ON A CLASS OF p-VALENT STARLIKE FUNCTIONS
OF ORDER α

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ABSTRACT.   Let Ω denote the class of functions $w(z)$, $w(o) = o$, $|w(z)| < 1$ analytic in the unit disc $U = \{z: |z| < 1\}$. For arbitrary fixed numbers A, B, $-1 < A \le 1$, $-1 \le B < 1$ and $o \le \alpha < p$, denote by $P(A, B, p, \alpha)$ the class of functions $P(z) = p +$ $\sum_{n=1}^{\infty} b_n z^n$ analytic in U such that $P(z) \in P(A, B, p, \alpha)$ if and only if $P(z) =$

$$\frac{p + [p \, B + (A-B) \, (p-\alpha)] \, w(z)}{1 + Bw(z)} , \quad w \in \Omega \, , \, z \in U.$$

Moreover, let $S(A, B, p, \alpha)$ denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ analytic in U and satisfying the condition that $f(z) \in S(A, B, p, \alpha)$ if and only if $\frac{zf'(z)}{f(z)} = P(z)$ for some $P(z) \in P(A, B, p, \alpha)$ and all z in U.

In this paper we determine the bounds for $|f(z)|$ and $|\arg \frac{f(z)}{z}|$ in $S(A, B, p, \alpha)$, we investigate the coefficient estimates for functions of the class $S(A, B, p, \alpha)$ and we study some properties of the class $P(A, B, p, \alpha)$.

1.   INTRODUCTION.

Let A_p (p a fixed integer greater than zero) denote the class of functions

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$$ which are analytic in $U = \{z : |z| < 1\}$. We use Ω to denote the class of bounded analytic functions $w(z)$ in U satisfies the conditions $w(o) = o$ and $|w(z)| \le |z|$ for $z \in U$.

Let $P(A, B)(-1 \le B < A \le 1)$ denote the class of functions having the form

$$P_1(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \qquad (1.1)$$

which are analytic in U and such that $P_1(z) \in P(A, B)$ if and only if

$$P_1(z) = \frac{1 + Aw(z)}{1 + Bw(z)} , \quad w \in \Omega, z \in U. \qquad (1.2)$$

The class $P(A, B)$ was introduced by Janowski [1].

For $-1 \le B < A \le 1$ and $o \le \alpha < p$, denote by $P(A, B, p, \alpha)$ the class of functions

$P(z) = p + \sum\limits_{k=1}^{\infty} c_k z^k$ which are analytic in U and which satisfy that $P(z) \in P(A, B, p, \alpha)$ if and only if

$$P(z) = (p - \alpha) P_1(z) + \alpha \ , \ P_1(z) \in P(A, B). \tag{1.3}$$

Using (1.2) in (1.3), one can show that $P(z) \in P(A, B, P, \alpha)$ if and only if

$$P(z) = \frac{p + [pB + (A - B)(p - \alpha)]w(z)}{1 + B \, w(z)} \ , \ w \in \Omega \ . \tag{1.4}$$

It was shown in [1] that

$P_1(z) \in P(A, B)$ if and only if

$$P(z) = \frac{(1 + A) \, p(z) + 1-A}{(1 + B) \, p(z) + 1 - B} \tag{1.5}$$

for some $p(z) \in P(1, -1) = P$ (the class of functions of form (1.1) which are analytic in U and have a positive real part in U). Thus, from (1.3) and (1.5), we have

$P(z) \in P(A, B, p, \alpha)$ if and only if

$$P(z) = \frac{[p+pB+(A-B)(p-\alpha)]p(z)+[p-pB-(A-B)(p-\alpha)]}{(1+B)p(z) + 1-B} \ , \ p(z) \in P. \tag{1.6}$$

Moreover, let $S(A, B, p, \alpha)$ denote the class of functions $f(z) \in A_p$ which satisfy

$$\frac{zf'(z)}{f(z)} = P(z) \tag{1.7}$$

for some $P(z)$ in $P(A, B, p, \alpha)$ and all z in U.

We note that $S(A, B, 1, o) = S*(A, B)$, is the class of functions $f_1(z) \in A_1$ which satisfy

$$\frac{zf'_1(z)}{f_1(z)} = P_1(z), P_1 \in P(A, B). \tag{1.8}$$

The class $S*(A, B)$ was introduced by Janowski [1]. Also, $S(1, -1, p, \alpha) = S_\alpha(p)$, is the class of p-valent starlike functions of order α, $o \le \alpha < p$, investigated by Goluzine [2].

From (1.3), (1.7) and (1.8), it is easy to show that $f \in S(A, B, p, \alpha)$ if and only if for $z \in U$

$$f(z) = z^p [\frac{f_1(z)}{z}]^{(p - \alpha)} \ , \ f_1 \in S*(A, B). \tag{1.9}$$

2.   THE ESTIMATION OF $|f(z)|$ AND $|\arg \frac{f(z)}{z}|$ FOR THE CLASS $S(A, B, p, \alpha)$.

LEMMA 1.   Let $P(z) \in P(A, B, p, \alpha)$. Then, for $|z| \le r$, we have

$$\left| P(z) - \frac{p-[pB+(A-B)(p-\alpha)]Br^2}{1-B^2r^2} \right| \le \frac{(A-B)(p-\alpha)r}{1 - B^2r^2} \quad .$$

PROOF. It is easy to see that the transformation

$$P_1(z) = \frac{1+Aw(z)}{1+Bw(z)} \quad \text{maps } |w(z)| \le r \text{ onto the circle}$$

$$\left| P_1(z) - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{1-B^2r^2} \ . \tag{2.1}$$

Then the result follows from (1.3) and (2.1).

THEOREM 1. If $f(z) \ \epsilon \ S(A, B, p, \alpha)$, then for $|z| = r$, $o \le r < 1$,

$$C(r; -A, -B, p, \alpha) \le |f(z)| \le C(r; A, B, p, \alpha), \tag{2.2}$$

where

$$C(r; A, B, p, \alpha) = \begin{cases} r^p(1 + Br)^{\frac{(A-B)(p - \alpha)}{B}} & \text{for } B \ne o, \\[2ex] r^p . e^{A(p - \alpha)r} & \text{for } B = o. \end{cases}$$

These bounds are sharp, being attained at the point $z = re^{i\phi}$, $o \le \phi \le 2\pi$, by

$$f_*(z) = z^p \ f_0 \ (z; -A, -B, p, \alpha) \tag{2.3}$$

and

$$f^*(z) = z^p \ f_0(z; A, B, p, \alpha), \tag{2.4}$$

respectively, where

$$f_0(z; A, B, p, \alpha) = \begin{cases} (1 + Be^{-i\phi}z)^{\frac{(A-B)(p - \alpha)}{B}} & \text{for } B \ne o \\[2ex] e^{A(p - \alpha)e^{-i\phi}z} & \text{for } B = o \ . \end{cases}$$

PROOF. From (1.9), we have $f(z) \ \epsilon \ S(A, B, p, \alpha)$ if and only if

$$f(z) = z^p \ [\frac{f_1(z)}{z}]^{(p - \alpha)}, \ f_1 \ \epsilon \ S*(A, B) \ . \tag{2.5}$$

It was shown by Janowski [1] that for $f_1(z) \ \epsilon \ S*(A, B)$

$$f_1(z) = z \exp (\int_0^z \frac{P_1(\zeta) - 1}{\zeta} \ d \zeta \ ), \ P_1(z) \ \epsilon \ P(A, B). \tag{2.6}$$

Thus from (2.5) and (2.6), we have for $f(z) \in S(A, B, p, \alpha)$

$$f(z) = z^p \exp \left(\int_o^z \frac{P(\zeta) - p}{\zeta} d\zeta \right), P(z) \in P(A, B, p, \alpha).$$

Therefore

$$|f(z)| = |z|^p \exp \left(\text{Re} \int_o^z \frac{P(\zeta) - p}{\zeta} d\zeta \right).$$

substituting $\zeta = zt$, we obtain

$$|f(z)| = |z|^p \exp \left(\text{Re} \int_o^1 \frac{P(zt) - p}{t} dt \right).$$

Hence

$$|f(z)| \leq |z|^p \exp \left(\int_o^1 \max_{|zt| = rt} \left(\text{Re} \frac{P(zt) - p}{t} \right) dt \right).$$

From Lemma 1, it follows that

$$\max_{|zt| = rt} \frac{P(zt) - p}{t} = \frac{(A-B)(p - \alpha)r}{1 + Brt} ;$$

then, after integration, we obtain the upper bounds in (2.2). Similarly, we obtain the bounds on the left-hand side of (2.2) which ends the proof.

REMARKS ON THEOREM 1.

1. Choosing A = 1 and B = -1 in Theorem 1, we get the result due to Goluzina [2].

2. Choosing P = 1 and α = o in Theorem 1, we get the result due to Janowski [1].

3. Choosing p = 1, A = 1 and B = -1 in Theorem 1, we get the result due to Robertson [3].

4. Choosing p = 1, A = 1 and B = α = o in Theorem 1, we get the result due to Singh [4].

THEOREM 2. If $f(z) \in S(A, B, p, \alpha)$, then for $|z| = r < 1$

$$\left| \arg \left(\frac{f(z)}{z^p} \right) \right| \leq \frac{(A - B)(p - \alpha)}{B} \sin^{-1} (Br), B \neq o, \tag{2.7}$$

$$\left| \arg \left(\frac{f(z)}{z^p} \right) \right| \leq A(p - \alpha)r, B = o. \tag{2.8}$$

These bounds are sharp, being attained by the function $f_o(z)$ defined by

$$f_o(z) = \begin{cases} z^p(1 + B \delta z)^{\frac{(A - B)(p - \alpha)}{B}} & , B \neq o, \\ \\ z^p \exp(A(p - \alpha) \delta z) & , B = o, |\delta| = 1. \end{cases} \tag{2.9}$$

PROOF. It was shown by Goel and Mehrok [5] that for $f_1 \epsilon$ S*(A, B)

$$\left| \arg \frac{f_1(z)}{z} \right| \leq \frac{(A - B)}{B} \sin^{-1} (Br), \ B \neq o, \tag{2.10}$$

$$\left| \arg \frac{f_1(z)}{z} \right| \leq Ar , \ \ B = o. \tag{2.11}$$

Therefore, the proof of Theorem 2 is an immediate consequence of (1.9), (2.10) and (2.11).

REMARK ON THEOREM 2.

Choosing $p = 1$, $A = 1$ and $B = -1$ in Theorem 2, we get the result due to Pinchuk [6].

3. COEFFICIENT ESTIMATES FOR THE CLASS S(A, B, P, α).

LEMMA 2.   If integers p and m are greater than zero; $o \leq \alpha < p$ and $-1 \leq B < A \leq 1$. Then

$$\prod_{j=0}^{m-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} = \frac{1}{m^2} \{ \ (B - A)^2(p - \alpha)^2 +$$

$$\sum_{k=1}^{m-1} [k^2(B^2 -1) + (B - A)^2(p - \alpha)^2 + 2kB(B - A)(p - \alpha)] \times$$

$$\prod_{j=0}^{k-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} \} \ . \tag{3.1}$$

PROOF. We prove the lemma by induction on m. For m = 1, the lemma is obvious.
Next, suppose that the result is true for m = q-1. We have

$$\frac{1}{q^2} \{ \ (B - A)^2(p - \alpha)^2 + \sum_{k=1}^{q-1} [k^2(B^2 - 1) +$$

$$(B - A)^2(p - \alpha)^2 + 2kB(B - A)(p - \alpha)] \times \prod_{j=0}^{k-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} \}$$

$$= \frac{1}{q^2} \{ \ (B - A)^2(p - \alpha)^2 + \sum_{k=1}^{q-2} [k^2(B^2 - 1) +$$

$$(B - A)^2(p - \alpha)^2 + 2kB(B - A)(p - \alpha)] \times \prod_{j=0}^{k-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2}$$

$$+ [(q - 1)^2 (B^2 - 1) + (B - A)^2 (p - \alpha)^2 + 2(q - 1)B(B - A)(p - \alpha)] \times$$

$$\prod_{j=0}^{q-2} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} \}$$

$$= \prod_{j=o}^{q-2} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} \times \{ \frac{(q - 1)^2 B^2 + (B - A)^2 (p - \alpha)^2 + 2(q - 1)B(B - A)(p - \alpha)}{q^2} \}$$

$$= \prod_{j=o}^{q-1} \frac{|(B - A)(p - \alpha) + Bj|^2}{(j + 1)^2} .$$

Showing that the result is valid for m = q. This proves the lemma.

THEOREM 3. If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \in S(A, B, p, \alpha)$, then

$$|a_n| \leq \prod_{k=o}^{n-(p+1)} \frac{|(B - A)(p - \alpha) + Bk|}{k + 1} \qquad (3.2)$$

for n ≥ p + 1, and these bounds are sharp for all admissible A, B and α and for each n.

PROOF. As f ∈ S(A, B, p, α), from (1.4) and (1.7), we have

$$\frac{zf'(z)}{f(z)} = \frac{p + [pB + (A - B)(p - \alpha)]w(z)}{1 + Bw(z)} , w \in \Omega .$$

This may be written as

$$\{ Bzf'(z) + [-pB + (B - A)(p - \alpha)]f(z) \} w(z) = pf(z) - zf'(z).$$

Hence

$$[B \{ pz^p + \sum_{k=1}^{\infty} (p + k)a_{p+k} z^{p+k}\} + [-pB + (B - A)(p - \alpha)] \{ z^p +$$

$$\sum_{k=1}^{\infty} a_{p+k} z^{p+k}\}]w(z) = p \{ z^p + \sum_{k=1}^{\infty} (p + k)a_{p+k} z^{p+k}\} -$$

$$\{ pz^p + \sum_{k=1}^{\infty} (p + k)a_{p+k} z^{p+k} \}$$

or

$$[pB + [-pB + (B - A)(p - \alpha)] + \sum_{k=1}^{\infty} \{ (p + k)B +$$

$$[-pB + (B - A)(p - \alpha)] \} a_{p+k} z^k] w(z) = (P - p) +$$

$$\sum_{k=1}^{\infty} \{ p - (p + k) \} a_{p+k} z^k$$

which may be written as

$$\sum_{k=o}^{\infty} [\{(p+k)B + [-pB + (B-A)(p-\alpha)]\} a_{p+k} z^k] w(z)$$

$$\sum_{k=o}^{\infty} \{-k\} a_{p+k} z^k \qquad\qquad (3.3)$$

where $a_p = 1$ and $w(z) = \sum_{k=o}^{\infty} b_{k+1} z^{k+1}$.

Equating coefficients of z^m on both sides of (3.3), we obtain

$$\sum_{k=o}^{m-1} \{(p+k)B + [-pB + (B-A)(p-\alpha)]\} a_{p+k} b_{m-k} =$$

$$\{-m\} a_{p+m} ;$$

which shows that a_{p+m} on right-hand side depends only on

$$a_p, a_{p+1}, \cdots, a_{p+(m-1)}$$

of left-hand side. Hence we can write

$$\sum_{k=o}^{m-1} [\{(p+k)B + [-pB + (B-A)(p-\alpha)]\} a_{p+k} z^k] w(z)$$

$$= \sum_{k=o}^{m} \{-k\} a_{p+k} z^k + \sum_{k=m+1}^{\infty} A_k z^k$$

for $m = 1, 2, 3, \ldots$, and a proper choise of $A_k (k \geq o)$.
Let $z = re^{i\theta}$, $o < r < 1$, $o \leq \theta \leq 2\pi$, then

$$\sum_{k=o}^{m-1} | (p+k)B + [-pB + (B-A)(p-\alpha)] |^2 | a_{p+k}|^2 r^{2k}$$

$$= \frac{1}{2\pi} \int_o^{2\pi} \left| \sum_{k=o}^{m-1} \{(p+k)B + [-pB + (B-A)(p-\alpha)]\} a_{p+k} r^k e^{i\theta k} \right|^2 d\theta$$

$$\geq \frac{1}{2\pi} \int_o^{2\pi} \left| \sum_{k=o}^{m-1} \{(p+k)B + [-pB + (B-A)(p-\alpha)]\} a_{p+k} r^k e^{i\theta k} \right|^2 |w(re^{i\theta})|^2 d\theta$$

$$\geq \frac{1}{2\pi} \int_o^{2\pi} \left| \sum_{k=o}^{m} \{-k\} a_{p+k} r^k e^{i\theta k} + \sum_{k=m+1}^{\infty} A_k r^k e^{i\theta k} \right|^2 d\theta$$

$$\geq \sum_{k=o}^{m} |-k|^2 |a_{p+k}|^2 r^{2k} + \sum_{k=m+1}^{\infty} |A_k|^2 r^{2k}$$

$$\geq \sum_{k=o}^{m} k^2 |a_{p+k}|^2 r^{2k} . \qquad\qquad (3.4)$$

Setting $r \to 1$ in (3.4), the inequality (3.4) may be written as

$$\sum_{k=0}^{m-1} \{ \, |(p+k) \, B + [-pB + (B - A)(p - \alpha)] \, |^2 - k^2 \} |a_{p+k}|^2$$

$$\geq m^2 \, |a_{p+m}|^2 \quad . \tag{3.5}$$

Simplification of (3.5) leads to

$$|a_{p+m}|^2 \leq \frac{1}{m^2} \sum_{k=0}^{m-1} \{ \, k^2(B^2 - 1) + (B - A)(p - \alpha)[(B - A)(p - \alpha) + 2kB] \} |a_{p+k}|^2 . \tag{3.6}$$

Replacing $p + m$ by n in (3.6), we are led to

$$|a_n|^2 \leq \frac{1}{(n-p)^2} \cdot \sum_{k=0}^{n-(p+1)} \{ k^2(B^2 - 1) + (B - A)(p - \alpha) \times$$

$$[B - A)(p - \alpha) + 2kB] \} \, |a_{p+k}|^2 , \tag{3.7}$$

where $n \geq p + 1$.
For $n = p + 1$, (3.7) reduces to

$$|a_{p+1}|^2 \leq (B - A)^2(p - \alpha)^2$$

or

$$|a_{p+1}| \leq (A - B)(p - \alpha) \tag{3.8}$$

which is equivalent to (3.2).
    To establish (3.2) for $n > p + 1$, we will apply induction argument.
    Fix n, $n \geq p + 1$, and suppose (3.2) holds for $k = 1, 2, \ldots , \ldots\ldots , \ldots$
$n-(p+1)$. Then

$$|a_n|^2 \leq \frac{1}{(n-p)^2} \{ (B - A)^2(p - \alpha)^2 + \sum_{k=1}^{n-(p+1)} \{ k^2(B^2 - 1) +$$

$$(B - A)(p - \alpha)[(B - A)(p - \alpha) + 2kB] \} \prod_{j=0}^{k-1} \frac{|(B-A)(p-\alpha) + Bj \, |^2}{(j + 1)^2} \} . \tag{3.9}$$

Thus, from (3.7), (3.9) and Lemma 2 with $m = n - p$, we obtain

$$|a_n|^2 \leq \prod_{j=0}^{n-(p+1)} \frac{|(B - A)(p - \alpha) + Bj \, |^2}{(j + 1)^2} \quad .$$

This completes the proof of (3.2). This proof is based on a technique found in Clunie [7].

For sharpness of (3.2) consider

$$f(z) = \frac{z^p}{(1 - B \delta z)^{\frac{(B-A)(p-\alpha)}{B}}} \quad , |\delta| = 1, \; B \neq 0.$$

REMARK ON THEOREM 3.

Choosing $p = 1$, $A = 1$, and $B = -1$ in Theorem 3, we get the result due to Robertson [3] and Schild [8].

4.   DISTORTION AND COEFFICIENT BOUNDS FOR FUNCTIONS IN $P(A, B, p, \alpha)$.

THEOREM 4.   If $P(z) \in P(A, B, p, \alpha)$, then

$$|\arg P(z)| \leq \sin^{-1} \frac{(A-B)(p-\alpha)r}{p-[pB + (A-B)(p-\alpha)]Br^2} \; , \; |z| = r.$$

The bound is sharp.

PROOF.   The proof follows from Lemma 1. To see that the result is sharp, let

$$P(z) = p \left\{ \frac{1 + [B + (A-B)(1 - \frac{\alpha}{p})]\delta_1 z}{1 + \delta_1 Bz} \right\} \; , \; |\delta_1| = 1. \qquad (4.1)$$

Putting

$$\delta_1 = \frac{r}{z} \left\{ \frac{-([B + (A-B)(1 - \frac{\alpha}{p})] + B)r}{1 + [B + (A-B)(1 - \frac{\alpha}{p})]Br^2} + \right.$$

$$\left. \frac{i \sqrt{1 - [B + (A-B)(1 - \frac{\alpha}{p})]^2 r^2} \sqrt{1 - B^2 r^2}}{1 + [B + (A-B)(1 - \frac{\alpha}{p})]Br^2} \right\} \quad , \; r = |z| \; ,$$

in (4.1), we have

$$\arg P(z) = \sin^{-1} \frac{(A - B)(p - \alpha)r}{p - [pB + (A - B)(p - \alpha)]Br^2} \qquad .$$

An immediate consequence of Lemma 1 is

COROLLARY 1.   If $P(z)$ is in $P(A, B, p, \alpha)$, then for $|z| \leq r < 1$

$$\frac{p - (A-B)(p - \alpha)r - [pB + (A-B)(p - \alpha)]Br^2}{1 - B^2 r^2} \quad \leq |P(z)| \leq$$

$$\frac{p + (A-B)(p - \alpha)r - [pB + (A-B)(p - \alpha)]Br^2}{1 - B^2 r^2} \quad ,$$

and

$$\frac{p - (A-B)(p - \alpha)r - [pB + (A-B)(p - \alpha)]Br^2}{1 - B^2 r^2} \quad \leq \text{Re} \{P(z)\} \leq$$

$$\frac{p + (A-B)(p - \alpha)r - [pB + (A-B)(p - \alpha)]Br^2}{1 - B^2 r^2} \qquad .$$

REMARK ON COROLLARY 1.

Choosing $p = 1$, $A = 1$ and $B = -1$ in the above corollary we get the following distortion bounds studied by Libera and Livingston [9] stated in the following corollary.

COROLLARY 2.   If $P(z)$ is in $P(1, -1, \alpha) = P(\alpha)$, $0 \le \alpha < 1$ (the class of functions $P(z)$ with positive real part of order α, $0 \le \alpha < 1$), then for $|z| \le r < 1$

$$\frac{1 - 2(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2} \le |P(z)| \le \frac{1 + 2(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2}.$$

and

$$\frac{1 - 2(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2} \le \mathrm{Re}\{P(z)\} \le \frac{1 + 2(1 - \alpha)r + (1 - 2\alpha)r^2}{1 - r^2} \quad .$$

The coefficient bounds which follow are derived by using the method of Clunie [7].

THEOREM 5.   If $P(z) = p + \sum\limits_{k=1}^{\infty} b_k z^k$ is in $P(A, B, p, \alpha)$, then

$$|b_n| \le (A - B)(p - \alpha), \quad n = 1, 2, \ldots ; \qquad (4.2)$$

these bounds are sharp.

PROOF.   The representation $P(z)$ in (1.4) is equivalent to

$$[B + (A - B)(p - \alpha) - B P(z)]w(z) = P(z) - p, \quad w \in \Omega . \qquad (4.3)$$

or

$$[B + (A - B)(p - \alpha) - B \sum\limits_{k=0}^{\infty} b_k z^k]w(z) = \sum\limits_{k=1}^{\infty} b_k z^k, \quad b_0 = p. \qquad (4.4)$$

This can be written as

$$[(A - B)(p - \alpha) - B \sum\limits_{k=1}^{n-1} b_k z^k]w(z) = \sum\limits_{k=1}^{n} b_k z^k + \sum\limits_{k=n+1}^{\infty} q_k z^k , \qquad (4.5)$$

the last term also being absolutely and uniformly convergent in compacta on U. Writing $z = re^{i\theta}$, performing the indicated integration and making use of the bound $|w(z)| \le |z| < 1$ for z in U gives

$$(A - B)^2(p - \alpha)^2 + B^2 \sum_{k=1}^{n-1} |b_k|^2 r^{2k} =$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| (A - B)(p - \alpha) + B \sum_{k=1}^{n-1} b_k r^k e^{ik\theta} \right|^2 d\theta$$

$$\geq \frac{1}{2\pi} \int_0^{2\pi} \left| \{(A - B)(p - \alpha) + B \sum_{k=1}^{n-1} b_k r^k e^{i\theta k}\} w(re^{i\theta}) \right|^2 d\theta$$

$$\geq \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{n} b_k r^k e^{ik\theta} + \sum_{k=n+1}^{\infty} q_k r^k e^{ik\theta} \right|^2 d\theta$$

$$\geq \sum_{k=1}^{n} |b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |q_k|^2 r^{2k} .$$

The last term is non-negative and $r < 1$, therefore

$$(A - B)^2(p - \alpha)^2 + B^2 \sum_{k=1}^{n-1} |b_k|^2 \geq \sum_{k=1}^{n} |b_k|^2 , \qquad (4.6)$$

or

$$|b_n|^2 \leq (A - B)^2(p - \alpha)^2 + (B^2 - 1) \sum_{k=1}^{n-1} |b_k|^2 . \qquad (4.7)$$

Since $-1 \leq B < 1$, we have $B^2 - 1 \leq 0$. Hence

$$|b_n| \leq (A - B)(p - \alpha) , \qquad (4.8)$$

and this is equivalent to (4.2). If $w(z) = z^n$, then

$$P(z) = p + (A - B)(p - \alpha)z^n + \ldots ,$$

which makes (4.2) sharp.

REMARK ON THEOREM 5.

Choosing $p = 1$, $A = 1$, and $B = -1$ in Theorem 5, we get the result due to Libera [10] stated in the following corollary.

COROLLARY 3. If $P(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \in P(1, -1, 1, \alpha) = P(\alpha)$, $0 \leq \alpha < 1$,

then

$$|b_n| \leq 2(1 - \alpha), n = 1, 2, 3, \ldots ;$$

these bounds are sharp.

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