

**BOUNDS ON THE CURVATURE FOR FUNCTIONS WITH  
 BOUNDED BOUNDARY ROTATION OF ORDER 1-b**

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**ABSTRACT.** Let  $V_k(1-b)$ ,  $k \geq 2$  and  $b \neq 0$  real, denotes the class of locally univalent analytic functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  in  $D = \{z: |z| < 1\}$  such that  $\int_0^{2\pi} \left| \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z f''(z)}{f'(z)} \right\} \right| d\theta < \pi k$ ,  $z = re^{i\theta} \in D$ . In this note sharp bounds on the curvature of the image of  $|z| = r$ ,  $0 < r < 1$ , under a mapping  $f$  belonging to the class  $V_k(1-b)$  have been obtained.

**KEY WORDS AND PHRASES.** Analytic Function, Univalent Functions, Functions with Bounded Boundary Rotation.

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1. INTRODUCTION.

Let  $A$  denote the class of functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  which are analytic in  $D = \{z: |z| < 1\}$ . For  $G \in A$ , we say  $G$  belongs to the class  $S(1-b)$  ( $b \neq 0$  complex) if and only if  $G(z)/z \neq 0$  in  $D$  and  $\operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{z G'(z)}{G(z)} - 1 \right) \right\} > 0$ ,  $z \in D$ .

The class  $S(1-b)$  was introduced by Nasr and Aouf in [1]. It is shown in [1] that  $G \in S(1-b)$  if and only if there is a function  $g \in S(0)$  such that

$$G(z) = z [g(z)/z]^b \tag{1.1}$$

and for  $b \neq 0$  real

$$(1+r)^{-2b} \leq |G(z)/z| \leq (1-r)^{-2b} \tag{1.2}$$

$$(1-r)^{-2b} \leq |G(z)/z| \leq (1+r)^{-2b} \tag{1.3}$$

Let  $V_k(1-b)$ ,  $k \geq 2$  and  $b \neq 0$  complex, denotes the class of functions  $f \in A$  which satisfy  $f(z) \neq 0$  in  $D$  and

$$\int_0^{2\pi} \left| \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z f''(z)}{f'(z)} \right\} \right| d\theta \leq \pi k, \quad z = re^{i\theta} \in D.$$

The class  $V_k(1-b)$  was introduced by Nasr [2]. It was shown in [2] that  $f \in V_k(1-b)$  if and only if there exist two functions  $g_1, g_2 \in S(0)$  such that

$$f'(z) = \{g_1(z)/z\}^{b(k+2)/4} / \{g_2(z)/z\}^{b(k-2)/4} \tag{1.4}$$

And from (1.1) and (1.4) we deduce immediately that  $f \in V_k(1-b)$  if and only if there exist two functions  $G_1, G_2 \in S(1-b)$  such that

$$f'(z) = \{G_1(z)/z\}^{(k+2)/4} / \{G_2(z)/z\}^{(k-2)/4}$$

The subclasses  $S(1-b)$ ,  $V_2(1-b)$  and  $V_k(1-b)$  are respectively, classes of functions starlike of order  $1-b$ , convex of order  $1-b$  and of bounded boundary rotation of order  $1-b$ . We shall denote the subclasses  $V_2(1-b)$  and  $V_k(1-b)$  respectively by  $C(1-b)$  and  $V_k$ .

For a locally univalent function  $f$  in  $D$  the curvature  $K_r^f(z)$  at the point  $w = f(z)$  for the level line, i.e. the image of the circle  $|z| = r$  under the mapping  $f$ , is given by

$$K_r^f(z) = \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)^2} \right\} / |zf'(z)| \tag{1.6}$$

Let  $\inf K_r^B$  and  $\sup K_r^B$  denote respectively, the infimum and supremum of  $K_r^f(z)$  for  $|z| = r$  when  $f$  belongs to a certain subclass  $B$  of locally univalent functions in  $A$ , which is normal and compact.

The problem of estimating  $K_r^f(z)$  for various classes of functions has attracted considerable attention (see [3, p.p. 599-601] for the history of this problem).

The purpose of this paper is to establish  $\inf K_r^{V_k(1-b)}$  and  $\sup K_r^{V_k(1-b)}$  for  $b \neq 0$  real.

2. STATEMENT OF RESULTS.

$$\text{Set } k_1 = (k-2)/4 \text{ and } H(r) = (1+r^2)/2r - \{\log(1+r)/(1-r)\}^{-1}, \quad 0 < r < 1.$$

A simple calculation shows that  $H(r)$  increases strictly with  $r$  and that  $0 < H(r) < 1$ .

THEOREM 1. If  $f \in V_k(1-b)$ ,  $b > 0$ , then

$$\inf K_r^{V_k(1-b)} = \begin{cases} (1-r^2)^b/r & \text{for } k = 2, \quad 0 < b < 1, \\ \frac{(1+r)^{2b(k_1+1)} - 1}{r(1-r)^{2bk_1+1}} \{1 - bkr + (2b-1)r\} & \text{otherwise} \end{cases} \tag{2.1}$$

and

$$\sup K_r^{V_k(1-b)} = \begin{cases} \frac{(1-r)^{r(1-b)} + b}{r(1+r)^{r(1-b)} - b} \left\{ \frac{e}{2r} \frac{(1-r)^{(1+r)^2/2r}}{(1+r)^{(1-r)^2/2r}} \log \left( \frac{1+r}{1-r} \right) \right\}^{1+bk_1} & \text{for } r - \frac{b(k_1+1)}{1+bk_1} < (1+r) < H(r) < r + \frac{b(k_1+1)}{1+bk_1} (1-r) \\ \frac{(1-r)^{2b(k_1+1)} - 1}{r(1+r)^{2bk_1+1}} \{1 + bkr + (2b-1)r^2\} & \text{otherwise} \end{cases} \tag{2.2}$$

These bounds are sharp for all  $0 < r < 1$ .

THEOREM 2. If  $t \in v_k(1-b)$ ,  $b < 0$ , then

$$\inf K_r^{V_k(1-b)} = \frac{(1-r)^{2b(k_1+1)-1}}{r(1+r)^{2bk_1+1}} \{1 + bkr + (2b-1)r^2\} \quad (2.5)$$

and

$$\sup K_r^{V_k(1-b)} = \begin{cases} \frac{(1-r)^{r(1-b)+b}}{r(1+r)^{r(1-b)-b}} \left\{ \frac{e}{2r} \frac{(1-r)^{(1+r)^2/2r}}{(1+r)^{(1-r)^2/2r}} \log\left(\frac{1+r}{1-r}\right) \right\}^{b(k_1+1)-1} & (2.6) \end{cases}$$

$$\sup K_r^{V_k(1-b)} = \begin{cases} \text{for } r - \frac{bk_1}{b(k_1+1)-1}(1+r) \leq H(r) \leq r + \frac{bk_1}{b(k_1+1)-1}(1-r) & (2.7) \end{cases}$$

$$\sup K_r^{V_k(1-b)} = \begin{cases} \frac{(1+r)^{2b(k_1+1)-1}}{r(1-r)^{2bk_1+1}} \{1 - bkr + (2b-1)r^2\} & \text{otherwise.} \quad (2.8) \end{cases}$$

These bounds are sharp for all  $0 < r < 1$ .

Indeed, if  $k = 2$ ,  $b > 0$  our results coincide with the results given for  $\inf K_r^{C(1-b)}$  and  $\sup K_r^{C(1-b)}$  by Singh [7], also for  $k = 2$ ,  $0 < b \leq 1$ , our results are reduced to those given for  $\inf K_r^{C(1-b)}$  and  $\sup K_r^{C(1-b)}$  by Zederkiewicz [4] and those given by Eenigenburg [5] for  $\inf K_r^{C(1-b)}$ . Moreover, for  $k = 2$ ,  $b = 1$ , coincide with those given for  $\inf K_r^{C(0)}$  and  $\sup K_r^{C(0)}$  by Zmorović [6] and those given by Keogh [7] for  $\inf K_r^{C(0)}$ . Finally, if  $b = 1$  our results agree with those reached by Noonan [8] and Singh [9] for  $\inf K_r^{V_k}$  and  $\sup K_r^{V_k}$ . But to the best of our knowledge the values of  $\inf K_r^{C(1-b)}$  and  $\sup K_r^{C(1-b)}$  for  $b < 0$  and also the values of  $\inf K_r^{V_k(1-b)}$  and  $\sup K_r^{V_k(1-b)}$  for  $b < 0$  and also the values of  $\inf K_r^{V_k(1-b)}$  and  $\sup K_r^{V_k(1-b)}$  for  $b \neq 1$  are not known as yet.

3. PROOFS.

We need the following:

LEMMA 1[9]: If  $g \in S(0)$ ,  $z = r e^{i\theta} \in D$ , then

$$(1-r^2) \left| \frac{g(z)}{z} \right| \leq \operatorname{Re} \frac{zg'(z)}{g(z)} \leq \frac{1+r}{1-r} + \frac{2r \log|(1-r^2)g(z)/z|}{(1-r^2) \log\{(1+r)/(1-r)\}} \quad (3.1)$$

Both sides of the above inequality are sharp.

COROLLARY 1. If  $G \in S(1-b)$ ,  $z = r e^{i\theta} \in D$ , then

$$B(r, G(z)/z) \leq A(r, G(z)/z) \quad \text{for } b > 0 \quad (3.2)$$

$$\leq \operatorname{Re} \frac{zG'(z)}{G(z)} \leq$$

$$A(r, G(z)/z) \leq B(r, G(z)/z) \quad \text{for } b < 0 \quad (3.3)$$

where

$$A(r, x) = \frac{1 + (2b - 1)r}{1 - r} + \frac{2r \log |(1-r)^{2b} x|}{(1-r^2) \log [(1+r)/(1-r)]}$$

and

$$B(r, x) = (1-b) + b(1-r^2) |x|^{1/b}$$

PROOF. The proof will follow immediately from (1.1) and (3.1)

PROOF OF THEOREM 1. Set

$$|G_1(z)/z| = u \quad \text{and} \quad |G_2(z)/z| = v. \quad (3.4)$$

Then from (2.2), we find that  $u$  and  $v$  lie in the interval

$$[1/(1+r)^{2b}, 1/(1-r)^{2b}]. \quad (3.5)$$

In view of (1.5) and (1.6) we need to find the extreme values of

$$K_r^f(z) = v^{k_1} \left[ (k_1 + 1) \frac{zG_1'(z)}{G_1(z)} - k_1 \frac{zG_2'(z)}{G_2(z)} \right] / ru^{k_1+1}, \quad (3.6)$$

In view of (3.2) and (3.6) we need to obtain the minimum of

$$F(u, v) = v^{k_1} [(k_1 + 1)B(r, u) - k_1 A(r, v)] / ru^{k_1+1} \quad (3.7)$$

and the maximum of

$$H(u, v) = v^{k_1} [(k_1 + 1)A(r, u) - k_1 B(r, v)] / ru^{k_1+1} \quad (3.8)$$

when  $u$  and  $v$  lie in the interval given by (3.5).

This reduces the problem to finding extreme values of functions of two variables.

It is easily seen that for  $0 < b \leq 1$  and  $k = 2$  ( $k_1 = 0$ ) the minimum is attained for

$$u = 1/(1-r^2)^b \quad (3.9)$$

and because the value of  $u$  lies within the interval given by (3.5) this gives the minimum. We thus obtain (2.1). If  $k = 2$ ,  $b > 1$  or  $k > 2$ ,  $b > 0$  it is readily confirmed that the roots of

$$\frac{\partial F}{\partial u} = 0 = \frac{\partial F}{\partial v}$$

do not give the minimum. Hence, the minimum is attained on the boundary for

$u = 1/(1-r)^{2b}$  and  $v = 1/(1-r)^{2b}$ . This yields (2.2).

In order to maximize  $H(u, v)$ , it is found that the equations

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial H}{\partial v}$$

give

$$v = \{2r/(1-r^2)^2 \log [(1+r)/(1-r)]\}^b \quad (3.10)$$

and

$$\log [(1-r)^{2b} u] = \frac{1 + bk_1}{1 + k_1} - \left[ \frac{1 + bk_1}{1 + k_1} + \frac{2br}{1-r} \right] \frac{(1-r^2)}{2r} \log \left( \frac{1-r}{1+r} \right) \quad (3.11)$$

The value of  $v$  given by (3.10) lies in the interval given by (3.5) because

$$2r/(1+r)^2 \leq \log [(1+r)/(1-r)] \leq 2r/(1-r)^2 \quad (3.12)$$

but the value of  $u$  given by (3.11) lies in the interval given by (3.5) if

$$r - [b(1+k_1)(1+r)/(1+bk_1)] \leq H(r) \leq r + [b(1+k_1)(1-r)/(1+bk_1)]. \tag{3.13}$$

This gives (2.3). When (3.13) does not hold, the maximum values of (3.8) is obtained on the boundary for  $u = 1/(1-r)^{2b}$  and  $v = 1/(1+r)^{2b}$ . This yields (2.4).

The above inequalities are sharp and the extremal functions are given below where equality in each case, is attained at  $z = r$ .

(i) For equality in (2.1)

$$f_1'(z) = 1/[(1 - ze^{it})^\lambda (1 - ze^{-it})(1 - \lambda)]^b, \quad 0 \leq \lambda \leq 1, \quad 0 < b \leq 1, \quad \text{where}$$

$$\cos t = r \quad \text{and} \quad \lambda b = \frac{1+r^2}{r} + (1-b)r - H(r) \tag{3.14}$$

(ii) For equality in (2.2)

$$f_2'(z) = (1-z)^{2bk_1} / (1+z)^{2b(1+k_1)} \tag{3.15}$$

(iii) For equality in (2.3)

$$f_3'(z) = (1-ze^{-it})^{2bk_1} \frac{(1-z)^{(1+bk_1)H(r)+r(b-1)-b(1+k_1)}}{(1+z)^{(1+bk_1)H(r)+(b-1)+b(1+k_1)}} \tag{3.16}$$

where

$$1+r^2 - 2r \cos t = (1-r^2)^2 / [1+r^2-2rH(r)], \tag{3.17}$$

(iv) For equality in (2.4)

$$f_4'(z) = (1+z)^{2bk_1} / (1-z)^{2b(k_1+2)} \tag{3.18}$$

PROOF OF THEOREM 2. Taking into consideration (1.3), (1.5), (1.6), (3.3) and using the notation  $G_1(z)/z = u$  and  $G_2(z)/z = v$ , we find that for  $u$  and  $v$  lie in the interval

$$[1/(1-r)^{2b}, 1/(1+r)^{2b}], \tag{3.19}$$

we need to obtain the minimum of

$$F_1(u,v) = v^{k_1} [(1+k_1)A(r,u) - k_1 B(r,v)] / ru^{1+k_1} \tag{3.20}$$

and the maximum of

$$H_1(u,v) = v^{k_1} (1+k_1)B(r,u) - k_1 A(r,v) / ru^{1+k_1} \tag{3.21}$$

It is readily verified in the case of (3.20) that the equations:

$$\frac{\partial F_1}{\partial u} = 0 = \frac{\partial F_1}{\partial v}$$

do not give the minimum and that minimum is attained for  $u = 1/(1-r)^{2b}$  and  $v = 1/(1+r)^{2b}$  and this value is given by (2.5). Simple calculation confirms the case of equality for the function  $f(z) = f_4(z)$  given by (3.18).

In order to maximize  $H_1(u,v)$  given by (3.21), it is found that the equations:

$$\frac{\partial H_1}{\partial u} = 0 = \frac{\partial H_1}{\partial v}$$

give

$$v = [2r/(1-r^2)^2 \log (1+r)/(1-r)]^b \quad (3.22)$$

and

$$\log (1-r)^{2b} u = \frac{1+bk_1}{1+k_1} - \left[ \frac{1+bk_1}{1+k_1} + \frac{2br}{1-r} \right] \frac{(1-r^2)}{2r} \log \left( \frac{1+r}{1-r} \right) \quad (3.23)$$

The value of  $v$  given by (3.22) lies in the interval (3.19) because,

$$1 > H(r) > 0$$

but the value of  $u$  given by (3.23) lies in the interval (3.19) if

$$r - [b(k-2)(1+r)/(b(k+2)-4)] < H(r) + [b(k-2)(1-r)/(b(k+2)-4)] \quad (3.25)$$

This proves (2.6). The case of equality can be directly confirmed by the function  $f(z)$  given by

$$f_5^1(z) = \frac{(1+z)^{H(r)b(1+k_1)+bk_1+r(1-b)}}{(1-z)^{H(r)b(1+k_1)-bk_1+r(1-b)}} (1-ze^{-it})^{-2b(1+k_1)} \quad (3.26)$$

where

$$1+r^2-2r \cos t = (1-r^2)^2/[1+r^2-2rH(r)] \quad (3.27)$$

when (3.15) does not hold, the maximum value of  $H_1(u,v)$  is attained for

$u = 1/(1+r)^{2b}$ ,  $v = 1/(1-r)^{2b}$  and the corresponding value of  $\sup K_r^{V_k(1-b)}$  is given

by (2.7). Simple calculations confirm the case of equality for the functions given by (3.15).

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