \sqrt[6]{1-y}$, var $g(x,\cdot) = \sqrt[6]{x}$ and

$$\int_0^1 \sqrt[6]{1-y}\, dy + \int_0^1 \sqrt[6]{x}\, dx \leq 2. \tag{13}$$

So g is of bounded variation in the sense of Tonelli.

Define for $n \geq 2$

$$K_n = [1 - \frac{1}{n-1}, 1 - \frac{1}{n}]^2, \qquad L_n = \{(x,y) \in K_n : y \leq x\} \tag{14}$$

and for every $n \geq 2$ construct a continuous $f_n : K_n \to R$ such that $f_n(x,y) = -f_n(y,x)$, f_n is equal to 0 on the boundary of K_n, nonnegative on L_n and

$$\int_{L_n} f_n(x,y)dxdy = \frac{1}{\sqrt{n}}, \tag{15}$$

and $f_n(x,y) = 0$ for every $(x,y) \in K_n$ such that $|x-y| < n^{-3}$. Then for f given by

$$f(x,y) = \begin{cases} f_n(x,y) & \text{for } (x,y) \in K_n \text{ for some } n \geq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

we have $f \in \mathcal{H}$, yet $fg \notin \mathcal{H}$.

7. REMARK.

It is shown in [8] that the space of Henstock integrable function of one variable is barrelled. We will show it to be true also in two dimensions.

8. DEFINITION.

If $\mathcal{E}$ is a topological vector space then a set $\mathcal{B} \subset \mathcal{E}$ is a barrel if $\mathcal{B}$ is closed, convex, circled and radial at 0. A locally convex space in which every barrel is a neighborhood of 0 is termed a barrelled space. It should be noted that each barrel in a space $\mathcal{E}$ which is of the second category in itself is necessarily a neighborhood of 0. In particular, every Banach space is barrelled. The importance of barrelled spaces lies in the following Barrel Theorem.

9. THEOREM.

Let $\mathcal{E}$ be a barrelled space and $\mathcal{F}$ be a pointwise bounded family of continuous linear functions on $\mathcal{E}$ into a locally convex space $\mathcal{K}$. Then the family $\mathcal{F}$ is equicontinuous. Consequently, in this case, $\mathcal{F}$ is uniformly bounded on each bounded subset of $\mathcal{E}$.

PROOF. See [5] (page 104).

This theorem implies in particular that the Banach-Steinhaus Theorem holds for barrelled spaces.

10. DEFINITION.

Let S stand for the space of real-valued additive functions F of interval $I = [a,b] \times [c,d] \subset I_0$ for which there is a continuous $f:I_0 \to \mathbb{R}$ such that

$$F(I) = f(a,c) - f(a,d) - f(b,c) + f(b,d) \tag{17}$$

Notice that if $F \in S$ then there is a unique $f \in C$, such that $f(x,y) = 0$ if $x = 0$ or $y = 0$, i.e., $f \in C_0$, defining it. Let

$$||F|| = \sup_{(x,y)\in I_0} | f(x,y) |$$

where $F \in S$, and $f \in C_0$ defines it. S is a Banach space isometric to C_0.

11. THEOREM.

Let $\mathcal{X}$ be a subspace of S satisfying the following two conditions:

(a) If $F \in \mathcal{X}$ and $J \subset I_0$, and

$$F_J(I) = F(I \cap J) \tag{18}$$

for $I \subset I_0$ then $F_J \in \mathcal{X}$;

(b) If $c \in I_0$, $F \in S$, and $F_J \in \mathcal{X}$ for every $J \subset I_0$ such that if ℓ_1,ℓ_2 are the vertical and the horizontal line segments through c then $J \cap \ell_1 = \phi$, $J \cap \ell_2 = \phi$, then $F \in \mathcal{X}$.

Then $\mathcal{X}$ is barrelled.

PROOF. In the proof we will denote, for $z_1,z_2 \in R^2$, by $[z_1,z_2]$ an interval for which z_1,z_2 are opposite vertices. Let $\mathcal{B}$ be a barrel in $\mathcal{X}$. If $\mathcal{B}$ is not a neighborhood of zero, then it is nowhere dense. To show that, suppose that a barrel $\mathcal{B}$ is not nowhere dense. There is an open set $\mathcal{U}$ such that $\mathcal{U} \subset \mathcal{B}$. Since $\mathcal{B}$ is convex and circled

$$\frac{1}{2} (\mathcal{U} - \mathcal{U}) \subset \frac{1}{2} (\mathcal{B} - \mathcal{B}) = \frac{1}{2} (\mathcal{B} + \mathcal{B}) \subset \mathcal{B}. \tag{19}$$

$\mathcal{U} - \mathcal{U}$ is a neighborhood of zero, and so is $\mathcal{B}$.

For every $I \subset I_0$ write

$$\mathcal{X}(I) = \{F_I : F \in X\} \tag{20}$$

and

$$\mathcal{B}(I) = \mathcal{B} \cap \mathcal{X}(I). \tag{21}$$

Then $\mathcal{B}(I)$ is a barrel in $\mathcal{X}(I)$.

Suppose $I = I_1 \cup \ldots\ldots \cup I_n$, where $I_1,\ldots,I_n$ are nonoverlapping. Then $\mathcal{B}(I_i) \subset \mathcal{B}(I)$ for $i = 1,\ldots,n$, so if $F_i \in \mathcal{B}(I_i)$, $i = 1,\ldots,n$, then $F_i \in \mathcal{B}(I)$, and, since $\mathcal{B}(I)$ is convex,

$$\frac{1}{n} (F_1 +\ldots+ F_n) \in \mathcal{B}(I). \tag{22}$$

Consequently, $\mathcal{B}(I_1) +\ldots+ \mathcal{B}(I_n) \subset n\,\mathcal{B}(I)$. The space $\mathcal{X}(I)$ is a topological direct sum of $\mathcal{X}(I_1),\ldots,\mathcal{X}(I_n)$. If $\mathcal{B}(I_1),\ldots,\mathcal{B}(I_n)$ are neighborhoods of zero in $\mathcal{X}(I_1),\ldots,\mathcal{X}(I_n)$ (respectively) then $\mathcal{B}(I)$ is a neighborhood of zero in $\mathcal{X}(I)$. Thus, if $\mathcal{B}(I)$ is nowhere dense in $\mathcal{X}(I)$ then at least one of $\mathcal{B}(I_i)$'s, $i = 1,\ldots,n$, is nowhere dense in the corresponding $\mathcal{X}(I_i)$.

Therefore, if we divide I_0 into four subintervals by splitting the sides into halves, among so obtained intervals there is at least one, call it I_1, such that $\mathcal{B}(I_1)$ is nowhere dense in $\mathcal{X}(I_1)$. Applying the same procedure to I_1, and then continuing it, we obtain a sequence of intervals I_n such that

$$\bigcap_{n \in N} I_n = \{c\}. \tag{23}$$

where c is a certain point in I_0, and $\mathcal{B}(I_n)$ is nowhere dense in $\mathcal{X}(I_n)$ for every $n \in N$.

For every $n \in N$, write

$$I_n = I_n^1 \cup I_n^2 \cup I_n^3 \cup I_n^4 . \tag{24}$$

where I_n^i, $i = 1,2,\dots,4$ are subintervals of I_n obtained from it by drawing lines parallel to its sides and going through c. We can assume that I_n^i's are numbered so that

$$I_{n+1}^i \subset I_n^i \tag{25}$$

for every n and i. Notice that since $\mathcal{B}(I_n)$ is nowhere dense in $\mathcal{X}(I_n)$ for every n, there is at least one i such that $\mathcal{B}(I_n^i)$ is nowhere dense in $\mathcal{X}(I_n^i)$.

Consider the four sequences $\{I_n^i\}_{n \in N}$, for $i = 1,2,3,4$. If in each of them there is only finitely many $n \in N$ such that $\mathcal{B}(I_n^i)$ is nowhere dense in $\mathcal{X}(I_n^i)$ then after passing those finitely many indices we will get all four $\mathcal{B}(I_n^i)$, $i = 1,2,3,4$, being neighborhoods of zero. This will force $\mathcal{B}(I_n)$ to be a neighborhood of zero, a contradiction. Therefore, among the four sequences $\{I_n^i\}_{n \in N}$ there has to be one which produces infinitely many $\mathcal{B}(I_n^i)$'s which are nowhere dense in the corresponding $\mathcal{X}(I_n^i)$'s.

Let $\{I_n^{i_0}\}_{n \in N}$ be that sequence, and let $\{I_{n_k}^{i_0}\}_{k \in N}$ be its subsequence such that $\mathcal{B}(I_{n_k}^{i_0})$ is nowhere dense in $\mathcal{X}(I_{n_k}^{i_0})$ for every $k \in N$. Write $J_k = I_{n_k}^{i_0}$ for $k \in N$, and let $J_k = [c,x_k]$.

Let $u_1 = x_1$. There exists a function $G_1 \in \mathcal{X}(J_1)$ such that $G_1 \notin \mathcal{B}$ and $||G_1|| < 1/2$. Then since $\mathcal{B}$ is closed and $\lim_{x \to c} G_{1[x,u_1]} = G_1$ (in $\mathcal{X}$) there is a $u_2 = x_{k_2}$ (for some $k_2 \in N$) such that if $F_1 = G_{1[u_2,u_1])}$, then $F_1 \in \mathcal{X}([u_2,u_1])$, $F_1 \notin \mathcal{B}$, and $||F_1|| < 1/2$.

Proceeding by induction, if $n \in N$, then we have a function $G_n \in \mathcal{X}(J_{k_n})$ such that $G_n \notin n\mathcal{B}$ and $||G_n|| < 1/2^n$. Since is closed and

$$\lim_{x \to c} G_{n[x,u_n]} = G_n \text{ (in } \mathcal{X}) \tag{26}$$

there is a $u_{n+1} = x_{k_{n+1}}$ (for some $k_{n+1} \in N$) such that if $F_n = G_{n[u_{n+1},u_n]}$ then $F_n \in \mathcal{X}([u_{n+1},u_n]$, $F_n \in n\mathcal{B}$, and $||F_n|| < 1/2^n$.

Consider the set $\mathcal{A}$ defined as the closed convex hull of the sequence $\{F_n\}$ in S. Every element of $\mathcal{A}$ is of the form

$$F = \sum_{n=1}^{+\infty} \lambda_n F_n \tag{27}$$

for some sequence of scalars $\{\lambda_n\}$ with $\sum_{n=1}^{+\infty} |\lambda_n| \leq 1$. Take a $u \in [c_1,u_1]$, $u \neq c$, $u \neq u_1$, and notice that

$$F_{[u,u_1]} = \sum_{n=1}^{+\infty} \lambda_n F_{n[u,u_1]} . \tag{28}$$

Now only finitely many terms on the right-hand side of (28) are nonzero. Therefore for every such u, $F_{[u,u_1]} \in X([u,u_1])$. Consequently, by the condition (b), $A \subset X$. Therefore B absorbs A^1 (B is a barrel). This, however, is a contradiction, since B does not even absorb the sequence $\{F_n\}$. The proof is ended.

12. REMARK.

It is well known, and shown in [3], that

$$\tilde{H} = \{\tilde{f}: f \in H(I_0)\} \tag{29}$$

equipped with the Alexiewicz norm is a subspace of S satisfying the conditions (a), (b) of theorem 9.

13. COROLLARY.

H is barrelled.

14. COROLLARY.

If T is a pointwise bounded family of continuous linear functionals on H then T is equicontinuous, and consequently, uniformly bounded on each bounded subset of H.

15. COROLLARY.

If $\{g_n\}$ is a sequence of functions of strongly bounded variation on I_0 such that for every $f \in H$

$$\lim_{n\to\infty} \int_{I_o} \tilde{f}(x,y)dg_n(x,y) \tag{30}$$

exists, then

$$T(f) = \lim_{n\to\infty} \int_{I_o} \tilde{f}(x,y)\, dg_n(x,y) \tag{31}$$

is a continuous linear functional on H.

We were not able to prove or disprove whether the functional (31) is itself generated by a certain function of strongly bounded variation. We do not know either whether all functionals on H are of the form (31).

16. REMARK.

[8] presents a Henstock-type integral in the plane for which the classical divergence theorem holds. The integral introduced by Pfeffer integrates divergence of every differentiable vector field (unlike the Lebesgue integral).

Applying the proposition 4.10 of [8], one can show that the integral of Pfeffer satisfies the conditions (a), (b) of Theorem 11. Indefinite integral is also continuous. Thus, the space of Pfeffer-integrable functions, equipped with the Alexiewicz norm, is also barrelled.

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