

# ON THE COMPUTATION OF THE CLASS NUMBERS OF SOME CUBIC FIELDS

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**ABSTRACT.** Class numbers are calculated for cubic fields of the form  $x^3 + 12Ax - 12 = 0$ ,  $A > 0$ , for  $1 \leq a \leq 17$ , and for some other values of  $A$ . These fields have a known unit, which under certain conditions is the fundamental unit, and are important in studying the Diophantine Equation  $x^3 + y^3 + z^3 = 3$ .

**KEY WORDS AND PHRASES.** *Class numbers, cubic fields, Diophantine equation.*

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## 1. INTRODUCTION AND SOME THEOREMS.

We consider the cubic fields defined by an equation of the form

$$f(x) = x^3 + 12Ax - 12 = 0, \quad (1.1)$$

where  $A > 0$ . The field defined by this equation is important because it is related to the Diophantine equation  $x^3 + y^3 + z^3 = 3$  when  $A = 9a^2$  [1]. Equation (1.1) is clearly irreducible, and as  $f(x)$  is increasing, it defines a real cubic field  $K$  (with two complex conjugates) with exactly one fundamental unit. Let  $\theta$  be the real root of (1.1). We write  $K = Q(\theta)$ . Note that  $0 < \theta < 1$ . Also  $\eta = \frac{\theta^3}{12} = 1 - A\theta$  defines a unit of  $K$ . As  $0 < \eta < 1$ , we have  $\theta < \frac{1}{A}$ . The discriminant of  $f(x)$  is  $D = -2^4 \cdot 3^3 (16A^3 + 9)$ . As  $f(x)$  is an Eisenstein polynomial with respect to 3, we have  $(3) = q^3$ . Also as  $\frac{6}{\theta}$  satisfies  $x^3 - 36Ax - 18 = 0$ , we see that for the same reason  $(2) = p^3$ , and as  $\frac{6}{\theta} = 6A + \frac{\theta^2}{2}$  we see that  $\frac{\theta^2}{2} \in \mathcal{O}_K$ , the ring of integers of  $K$ . Thus the discriminant,  $D$ , of  $K$ , divides  $-2^2 \cdot 3^3 (16A^3 + 9)$ . We now state:

**THEOREM 1.** In  $K$ , the discriminant  $D = \frac{-2^2 \cdot 3^3 (16A^3 + 9)}{q^2}$  where  $q^2$  is the largest square, prime to 3, dividing  $D$ . The unit  $\eta$  is never a cube, and if  $q=1$  or  $q=5$  then  $\eta$  is the fundamental unit except when  $A = 1$ . The class-number  $h$ , of  $K$ , is divisible by 3. The primes  $p_i$  dividing  $D$  (except for 2 and 3) ramify as

$(p_i) = p_i^2 q_i$ . A basis for  $O_K$  is given by  $\theta_0 = 1$ ,  $\theta_1 = \frac{\theta^2}{2}$ ,  $\theta_2 = \frac{16A^2 + 3\theta + 2A\theta^2}{3^i q}$ ,  
 $(\beta^i = (3, A))$ .

As the proof is similar to the proof of the corresponding theorem in [1], we omit it, as well as the proof of the following two theorems, also in [1].

THEOREM 2. If the 3-component of the class-group of  $K$  is a direct product of cyclic groups of order 3, then

$$x^3 + 12Ax - 12 = 4z^3 \quad (1.2)$$

has no solutions.

Corollary: If  $3 \nmid h$ , then (1.2) has no solutions.

THEOREM 3. If  $(h, 2) = 1$ , and  $q = 1$ , then solving  $x^3 + 12Ax - 12 = y^2$  is equivalent to solving  $-AG^4 - 2G^3H + 3H^4 = -1$  (This has no solutions (mod  $p$ ) for small primes  $p$ , e.g.  $A = 14$ ,  $p = 5$ ).

## 2. NUMERICAL COMPUTATIONS.

$$\text{We note that } \lim_{s \rightarrow 1+} \frac{\zeta_k(s)}{\zeta(s)} = \frac{4\pi \log \epsilon \cdot h}{2\sqrt{2^2 \cdot 3^3} (16A^3 + 9)/q^2} \quad (2.1)$$

where  $\epsilon > 1$  is the fundamental unit of  $K$ . As in [2], the left-hand side of (2.1) can

be expressed as  $f = \lim_{P \rightarrow \infty} f_P = \lim_{P \rightarrow \infty} \prod_{p=5}^P f(p)$  where

$$f(p) = \begin{cases} \frac{p}{p-1} & \text{if } p \text{ ramifies } ((p_i) = p_i^2 q_i) \\ \frac{p^2}{p^2 + p + 1} & \text{if } p \text{ remains inert} \\ \frac{p^2}{p^2 - 1} & \text{if } (p) = p q \\ \left(\frac{p}{p-1}\right)^2 & \text{if } p \text{ splits completely} \end{cases}$$

Hence (2.1) implies that approximately,

$$h = \frac{\sqrt{27(16A^3 + 9)}}{\pi \cdot q \cdot \log \epsilon} f_P \quad (2.2)$$

for  $P$  sufficiently large.

For Table 1, the product in (2.2) was calculated for  $P = P(2027)$ , (at intervals of 50), where  $P(i)$  indicates the  $i^{\text{th}}$  prime, and  $1 \leq A \leq 36$ :

**TABLE 1**

<u>A</u>	<u><math>-D/2^2 \cdot 3^3</math></u>	<u>h</u>	<u>A</u>	<u><math>-D/2^2 \cdot 3^3</math></u>	<u>h</u>
1	5 <sup>2</sup>	6	19	7·15679	39
2	137	3	20	7·18287	72
3	3 <sup>2</sup> ·7 <sup>2</sup>	6	21	3 <sup>2</sup> ·5·37·89	54
4	1033	6	22	347·491	36
5	7 <sup>2</sup> ·41	3	23	194681	72
6	3 <sup>2</sup> ·5·7·11	21	24	3 <sup>2</sup> ·7·3511	54

<u>A</u>	<u><math>-D/2^2 \cdot 3^3</math></u>	<u>h</u>	<u>A</u>	<u><math>-D/2^2 \cdot 3^3</math></u>	<u>h</u>
7	23·239	9	25	29·37·233	72
8	59·139	18	26	$5^2 \cdot 7 \cdot 1607$	15
9	$3^2 \cdot 1297$	12	27	$3^2 \cdot 7 \cdot 4999$	54
10	7·2287	27	28	11·37·863	78
11	5·4261	24	29	359·1087	48
12	$3^2 \cdot 7 \cdot 439$	24	30	$3^2 \cdot 23 \cdot 2087$	72
13	7·5023	48	31	5·7·13619	162
14	43913	21	32	17·3084	78
15	$3^2 \cdot 17 \cdot 353$	36	33	7·82143	114
16	5·13109	36	34	7·89839	87
17	7·11·1021	48	35	686009	75
18	$3^2 \cdot 10369$	36	36	$3^2 \cdot 5 \cdot 53 \cdot 313$	156

In all the cases above except when  $A = 1$  or  $A = 5$ ,  $\eta = \frac{1}{\epsilon}$  is the fundamental unit of  $K$ . When  $A = 1, 5$ ,  $\eta = \epsilon^{-2}$ .  $K$  is a pure cubic field if and only if  $A = 1$  or  $A = 3$ .

Also because of the equivalence of (1.2) with the Diophantine equation  $x^3 + y^3 + z^3 = 3$  when  $A = 9a^2$ , the class-numbers of  $K$  were calculated using (2.2) for  $1 \leq a \leq 17$  (Actually Cassels has shown that for solutions of (1.2) to exist in this case, one must have  $3|a$  [3]). While most of the values obtained in this way were approximate, perhaps congruence conditions may be used to find them exactly, or perhaps they may be of use in regards to Brauer-Siegel Theorem, so we list them in Table 2. (The Brauer-Siegel Theorem applied here states  $\log h \sim \frac{3}{2} \log A - \log q$ ).

TABLE 2

<u>a</u>	<u><math>-D/2^2 \cdot 3^5</math></u>	<u>I (P(I) is the I<sup>th</sup> prime)</u>	<u>h</u>
1	1297	8303	12
2	5·53·313	14903	156
3	5·188957	10803	216
4	5380417	4303	420
5	3557·5693	2201	789 (*)
6	37·241·6781	3003	1410
7	5·30494621	1002	3285
8	$5^3 \cdot 17 \cdot 29 \cdot 37 \cdot 149$	1002	873
9	17·40514561	1002	3549
10	181·1361·5261	1002	6999
11	89·25797113	212	6753
12	5·23761·32573	212	15999
13	5·8821·141833	212	21864

<u>a</u>	<u><math>-D/2^2 \cdot 3^5</math></u>	<u>I (P(I) is the I<sup>th</sup> prime)</u>	<u>h</u>
14	37·263737261	212	10062
15	1193·2381·5197	212	22653
16	35801·607337	212	16764
17	5 <sup>2</sup> ·1251291577	212	4644

The second column gives the factorization of  $-D/2^2 \cdot 3^5$ .

(\*) For  $a \geq 5$ , the values of  $h$  should be considered as estimates, but are probably accurate within  $\frac{1}{2}\%$ .

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