

## STRONG LAWS OF LARGE NUMBERS FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS

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**ABSTRACT.** Let  $\{X_{nk}\}$  be an array of rowwise independent random elements in a separable Banach space of type  $p + \delta$  with  $EX_{nk} = 0$  for all  $k, n$ . The complete convergence (and hence almost sure convergence) of  $n^{-1/p} \sum_{k=1}^n X_{nk}$  to 0,  $1 \leq p < 2$ , is obtained when  $\{X_{nk}\}$  are uniformly bounded by a random variable  $X$  with  $E|X|^{2p} < \infty$ . When the array  $\{X_{nk}\}$  consists of i.i.d. random elements, then it is shown that  $n^{-1/p} \sum_{k=1}^n X_{nk}$  converges completely to 0 if and only if  $E\|X_{11}\|^{2p} < \infty$ .

**KEY WORDS AND PHRASES.** Random elements, Strong laws of large numbers, Complete Convergence, Rademacher type  $p + \delta$  spaces.

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### 1. INTRODUCTION AND PRELIMINARIES.

Let  $(E, \|\cdot\|)$  be a real separable Banach space. Let  $(\Omega, \mathcal{A}, P)$  denote a probability space. A random element  $X$  in  $E$  is a function from  $\Omega$  into  $E$  which is  $\mathcal{A}$ -measurable with respect to the Borel subsets  $\mathcal{B}(E)$ . The  $p^{\text{th}}$  absolute moment of a random element  $X$  is  $E\|X\|^p$  where  $E$  is the expected value of the random variable  $\|X\|^p$ . The expected value of  $X$  is defined to be the Bochner integral (when  $E\|X\| < \infty$ ) and is denoted by  $EX$ . The concepts of independence and identical distributions have direct extensions to  $E$ . A separable Banach space is said to be of (Rademacher) type  $p$ ,  $1 \leq p \leq 2$ , if there exists a constant  $C$  such that

$$E \left\| \sum_{k=1}^n X_k \right\|^p \leq C \sum_{k=1}^n E \|X_k\|^p$$

for all independent random elements  $X_1, \dots, X_n$  with zero means and finite  $p^{\text{th}}$  moments. Every separable Hilbert space and finite-dimensional Banach space is of type 2. Every separable Banach space is at least type 1 while the  $\ell^p$  and  $L^p$  spaces are of type  $\min\{2, p\}$  for  $p \geq 1$ .

Throughout this paper  $\{X_{nk}: 1 \leq k \leq n, n \geq 1\}$  will denote rowwise independent random elements in  $E$  such that

$$EX_{nk} = 0 \quad \text{for all } n \text{ and } k \tag{1.1}$$

and such that  $\{X_{nk}\}$  are uniformly bounded by a random variable  $X$  with

$$E|X|^{2p} < \infty \quad \text{for some } 1 \leq p < 2. \tag{1.2}$$

Recall that an array  $\{X_{nk}\}$  of random elements is said to be uniformly bounded by a random variable  $X$  if for all  $n$  and  $k$  and for every real number  $t > 0$

$$P[\|X_{nk}\| > t] \leq P[|X| > t]. \quad (1.3)$$

Note that i.i.d. random elements are uniformly bounded by  $\|X_{11}\|$ . The major results of this paper show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \text{ completely} \quad (1.4)$$

where complete convergence is defined (as in Hsu and Robbins [1]) by

$$\sum_{n=1}^{\infty} P\left[\left\|\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk}\right\| > \varepsilon\right] < \infty \quad (1.5)$$

for each  $\varepsilon > 0$ .

Erdős [2] showed that for an array of i.i.d. random variables  $\{X_{nk}\}$ , (1.4) holds if and only if  $E|X_{11}|^{2p} < \infty$ . Jain [3] obtained a uniform strong law of large numbers for sequences of i.i.d. random elements in separable Banach spaces of type 2 which would yield (1.4) with  $p = 1$  for an array of i.i.d. random elements  $\{X_{nk}\}$  in a type 2 space. Woyczyński [4] showed that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_k \rightarrow 0 \text{ completely} \quad (1.6)$$

for any sequence  $\{X_n\}$  of independent random elements in a type  $p + \delta$ ,  $1 \leq p < 2$  and  $\delta > 0$ , with  $E X_n = 0$  for all  $n$  which is uniformly bounded by a random variable  $X$  satisfying  $E|X|^p < \infty$ . Móricz, Hu and Taylor [5] showed that Erdős' result could be obtained by replacing the i.i.d. condition by the uniformly bounded condition (1.3). In addition, they showed that Jain's result for i.i.d. random elements with  $p = 1$  did not require the space to be type 2 but held in all separable Banach spaces. In this paper, (1.4) is established in type  $p + \delta$  spaces,  $1 \leq p < 2$  and  $\delta > 0$ , for uniformly bounded rowwise independent random elements. For i.i.d. random elements in type  $p + \delta$  spaces, it is shown that (1.4) holds if and only if  $E\|X_{11}\|^{2p} < \infty$ . Thus, no sharper moment conditions are possible.

## 2. MAJOR RESULTS.

Many authors (starting with Beck [6]) have related the strong law of large numbers for non-identically distributed, independent random elements in separable Banach spaces to the necessity of the space being of type  $p + \delta$  for  $1 \leq p < 2$  and some  $\delta > 0$ . Consequently, attention is restricted to type  $p + \delta$  spaces in this paper. Three lemmas will be used in obtaining the major results. They are stated here without proof. Lemma 1 with  $r = 1$  is in most textbooks while Lemma 2 is accomplished using integration by parts. Lemma 3 is in Woyczyński [4].

LEMMA 1. For any  $r \geq 1$ ,  $E|X|^r < \infty$  if and only if

$$\sum_{n=1}^{\infty} n^{r-1} P[|X| > n] < \infty.$$

More precisely,  $r 2^{-r} \sum_{n=1}^{\infty} n^{r-1} P[|X| > n]$

$$\leq E|X|^r \leq 1 + r 2^r \sum_{n=1}^{\infty} n^{r-1} P[|X| > n].$$

LEMMA 2. If  $r \geq 1$ , then for any  $p > 0$

$$E\left(\left|X\right|^r I_{\left[\left|X\right| \leq n^{1/p}\right]}\right) \leq r \int_0^{n^{1/p}} t^{r-1} P[\left|X\right| > t] dt$$

and

$$E\left(\left|X\right|^r I_{\left[\left|X\right| > n^{1/p}\right]}\right) = n^{1/p} P[\left|X\right| > n^{1/p}] + \int_{n^{1/p}}^{\infty} P[\left|X\right| > t] dt.$$

LEMMA 3. Let  $1 \leq p \leq 2$  and  $q \geq 1$ . The following properties are equivalent:

- (i)  $E$  is of type  $p$ .
- (ii) There exists a  $C$  such that for all independent random elements  $X_1, \dots, X_n$  in  $E$  with  $EX_k = 0, k = 1, \dots, n$ ,

$$E \left\| \sum_{k=1}^n X_k \right\|^q \leq C E \left( \sum_{k=1}^n \|X_k\|^p \right)^{q/p}.$$

THEOREM 4. If  $\{X_{nk}\}$  is an array of rowwise independent random elements in a type  $p + \delta$  space,  $1 \leq p < 2$  and  $\delta > 0$ , which are uniformly bounded by a random variable  $X$  such that (1.1) and (1.2) holds, then

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \rightarrow 0 \text{ completely.}$$

PROOF. Define

$$Y_{nk} = X_{nk} I_{\left[\|X_{nk}\| \leq n^{1/p}\right]} \quad 1 \leq k \leq n, n \geq 1. \quad (2.1)$$

Then, by Lemma 1 (with  $r = 2$ ),

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{k=1}^n P[X_{nk} \neq Y_{nk}] &= \sum_{n=1}^{\infty} \sum_{k=1}^n P[\|X_{nk}\| > n^{1/p}] \\ &\leq \sum_{n=1}^{\infty} n P[\|X\| > n^{1/p}] \\ &= \sum_{n=1}^{\infty} n P[\|X\|^p > n] \leq 2E\|X\|^{2p} < \infty. \end{aligned}$$

Next, for any  $\epsilon > 0$ ,

$$\begin{aligned} &\sum_{n=1}^{\infty} P\left[\left\|\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} - \frac{1}{n^{1/p}} \sum_{k=1}^n Y_{nk}\right\| > \epsilon\right] \\ &\leq \sum_{n=1}^{\infty} P\left[\bigcup_{k=1}^n [X_{nk} \neq Y_{nk}]\right] \\ &\leq \sum_{n=1}^{\infty} \sum_{k=1}^n P[X_{nk} \neq Y_{nk}] < \infty. \end{aligned}$$

Therefore,

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} - \frac{1}{n^{1/p}} \sum_{k=1}^n Y_{nk} \right\| \rightarrow 0 \text{ completely,}$$

and it suffices to prove that

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n Y_{nk} \right\| \rightarrow 0 \text{ completely.} \quad (2.2)$$

To this end, let

$$Z_{nk} = Y_{nk} - EY_{nk} \quad (k=1, 2, \dots, n; \quad n=1, 2, \dots).$$

Then for  $1 \leq q \leq 2p$  it follows by Hölder's inequality that

$$\begin{aligned} (E\|Z_{nk}\|^q)^{1/q} &\leq 2(E\|Y_{nk}\|^q)^{1/q} \\ &\leq 2(E\|Y_{nk}\|^{2p})^{1/(2p)} \leq 2(E|X|^{2p})^{1/(2p)}, \end{aligned}$$

so that

$$\begin{aligned} E\|Z_{nk}\|^q &\leq 2^q (E|X|^{2p})^{q/(2p)} \\ &\leq 2^{2p} (1 + E|X|^{2p}) = C_1, \text{ say.} \end{aligned} \quad (2.3)$$

Furthermore,

$$\|Z_{nk}\| \leq \|Y_{nk}\| + \|EY_{nk}\| \leq 2n^{1/p}. \quad (2.4)$$

Following the techniques of Taylor [7] in expanding a high power of a sum, let  $r = p + \delta$  and  $v$  be chosen so that

$$s = \frac{v}{r} \text{ is an integer and } v > \left(\frac{1}{p} - \frac{1}{r}\right)^{-1}. \quad (2.5)$$

It is readily seen that  $E\left(\sum_{k=1}^n \|Z_{nk}\|\right)^v < \infty$ , so that, by Lemma 3,

$$\begin{aligned} E\left(\left(\sum_{k=1}^n Z_{nk}\right)^v\right) &\leq C E\left(\sum_{k=1}^n \|Z_{nk}\|^r\right)^s \\ &= C \sum_{k_1, \dots, k_s} E\left(\prod_{j=1}^s \|Z_{nk_j}\|^r\right) \end{aligned} \quad (2.6)$$

where the sum is extended for all  $s$ -tuples  $(k_1, \dots, k_s)$  with  $k_j = 1, 2, \dots, n$  for each  $j$ . The general term to be considered then will have

$$q_1 \text{ of the } k \text{'s} = \xi_1, \dots, q_m \text{ of the } k \text{'s} = \xi_m;$$

$$r_1 \text{ of the } k \text{'s} = \eta_1, \dots, r_\ell \text{ of the } k \text{'s} = \eta_\ell;$$

where

$$r \leq rq_i \leq 2p, \quad rr_j > 2p, \text{ and} \quad (2.7)$$

$$\sum_{i=1}^m q_i + \sum_{j=1}^\ell r_j = s. \quad (2.8)$$

Clearly,  $q_i = 1$ . Then, using (2.3) and (2.4), we can conclude that

$$\begin{aligned} &E\left(\prod_{i=1}^m \|Z_{n\xi_i}\|^r\right) \prod_{j=1}^\ell \|Z_{n\eta_j}\|^{rr_j} \\ &= \prod_{i=1}^m E\|Z_{n\xi_i}\|^{rq_i} \prod_{j=1}^\ell E\left(\|Z_{n\eta_j}\|^{2p} \|Z_{n\eta_j}\|^{rr_j-2p}\right) \\ &\leq C_1^{m+\ell} \prod_{j=1}^\ell (2n^{1/p})^{rr_j-2p} \\ &= C_1^{m+\ell} 2^{\sum_{j=1}^\ell (rr_j - 2p)} n^{\sum_{j=1}^\ell (rr_j/p) - 2\ell} \end{aligned} \quad (2.9)$$

$$\leq C_1^v 2^v n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell}$$

$$= C_2 n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell}, \text{ say.}$$

Combining all possible terms of form (2.9), we can write

$$\begin{aligned} & E \left( \sum_{k=1}^n \|z_{nk}\|^r \right)^s \\ & \leq C_3 \sum_{q_1, \dots, q_m; r_1, \dots, r_\ell}^* \sum_{\xi_1, \dots, \xi_m; \eta_1, \dots, \eta_\ell}^{**} E \left( \prod_{i=1}^m \|z_{n\xi_i}\|^{rq_i} \prod_{j=1}^{\ell} \|z_{n\eta_j}\|^{rr_j} \right) \\ & = C_3 \sum_{q_1, \dots, q_m; r_1, \dots, r_\ell}^* s_{q_1, \dots, q_m; r_1, \dots, r_\ell}, \text{ say,} \end{aligned} \quad (2.10)$$

where  $\sum^*$  is extended over all  $m$ -tuples  $(q_1, \dots, q_m)$  and  $\ell$ -tuples  $(r_1, \dots, r_\ell)$  such that Conditions (2.7) and (2.8) are satisfied (the cases  $m = 0$  or  $\ell = 0$  may also occur), while  $\sum^{**}$  is extended over all  $(m + \ell)$ -tuples  $(\xi_1, \dots, \xi_m; \eta_1, \dots, \eta_\ell)$  of different integers between 1 and  $n$  and  $C$  is a constant independent of  $n$ . Let  $m + \ell = t$ . Obviously,  $1 \leq t \leq s$ . We distinguish two cases according to  $t \geq 2$  or  $t = 1$ .

Case  $t \geq 2$ . By (2.9)

$$\begin{aligned} & s_{q_1, \dots, q_m; r_1, \dots, r_\ell} \\ & \leq C_2 \sum_{\xi_1, \dots, \xi_m; \eta_1, \dots, \eta_\ell}^{**} n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell} \\ & \leq C_2 n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell + t}. \end{aligned} \quad (2.11)$$

Now, the power to which  $n$  is raised here can be estimated by means of (2.8) and  $q_i = 1$  as follows

$$\begin{aligned} & \frac{1}{p} \sum_{j=1}^{\ell} rr_j - 2\ell + t \\ & = \frac{1}{p} \left( rs - \sum_{i=1}^m rq_i \right) - 2(t - m) + t \\ & = \frac{v}{p} - \frac{rm}{p} - 2(t - m) + t \\ & = \frac{v}{p} - t - m \left( \frac{r}{p} - 2 \right). \end{aligned} \quad (2.12)$$

We distinguish two further subcases according to  $m = t$  or  $m \leq t - 1$ .

Subcase  $m = t$ . By assumption  $1 \leq p < 2$ . Also,  $q_i = 1$  for each  $i$ .

Thus,  $m = s$  and, by (2.5),

$$\begin{aligned}
t + m \left( \frac{r}{p} - 2 \right) &= s \left( \frac{r}{p} - 1 \right) \\
&= \sqrt{\frac{1}{p} - \frac{1}{r}} > 1.
\end{aligned} \tag{2.13}$$

Subcase  $m \leq t - 1$ . Then  $t - m \geq 1$  and even  $t - m \geq 2$  in the particular case where  $m = 0$ . Thus, again

$$t + m \left( \frac{r}{p} - 2 \right) = (t - m) + m \left( \frac{r}{p} - 1 \right) > 1. \tag{2.14}$$

Now we turn to

Case  $t = 1$ . In this case necessarily  $m = 0$  and  $\ell = 1$ , consequently  $r_1 = s$  and

$$S_{q_1, \dots, q_m; r_1, \dots, r_\ell} = S_{0; s} = \sum_{k=1}^n E \|Z_{nk}\|^{rs}.$$

Using Lemma 2, we obtain that

$$\begin{aligned}
\Sigma_1 &= \sum_{n=1}^{\infty} \frac{1}{n^{v/p}} \sum_{k=1}^n E \|Z_{nk}\|^v \\
&\leq 2^v \sum_{n=1}^{\infty} \frac{1}{n^{v/p}} \sum_{k=1}^n E \|Y_{nk}\|^v \\
&\leq 2^v \sum_{n=1}^{\infty} \frac{1}{n^{v/p}} \sum_{k=1}^n v \int_0^{n^{1/p}} t^{v-1} P\{ \|X_{nk}\| > t \} dt \\
&\leq 2^v \sum_{n=1}^{\infty} \frac{1}{n^{v/p}} v n \int_0^{n^{1/p}} t^{v-1} P\{ |X| > t \} dt.
\end{aligned}$$

Letting  $t = n^{1/p} s^{1/v}$  and applying Lemma 1 (with  $r = 2$ ), it follows that

$$\begin{aligned}
\Sigma_1 &\leq 2^v \sum_{n=1}^{\infty} n \int_0^1 P\{ |X| > n^{1/p} s^{1/v} \} ds \\
&= 2^v \int_0^1 \sum_{n=1}^{\infty} n P\{ |s^{-1/v} X|^p > n \} ds \\
&\leq 2^{v+1} \int_0^1 s^{-2p/v} E|X|^{2p} ds \\
&= 2^{v+1} \frac{v}{v-2p} E|X|^{2p} < \infty.
\end{aligned} \tag{2.15}$$

Using Markov's inequality, (2.7) and (2.10) - (2.15) we have, for any  $\varepsilon > 0$ ,

$$\begin{aligned}
\Sigma_2(\varepsilon) &= \sum_{n=1}^{\infty} P\left\{ \left| \frac{1}{n^{1/p}} \sum_{k=1}^n Z_{nk} \right| > \varepsilon \right\} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{(\varepsilon n^{1/p})^v} E \left( \left\| \sum_{k=1}^n Z_{nk} \right\|^v \right) \\
&\leq \sum_{n=1}^{\infty} \frac{C}{(\varepsilon n^{1/p})^v} E \left( \sum_{k=1}^n \|Z_{nk}\|^{rs} \right)^s
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{CC_3}{\varepsilon^\nu} \left[ \sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} \sum_{k=1}^n E \|Z_{nk}\|^\nu \right. \\
& \quad \left. + \sum_{n=1}^{\infty} \frac{C_2}{n^{\nu/p}} \sum_{t=2}^s \sum_{q_1, \dots, q_m; r_1, \dots, r_\ell}^{(t)} \sum_{n=1}^{\infty} n^{-t-m(\frac{r}{p} - 2)} \right] \\
& = \frac{CC_3}{\varepsilon^\nu} \left[ \Sigma_1 + C_2 \sum_{t=2}^s \sum_{q_1, \dots, q_m; r_1, \dots, r_\ell}^{(t)} \sum_{n=1}^{\infty} n^{-t-m(\frac{r}{p} - 2)} \right],
\end{aligned}$$

where  $\Sigma^{(t)}$  means that the sum is extended over all  $m$ -tuples  $(q_1, \dots, q_m)$  and  $\ell$ -tuples  $(r_1, \dots, r_\ell)$  with Conditions (2.7) and (2.8) such that  $m + \ell = t$ . Since the number of terms in each of  $\Sigma^{(t)}$  is finite and the exponent of  $n$  is less than  $-1$ , for every  $\varepsilon > 0$ , we have  $\Sigma_2(\varepsilon) < \infty$ . Thus, we have proved that

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n Z_{nk} \right\| \rightarrow 0 \text{ completely } (n \rightarrow \infty).$$

In order to prove (2.2), we need to establish

$$\Sigma_3 = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^n \|E Y_{nk}\| < \infty. \quad (2.16)$$

To achieve this goal, we will proceed as follows. By (2.1),

$$\begin{aligned}
Y_{nk} &= X_{nk} \mathbb{I}_{\{\|X_{nk}\| \leq n^{1/p}\}} \\
&= X_{nk} - X_{nk} \mathbb{I}_{\{\|X_{nk}\| > n^{1/p}\}}.
\end{aligned}$$

Since  $E X_{nk} = 0$ , hence

$$\|E Y_{nk}\| \leq E \left( \|X_{nk}\| \mathbb{I}_{\{\|X_{nk}\| > n^{1/p}\}} \right).$$

Thus, using Lemma 2,

$$\begin{aligned}
\Sigma_3 &\leq \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^n E \left( \|X_{nk}\| \mathbb{I}_{\{\|X_{nk}\| > n^{1/p}\}} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^n (n^{1/p} P \{\|X_{nk}\| > n^{1/p}\}) + \int_{n^{1/p}}^{\infty} P \{\|X_{nk}\| > t\} dt \\
&\leq \sum_{n=1}^{\infty} (n P \{|X| > n^{1/p}\}) + \frac{n}{n^{1/p}} \int_{n^{1/p}}^{\infty} P \{|X| > t\} dt.
\end{aligned}$$

Letting  $t = n^{1/p} s$  and applying Lemma 1, we can conclude that

$$\begin{aligned}
\Sigma_3 &\leq \sum_{n=1}^{\infty} n P \{|X|^p > n\} + \sum_{n=1}^{\infty} n \int_1^{\infty} P \{|X| > n^{1/p} s\} ds \\
&\leq 2 E |X|^{2p} + \int_1^{\infty} \sum_{n=1}^{\infty} P \{|s^{-1} X|^p > n\} ds
\end{aligned}$$

$$\begin{aligned} &\leq 2 E|X|^{2p} + \int_1^\infty s^{-2p} E|X|^{2p} ds \\ &= \frac{4p-1}{2p-1} E|X|^{2p} < \infty, \end{aligned}$$

proving (2.2) through (2.16), and thereby completing the proof of Theorem 4.  $\rule{0pt}{10pt}$

Note that if  $\sup_{nk} E\|X_{nk}\|^{2p+\alpha} < \infty$ , for some  $\alpha > 0$ , then there exists a r.v.  $X$  such that  $\{X_{nk}\}$  are uniformly bounded by  $X$  and  $E|X|^{2p} < \infty$ . Therefore, Corollary 5 follows.

COROLLARY 5. Let  $E$  be a type  $p + \delta$  separable Banach space for  $1 \leq p < 2$  and  $\delta > 0$ . If  $\sup_{nk} E\|X_{nk}\|^{2p+\alpha} < \infty$  for some  $\alpha > 0$ , then

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \right\| \rightarrow 0 \text{ completely.}$$

For type  $1 + \delta$  spaces, Taylor [7] obtained

$$\sum_{k=1}^\infty a_{nk} X_{nk} \rightarrow 0 \text{ completely} \quad (2.17)$$

where  $\{X_{nk}\}$  is uniformly bounded by  $X$  with  $E|X|^{1+1/r} < \infty$  and  $\{a_{nk}\}$  are Toeplitz weights with  $\max_k |a_{nk}| = O(n^{-r})$ . In the special case of uniform weights  $a_{nk} = \frac{1}{n}$ ,  $1 \leq k \leq n$ , then  $r = 1$  and Theorem 4 can be thought of as an extension of this result. Extension of Theorem 4 to infinite arrays and general weights  $\{a_{nk}\}$  are possible but the detailed verification of their proofs are not included here. However, it will be shown next that the moment condition  $E|X|^{2p} < \infty$  cannot be reduced in Theorem 4. In particular, for an array  $\{X_{nk}\}$  of i.i.d. random elements in a type  $p + \delta$  space with  $EX_{11} = 0$ , it will be shown that the SLLN holds if and only if  $E\|X_{11}\|^{2p} < \infty$ .

THEOREM 6. Let  $\{X_{nk}\}$  be an array of i.i.d. random elements in a type  $p + \delta$  space,  $1 \leq p < 2$  and  $\delta > 0$ , with  $EX_{11} = 0$ . Then  $E\|X_{11}\|^{2p} < \infty$  if and only if

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \right\| \rightarrow 0 \text{ completely.} \quad (2.18)$$

PROOF: From Theorem 4, we know that  $E\|X_{11}\|^{2p} < \infty$  implies (2.18) since the array  $\{X_{nk}\}$  is uniformly bounded by  $\|X_{11}\|$ .

Now, assume that (2.18) holds. Since  $\{X_{nk}\}$  are i.i.d., for every  $n$  and  $\epsilon > 0$

$$P \left[ \left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{kk} \right\| > \epsilon \right] = P \left[ \left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk} \right\| > \epsilon \right].$$

By (2.18), for every  $\epsilon > 0$ ,

$$\sum_{n=1}^\infty P \left[ \left\| \frac{1}{n^{1/p}} \sum_{k=1}^n X_{kk} \right\| > \epsilon \right] < \infty, \quad (2.19)$$

which says  $\frac{1}{n^{1/p}} \sum_{k=1}^n X_{kk} \rightarrow 0$  a.s..

As a consequence,

$$\frac{1}{n^{1/p}} X_{nn} = \frac{1}{n^{1/p}} \sum_{k=1}^n X_{kk} - \left( \frac{n-1}{n} \frac{1}{n-1} \right)^{1/p} \sum_{k=1}^{n-1} X_{kk} \rightarrow 0 \text{ a.s.}$$

Let  $\epsilon = 1$ . It follows from Lemma 1 (with  $r = 1$ ) and the Borel-Cantelli lemma that

$$\begin{aligned} E\|X_{11}\|^p &\leq 1 + 2 \sum_{n=1}^{\infty} P[\|X_{11}\|^p > n] \\ &= 1 + 2 \sum_{n=1}^{\infty} P\left[\left\|\frac{1}{n^{1/p}} X_{nn}\right\| > 1\right] < \infty. \end{aligned}$$

Hence,

$$n P[\|X_{11}\|^p > n] \rightarrow 0. \quad (2.20)$$

By (2.18),

$$P\left[\left\|\sum_{k=1}^n X_{nk}\right\| < n^{1/p}\right] \rightarrow 1. \quad (2.21)$$

Therefore, from (2.20) and (2.21) there exists  $N$  such that if  $n \geq N$  then

$$n P[\|X_{11}\|^p > n] < \frac{1}{4} \quad \text{and} \quad P\left[\left\|\sum_{k=1}^n X_{nk}\right\|^p < n\right] > \frac{1}{2}. \quad (2.22)$$

Next, define the events

$$A_{nk} = \left[ \max_{1 \leq i < k} \|X_{ni}\| \leq 2n^{1/p}, \|X_{nk}\| > 2n^{1/p}, \text{ and } \left\|\sum_{\substack{i=1 \\ i \neq k}}^n X_{ni}\right\| < n^{1/p} \right] \quad (k = 1, 2, \dots, n: n = 1, 2, \dots).$$

Clearly,  $\{A_{nk}: k = 1, 2, \dots, n\}$  are disjoint subsets of the event  $\left[\left\|\sum_{k=1}^n X_{nk}\right\| > n^{1/p}\right]$  for each  $n = 1, 2, \dots$ . A familiar reasoning yields that

$$\begin{aligned} P\left[\left\|\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk}\right\| > 1\right] &\geq \sum_{k=1}^n P(A_{nk}) \\ &= \sum_{k=1}^n P[\|X_{nk}\| > 2n^{1/p}] P\left[\bigcap_{i=1}^{k-1} [\|X_{ni}\| \leq 2n^{1/p}] \cap [\left\|\sum_{\substack{i=1 \\ i \neq k}}^n X_{ni}\right\| < n^{1/p}] \right] \\ &\geq \sum_{k=1}^n P[\|X_{nk}\| > 2n^{1/p}] \left( P\left[\left\|\sum_{i=1}^n X_{ni}\right\| < n^{1/p}\right] - P\left[\bigcup_{i=1}^n [\|X_{ni}\| > 2n^{1/p}]\right] \right) \\ &\geq \sum_{k=1}^n P[\|X_{11}\| > 2n^{1/p}] \left( P\left[\left\|\sum_{i=1}^{n-1} X_{ni}\right\| < (n-1)^{1/p}\right] - n P[\|X_{11}\| > 2n^{1/p}] \right). \end{aligned}$$

Hence, by (2.22), for  $n \geq N$ ,

$$P\left[\left\|\frac{1}{n^{1/p}} \sum_{k=1}^n X_{nk}\right\| > 1\right] \geq \frac{1}{4} n P[\|X_{11}\|^p > 2^p n].$$

Therefore,  $\sum_{n=1}^{\infty} n P[\|X_{11}\|^p > 2^p n] < \infty$ . Thus, Lemma 1 yields  $E\|X_{11}\|^{2p} < \infty$ . //

## CONCLUDING REMARKS.

1. It should be noted that the case  $p = 1$  in Theorem 6 is obtainable in a type 1 space (cf: Theorem 4 of Hu, Moricz and Taylor [5]). In which case type  $1 + \delta$  is not needed.

2. For sequences of independent random elements which are uniformly bounded by a random variable  $X$  with  $E|X|^p < \infty$ , (1.6) holding necessitates the space being of type  $p + \delta$  (cf: Woyczyński [4] and Maurey and Pisier [8]). Thus, the necessity of type  $p + \delta$  follows for Theorem 4.

3. Theorem 6 shows that Theorem 4 is the best possible moment condition when no conditions on possible relations between the rows of the array are assumed.

4. In [4] it is mentioned that  $n^{-1/p} \sum_{k=1}^n X_k \rightarrow 0$  a.s. for i.i.d. random elements  $\{X_n\}$  with  $EX_1 = 0$  and  $E\|X_1\|^p < \infty$  apparently is equivalent to the space being of type  $p$ . Thus, it is interesting to conjecture whether Theorem 6 remains valid for only type  $p$  spaces  $1 \leq p < 2$ . Certainly, the "if part" is true for type  $p$  spaces, and Remark 1 indicates that it is true  $p = 1$ .

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