

STRONG LAWS OF LARGE NUMBERS FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS

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ABSTRACT. Let $\{X_{nk}\}$ be an array of rowwise independent random elements in a separable Banach space of type $p + \delta$ with $EX_{nk} = 0$ for all k, n . The complete convergence (and hence almost sure convergence) of $n^{-1/p} \sum_{k=1}^n X_{nk}$ to 0, $1 \leq p < 2$, is obtained when $\{X_{nk}\}$ are uniformly bounded by a random variable X with $E|X|^{2p} < \infty$. When the array $\{X_{nk}\}$ consists of i.i.d. random elements, then it is shown that $n^{-1/p} \sum_{k=1}^n X_{nk}$ converges completely to 0 if and only if $E\|X_{11}\|^{2p} < \infty$.

KEY WORDS AND PHRASES. Random elements, Strong laws of large numbers, Complete Convergence, Rademacher type $p + \delta$ spaces.

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1. INTRODUCTION AND PRELIMINARIES.

Let $(E, \|\cdot\|)$ be a real separable Banach space. Let (Ω, \mathcal{A}, P) denote a probability space. A random element X in E is a function from Ω into E which is \mathcal{A} -measurable with respect to the Borel subsets $\mathcal{B}(E)$. The p^{th} absolute moment of a random element X is $E\|X\|^p$ where E is the expected value of the random variable $\|X\|^p$. The expected value of X is defined to be the Bochner integral (when $E\|X\| < \infty$) and is denoted by EX . The concepts of independence and identical distributions have direct extensions to E . A separable Banach space is said to be of (Rademacher) type p , $1 \leq p \leq 2$, if there exists a constant C such that

$$E \left\| \sum_{k=1}^n X_k \right\|^p \leq C \sum_{k=1}^n E \|X_k\|^p$$

for all independent random elements X_1, \dots, X_n with zero means and finite p^{th} moments. Every separable Hilbert space and finite-dimensional Banach space is of type 2. Every separable Banach space is at least type 1 while the ℓ^p and L^p spaces are of type $\min\{2, p\}$ for $p \geq 1$.

Throughout this paper $\{X_{nk}: 1 \leq k \leq n, n \geq 1\}$ will denote rowwise independent random elements in E such that

$$EX_{nk} = 0 \quad \text{for all } n \text{ and } k \tag{1.1}$$

and such that $\{X_{nk}\}$ are uniformly bounded by a random variable X with

$$E|X|^{2p} < \infty \quad \text{for some } 1 \leq p < 2. \tag{1.2}$$

Recall that an array $\{\lambda_{nk}\}$ of random elements is said to be uniformly bounded by a random variable X if for all n and k and for every real number $t > 0$

$$P[\|X_{nk}\| > t] \leq P[|X| > t]. \quad (1.3)$$

Note that i.i.d. random elements are uniformly bounded by $\|X_{11}\|$. The major results of this paper show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \text{ completely} \quad (1.4)$$

where complete convergence is defined (as in Hsu and Robbins [1]) by

$$\sum_{n=1}^{\infty} P\left[\left\|\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk}\right\| > \varepsilon\right] < \infty \quad (1.5)$$

for each $\varepsilon > 0$.

Erdős [2] showed that for an array of i.i.d. random variables $\{X_{nk}\}$, (1.4) holds if and only if $E|X_{11}|^{2p} < \infty$. Jain [3] obtained a uniform strong law of large numbers for sequences of i.i.d. random elements in separable Banach spaces of type 2 which would yield (1.4) with $p = 1$ for an array of i.i.d. random elements $\{X_{nk}\}$ in a type 2 space. Woyczynski [4] showed that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_k \rightarrow 0 \text{ completely} \quad (1.6)$$

for any sequence $\{X_n\}$ of independent random elements in a type $p + \delta$, $1 \leq p < 2$ and $\delta > 0$, with $EX_n = 0$ for all n which is uniformly bounded by a random variable X satisfying $E|X|^p < \infty$. Móricz, Hu and Taylor [5] showed that Erdős' result could be obtained by replacing the i.i.d. condition by the uniformly bounded condition (1.3). In addition, they showed that Jain's result for i.i.d. random elements with $p = 1$ did not require the space to be type 2 but held in all separable Banach spaces. In this paper, (1.4) is established in type $p + \delta$ spaces, $1 \leq p < 2$ and $\delta > 0$, for uniformly bounded rowwise independent random elements. For i.i.d. random elements in type $p + \delta$ spaces, it is shown that (1.4) holds if and only if $E\|X_{11}\|^{2p} < \infty$. Thus, no sharper moment conditions are possible.

2. MAJOR RESULTS.

Many authors (starting with Beck [6]) have related the strong law of large numbers for non-identically distributed, independent random elements in separable Banach spaces to the necessity of the space being of type $p + \delta$ for $1 \leq p < 2$ and some $\delta > 0$. Consequently, attention is restricted to type $p + \delta$ spaces in this paper. Three lemmas will be used in obtaining the major results. They are stated here without proof. Lemma 1 with $r = 1$ is in most textbooks while Lemma 2 is accomplished using integration by parts. Lemma 3 is in Woyczynski [4].

LEMMA 1. For any $r \geq 1$, $E|X|^r < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{r-1} P[|X| > n] < \infty.$$

More precisely, $r 2^{-r} \sum_{n=1}^{\infty} n^{r-1} P[|X| > n]$

$$\leq E|X|^r \leq 1 + r 2^r \sum_{n=1}^{\infty} n^{r-1} P[|X| > n].$$

LEMMA 2. If $r \geq 1$, then for any $p > 0$

$$E \left(|X|^r I_{[|X| \leq n^{1/p}]} \right) \leq r \int_0^{n^{1/p}} t^{r-1} P[|X| > t] dt$$

and

$$E \left(|X|^r I_{[|X| > n^{1/p}]} \right) = n^{1/p} P[|X| > n^{1/p}] + \int_{n^{1/p}}^{\infty} P[|X| > t] dt.$$

LEMMA 3. Let $1 \leq p \leq 2$ and $q \geq 1$. The following properties are equivalent:

(i) E is of type p .

(ii) There exists a C such that for all independent random elements X_1, \dots, X_n in E with $EX_k = 0, k = 1, \dots, n$,

$$E \left\| \sum_{k=1}^n X_k \right\|^q \leq C E \left(\sum_{k=1}^n \|X_k\|^p \right)^{q/p}.$$

THEOREM 4. If $\{X_{nk}\}$ is an array of rowwise independent random elements in a type $p + \delta$ space, $1 \leq p < 2$ and $\delta > 0$, which are uniformly bounded by a random variable X such that (1.1) and (1.2) holds, then

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \text{ completely.}$$

PROOF. Define

$$Y_{nk} = X_{nk} I_{[\|X_{nk}\| \leq n^{1/p}]} \quad 1 \leq k \leq n, n \geq 1. \quad (2.1)$$

Then, by Lemma 1 (with $r = 2$),

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{k=1}^n P[X_{nk} \neq Y_{nk}] &= \sum_{n=1}^{\infty} \sum_{k=1}^n P[\|X_{nk}\| > n^{1/p}] \\ &\leq \sum_{n=1}^{\infty} n P[|X| > n^{1/p}] \\ &= \sum_{n=1}^{\infty} n P[|X|^p > n] \leq 2E|X|^{2p} < \infty. \end{aligned}$$

Next, for any $\varepsilon > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} P \left[\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} - \frac{1}{n^{1/p}} \sum_{k=1}^n Y_{nk} \right\| > \varepsilon \right] \\ \leq \sum_{n=1}^{\infty} P \left[\bigcup_{k=1}^n [X_{nk} \neq Y_{nk}] \right] \\ \leq \sum_{n=1}^{\infty} \sum_{k=1}^n P[X_{nk} \neq Y_{nk}] < \infty. \end{aligned}$$

Therefore,

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} - \frac{1}{n^{1/p}} \sum_{k=1}^n Y_{nk} \right\| \rightarrow 0 \text{ completely,}$$

and it suffices to prove that

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n Y_{nk} \right\| \rightarrow 0 \text{ completely.} \quad (2.2)$$

To this end, let

$$Z_{nk} = Y_{nk} - EY_{nk} \quad (k=1,2,\dots,n; \quad n=1,2,\dots).$$

Then for $1 \leq q \leq 2p$ it follows by Hölder's inequality that

$$\begin{aligned} (E\|Z_{nk}\|^q)^{1/q} &\leq 2(E\|Y_{nk}\|^q)^{1/q} \\ &\leq 2(E\|Y_{nk}\|^{2p})^{1/(2p)} \leq 2(E|X|^{2p})^{1/(2p)}, \end{aligned}$$

so that

$$\begin{aligned} E\|Z_{nk}\|^q &\leq 2^q(E|X|^{2p})^{q/(2p)} \\ &\leq 2^{2p}(1 + E|X|^{2p}) = C_1, \text{ say.} \end{aligned} \quad (2.3)$$

Furthermore,

$$\|Z_{nk}\| \leq \|Y_{nk}\| + \|EY_{nk}\| \leq 2n^{1/p}. \quad (2.4)$$

Following the techniques of Taylor [7] in expanding a high power of a sum, let $r = p + \delta$ and v be chosen so that

$$s = \frac{v}{r} \text{ is an integer and } v > \left(\frac{1}{p} - \frac{1}{r}\right)^{-1}. \quad (2.5)$$

It is readily seen that $E\left(\sum_{k=1}^n \|Z_{nk}\|\right)^v < \infty$, so that, by Lemma 3,

$$\begin{aligned} E\left(\sum_{k=1}^n \|Z_{nk}\|\right)^v &\leq C E\left(\sum_{k=1}^n \|Z_{nk}\|^r\right)^s \\ &= C \sum_{k_1, \dots, k_s} E\left(\prod_{j=1}^s \|Z_{nk_j}\|^r\right) \end{aligned} \quad (2.6)$$

where the sum is extended for all s -tuples (k_1, \dots, k_s) with $k_j = 1, 2, \dots, n$ for each j . The general term to be considered then will have

$$q_1 \text{ of the } k\text{'s} = \xi_1, \dots, q_m \text{ of the } k\text{'s} = \xi_m;$$

$$r_1 \text{ of the } k\text{'s} = \eta_1, \dots, r_\ell \text{ of the } k\text{'s} = \eta_\ell;$$

where

$$r \leq rq_i \leq 2p, \quad rr_j > 2p, \text{ and} \quad (2.7)$$

$$\sum_{i=1}^m q_i + \sum_{j=1}^{\ell} r_j = s. \quad (2.8)$$

Clearly, $q_i = 1$. Then, using (2.3) and (2.4), we can conclude that

$$\begin{aligned} &E\left(\prod_{i=1}^m \|Z_{n\xi_i}\|^{rq_i} \prod_{j=1}^{\ell} \|Z_{n\eta_j}\|^{rr_j}\right) \\ &= \prod_{i=1}^m E\|Z_{n\xi_i}\|^{rq_i} \prod_{j=1}^{\ell} E\left(\|Z_{n\eta_j}\|^{2p} \|Z_{n\eta_j}\|^{rr_j-2p}\right) \\ &\leq C_1^{m+\ell} \prod_{j=1}^{\ell} (2n^{1/p})^{rr_j-2p} \\ &= C_1^{m+\ell} 2^{\sum_{j=1}^{\ell} (rr_j - 2p)} n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell} \end{aligned} \quad (2.9)$$

$$\leq C_1^v 2^v n^{\sum_{j=1}^x (rr_j/p) - 2\ell}$$

$$= C_2 n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell}, \text{ say.}$$

Combining all possible terms of form (2.9), we can write

$$E \left(\sum_{k=1}^n \|Z_{nk}\|^r \right)^s \quad (2.10)$$

$$\leq C_3 \sum_{q_1, \dots, q_m; r_1, \dots, r_\ell}^* \sum_{\xi_1, \dots, \xi_m; \eta_1, \dots, \eta_\ell}^{**} E \left(\prod_{i=1}^m \|Z_{n\xi_i}\|^{rq_i} \prod_{j=1}^{\ell} \|Z_{n\eta_j}\|^{rr_j} \right)$$

$$= C_3 \sum_{q_1, \dots, q_m; r_1, \dots, r_\ell}^* S \quad \text{say,}$$

where Σ^* is extended over all m -tuples (q_1, \dots, q_m) and ℓ -tuples (r_1, \dots, r_ℓ) such that Conditions (2.7) and (2.8) are satisfied (the cases $m = 0$ or $\ell = 0$ may also occur), while Σ^{**} is extended over all $(m + \ell)$ -tuples $(\xi_1, \dots, \xi_m; \eta_1, \dots, \eta_\ell)$ of different integers between 1 and n and C is a constant independent of n . Let $m + \ell = t$. Obviously, $1 \leq t \leq s$. We distinguish two cases according to $t \geq 2$ or $t = 1$.

Case $t \geq 2$. By (2.9)

$$S_{q_1, \dots, q_m; r_1, \dots, r_\ell}$$

$$\leq C_2 \sum_{\xi_1, \dots, \xi_m; \eta_1, \dots, \eta_\ell}^{**} n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell}$$

$$\leq C_2 n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell + t}. \quad (2.11)$$

Now, the power to which n is raised here can be estimated by means of (2.8) and $q_i = 1$ as follows

$$\frac{1}{p} \sum_{j=1}^{\ell} rr_j - 2\ell + t$$

$$= \frac{1}{p} \left(rs - \sum_{i=1}^m rq_i \right) - 2(t - m) + t$$

$$= \frac{v}{p} - \frac{rm}{p} - 2(t - m) + t$$

$$= \frac{v}{p} - t - m \left(\frac{r}{p} - 2 \right). \quad (2.12)$$

We distinguish two further subcases according to $m = t$ or $m \leq t - 1$.

Subcase $m = t$. By assumption $1 \leq p < 2$. Also, $q_i = 1$ for each i .

Thus, $m = s$ and, by (2.5),

$$\begin{aligned}
 t + m \left(\frac{r}{p} - 2 \right) &= s \left(\frac{r}{p} - 1 \right) \\
 &= v \left(\frac{1}{p} - \frac{1}{r} \right) > 1.
 \end{aligned}
 \tag{2.13}$$

Subcase $m \leq t - 1$. Then $t - m \geq 1$ and even $t - m \geq 2$ in the particular case where $m = 0$. Thus, again

$$t + m \left(\frac{r}{p} - 2 \right) = (t - m) + m \left(\frac{r}{p} - 1 \right) > 1. \tag{2.14}$$

Now we turn to

Case $t = 1$. In this case necessarily $m = 0$ and $\ell = 1$, consequently $r_1 = s$ and

$$S_{q_1, \dots, q_m; r_1, \dots, r_\ell} = S_{0; s} = \sum_{k=1}^n E \|Z_{nk}\|^{rs}.$$

Using Lemma 2, we obtain that

$$\begin{aligned}
 \Sigma_1 &= \sum_{n=1}^{\infty} \frac{1}{n^{v/p}} \sum_{k=1}^n E \|Z_{nk}\|^v \\
 &\leq 2^v \sum_{n=1}^{\infty} \frac{1}{n^{v/p}} \sum_{k=1}^n E \|Y_{nk}\|^v \\
 &\leq 2^v \sum_{n=1}^{\infty} \frac{1}{n^{v/p}} \sum_{k=1}^n v \int_0^{n^{1/p}} t^{v-1} P[\|X_{nk}\| > t] dt \\
 &\leq 2^v \sum_{n=1}^{\infty} \frac{1}{n^{v/p}} v n \int_0^{n^{1/p}} t^{v-1} P[|X| > t] dt.
 \end{aligned}$$

Letting $t = n^{1/p} s^{1/v}$ and applying Lemma 1 (with $r = 2$), it follows that

$$\begin{aligned}
 \Sigma_1 &\leq 2^v \sum_{n=1}^{\infty} n \int_0^1 P[|X| > n^{1/p} s^{1/v}] ds \\
 &= 2^v \int_0^1 \sum_{n=1}^{\infty} n P[|s^{-1/v} X|^p > n] ds \\
 &\leq 2^{v+1} \int_0^1 s^{-2p/v} E|X|^{2p} ds \\
 &= 2^{v+1} \frac{v}{v-2p} E|X|^{2p} < \infty.
 \end{aligned}
 \tag{2.15}$$

Using Markov's inequality, (2.7) and (2.10) - (2.15) we have, for any $\epsilon > 0$,

$$\begin{aligned}
 \Sigma_2(\epsilon) &= \sum_{n=1}^{\infty} P \left[\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n Z_{nk} \right\| > \epsilon \right] \\
 &\leq \sum_{n=1}^{\infty} \frac{1}{(\epsilon n^{1/p})^v} E \left(\left\| \sum_{k=1}^n Z_{nk} \right\|^v \right) \\
 &\leq \sum_{n=1}^{\infty} \frac{C}{(\epsilon n^{1/p})^v} E \left(\sum_{k=1}^n \|Z_{nk}\|^r \right)^s
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{CC_3}{\varepsilon^\nu} \left[\sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} \sum_{k=1}^n E \|Z_{nk}\|^\nu \right. \\
&\quad \left. + \sum_{n=1}^{\infty} \frac{C_2}{n^{\nu/p}} \sum_{t=2}^s \sum_{q_1, \dots, q_m; r_1, \dots, r_\ell} \Sigma^{(t)} n^{\frac{\nu}{p} - t - m(\frac{\nu}{p} - 2)} \right] \\
&= \frac{CC_3}{\varepsilon^\nu} \left[\Sigma_1 + C_2 \sum_{t=2}^s \sum_{q_1, \dots, q_m; r_1, \dots, r_\ell} \Sigma^{(t)} \sum_{n=1}^{\infty} n^{-t - m(\frac{\nu}{p} - 2)} \right],
\end{aligned}$$

where $\Sigma^{(t)}$ means that the sum is extended over all m -tuples (q_1, \dots, q_m) and ℓ -tuples (r_1, \dots, r_ℓ) with Conditions (2.7) and (2.8) such that $m + \ell = t$. Since the number of terms in each of $\Sigma^{(t)}$ is finite and the exponent of n is less than -1 , for every $\varepsilon > 0$, we have $\Sigma_2(\varepsilon) < \infty$. Thus, we have proved that

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n Z_{nk} \right\| \rightarrow 0 \quad \text{completely } (n \rightarrow \infty).$$

In order to prove (2.2), we need to establish

$$\Sigma_3 = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^n \|E Y_{nk}\| < \infty. \quad (2.16)$$

To achieve this goal, we will proceed as follows. By (2.1),

$$\begin{aligned}
Y_{nk} &= X_{nk} I_{\left[\|X_{nk}\| \leq n^{1/p} \right]} \\
&= X_{nk} - X_{nk} I_{\left[\|X_{nk}\| > n^{1/p} \right]}.
\end{aligned}$$

Since $EX_{nk} = 0$, hence

$$\|E Y_{nk}\| \leq E \left(\|X_{nk}\| I_{\left[\|X_{nk}\| > n^{1/p} \right]} \right).$$

Thus, using Lemma 2,

$$\begin{aligned}
\Sigma_3 &\leq \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^n E \left(\|X_{nk}\| I_{\left[\|X_{nk}\| > n^{1/p} \right]} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^n \left(n^{1/p} P \left[\|X_{nk}\| > n^{1/p} \right] + \int_{n^{1/p}}^{\infty} P \left[\|X_{nk}\| > t \right] dt \right) \\
&\leq \sum_{n=1}^{\infty} \left(n P \left[|X| > n^{1/p} \right] + \frac{n}{n^{1/p}} \int_{n^{1/p}}^{\infty} P \left[|X| > t \right] dt \right).
\end{aligned}$$

Letting $t = n^{1/p} s$ and applying Lemma 1, we can conclude that

$$\begin{aligned}
\Sigma_3 &\leq \sum_{n=1}^{\infty} n P \left[|X|^p > n \right] + \sum_{n=1}^{\infty} n \int_1^{\infty} P \left[|X| > n^{1/p} s \right] ds \\
&\leq 2 E |X|^{2p} + \int_1^{\infty} \sum_{n=1}^{\infty} P \left[|s^{-1} X|^p > n \right] ds
\end{aligned}$$

$$\leq 2 E|X|^{2p} + \int_1^\infty s^{-2p} E|X|^{2p} ds$$

$$= \frac{4p-1}{2p-1} E|X|^{2p} < \infty,$$

proving (2.2) through (2.16), and thereby completing the proof of Theorem 4. ///

Note that if $\sup_{nk} E \|X_{nk}\|^{2p+\alpha} < \infty$, for some $\alpha > 0$, then there exists a r.v. X such that $\{X_{nk}\}$ are uniformly bounded by X and $E|X|^{2p} < \infty$. Therefore, Corollary 5 follows.

COROLLARY 5. Let E be a type $p+\delta$ separable Banach space for $1 \leq p < 2$ and $\delta > 0$. If $\sup_{nk} E \|X_{nk}\|^{2p+\alpha} < \infty$ for some $\alpha > 0$, then

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \right\| \rightarrow 0 \text{ completely.}$$

For type $1+\delta$ spaces, Taylor [7] obtained

$$\sum_{k=1}^\infty a_{nk} X_{nk} \rightarrow 0 \text{ completely} \quad (2.17)$$

where $\{X_{nk}\}$ is uniformly bounded by X with $E|X|^{1+1/r} < \infty$ and $\{a_{nk}\}$ are Toeplitz weights with $\max_k |a_{nk}| = O(n^{-r})$. In the special case of uniform weights $a_{nk} = \frac{1}{n}$, $1 \leq k \leq n$, then $r = 1$ and Theorem 4 can be thought of as an extension of this result. Extension of Theorem 4 to infinite arrays and general weights $\{a_{nk}\}$ are possible but the detailed verification of their proofs are not included here. However, it will be shown next that the moment condition $E|X|^{2p} < \infty$ cannot be reduced in Theorem 4. In particular, for an array $\{X_{nk}\}$ of i.i.d. random elements in a type $p+\delta$ space with $EX_{11} = 0$, it will be shown that the SLLN holds if and only if $E\|X_{11}\|^{2p} < \infty$.

THEOREM 6. Let $\{X_{nk}\}$ be an array of i.i.d. random elements in a type $p+\delta$ space, $1 \leq p < 2$ and $\delta > 0$, with $EX_{11} = 0$. Then $E\|X_{11}\|^{2p} < \infty$ if and only if

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \text{ completely.} \quad (2.18)$$

PROOF: From Theorem 4, we know that $E\|X_{11}\|^{2p} < \infty$ implies (2.18) since the array $\{X_{nk}\}$ is uniformly bounded by $\|X_{11}\|$.

Now, assume that (2.18) holds. Since $\{X_{nk}\}$ are i.i.d., for every n and $\varepsilon > 0$

$$P \left[\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{kk} \right\| > \varepsilon \right] = P \left[\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \right\| > \varepsilon \right].$$

By (2.18), for every $\varepsilon > 0$,

$$\sum_{n=1}^\infty P \left[\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{kk} \right\| > \varepsilon \right] < \infty, \quad (2.19)$$

which says $\frac{1}{n^{1/p}} \sum_{k=1}^n X_{kk} \rightarrow 0$ a.s..

As a consequence,

$$\frac{1}{n^{1/p}} X_{nn} = \frac{1}{n^{1/p}} \sum_{k=1}^n X_{kk} - \left(\frac{n-1}{n} \frac{1}{n-1} \right)^{1/p} \sum_{k=1}^{n-1} X_{kk} \rightarrow 0 \quad \text{a.s.}$$

Let $\varepsilon = 1$. It follows from Lemma 1 (with $r = 1$) and the Borel-Cantelli lemma that

$$\begin{aligned} E \|X_{11}\|^p &\leq 1 + 2 \sum_{n=1}^{\infty} P [\|X_{11}\|^p > n] \\ &= 1 + 2 \sum_{n=1}^{\infty} P \left[\left\| \frac{1}{n^{1/p}} X_{nn} \right\| > 1 \right] < \infty. \end{aligned}$$

Hence,

$$nP [\|X_{11}\|^p > n] \rightarrow 0. \quad (2.20)$$

By (2.18),

$$P \left[\left\| \sum_{k=1}^n X_{nk} \right\| < n^{1/p} \right] \rightarrow 1. \quad (2.21)$$

Therefore, from (2.20) and (2.21) there exists N such that if $n \geq N$ then

$$nP [\|X_{11}\|^p > n] < \frac{1}{4} \quad \text{and} \quad P \left[\left\| \sum_{k=1}^n X_{nk} \right\|^p < n \right] > \frac{1}{2}. \quad (2.22)$$

Next, define the events

$$A_{nk} = \left[\max_{1 \leq i < k} \|X_{ni}\| \leq 2n^{1/p}, \|X_{nk}\| > 2n^{1/p}, \text{ and } \left\| \sum_{\substack{i=1 \\ i \neq k}}^n X_{ni} \right\| < n^{1/p} \right]$$

$$(k = 1, 2, \dots, n; n = 1, 2, \dots).$$

Clearly, $\{A_{nk} : k = 1, 2, \dots, n\}$ are disjoint subsets of the event $\left[\left\| \sum_{k=1}^n X_{nk} \right\| > n^{1/p} \right]$

for each $n = 1, 2, \dots$. A familiar reasoning yields that

$$\begin{aligned} P \left[\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \right\| > 1 \right] &\geq \sum_{k=1}^n P(A_{nk}) \\ &= \sum_{k=1}^n P [\|X_{nk}\| > 2n^{1/p}] P \left[\bigcap_{i=1}^{k-1} [\|X_{ni}\| \leq 2n^{1/p}] \cap \left[\left\| \sum_{\substack{i=1 \\ i \neq k}}^n X_{ni} \right\| < n^{1/p} \right] \right] \\ &\geq \sum_{k=1}^n P [\|X_{nk}\| > 2n^{1/p}] \left(P \left[\left\| \sum_{\substack{i=1 \\ i \neq k}}^n X_{ni} \right\| < n^{1/p} \right] - P \left[\bigcup_{i=1}^n [\|X_{ni}\| > 2n^{1/p}] \right] \right) \\ &\geq \sum_{k=1}^n P [\|X_{11}\| > 2n^{1/p}] \left(P \left[\left\| \sum_{i=1}^{n-1} X_{ni} \right\| < (n-1)^{1/p} \right] - nP [\|X_{11}\| > 2n^{1/p}] \right). \end{aligned}$$

Hence, by (2.22), for $n \geq N$,

$$P \left[\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \right\| > 1 \right] \geq \frac{1}{4} nP [\|X_{11}\|^p > 2^p n].$$

Therefore, $\sum_{n=1}^{\infty} nP [\|X_{11}\|^p > 2^p n] < \infty$. Thus, Lemma 1 yields $E \|X_{11}\|^{2p} < \infty$. ///

CONCLUDING REMARKS.

1. It should be noted that the case $p = 1$ in Theorem 6 is obtainable in a type 1 space (cf: Theorem 4 of Hu, Moricz and Taylor [5]). In which case type $1 + \delta$ is not needed.

2. For sequences of independent random elements which are uniformly bounded by a random variable X with $E|X|^p < \infty$, (1.6) holding necessitates the space being of type $p + \delta$ (cf: Woyczynski [4] and Maurey and Pisier [8]). Thus, the necessity of type $p + \delta$ follows for Theorem 4.

3. Theorem 6 shows that Theorem 4 is the best possible moment condition when no conditions on possible relations between the rows of the array are assumed.

4. In [4] it is mentioned that $n^{-1/p} \sum_{k=1}^n X_k \rightarrow 0$ a.s. for i.i.d. random elements $\{X_n\}$ with $EX_1 = 0$ and $E\|X_1\|^p < \infty$ apparently is equivalent to the space being of type p . Thus, it is interesting to conjecture whether Theorem 6 remains valid for only type p spaces $1 \leq p < 2$. Certainly, the "if part" is true for type p spaces, and Remark 1 indicates that it is true $p = 1$.

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