

INTEGRAL OPERATORS OF CERTAIN UNIVALENT FUNCTIONS

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ABSTRACT. A function f , analytic in the unit disc Δ , is said to be in the family $R_n(\alpha)$ if $\operatorname{Re}\{(z^n f(z))^{(n+1)} / (z^{n-1} f(z))^{(n)}\} > (n+\alpha)/(n+1)$ for some α ($0 \leq \alpha < 1$) and for all z in Δ , where $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. The class $R_n(\alpha)$ contains the starlike functions of order α for $n \geq 0$, and the convex functions of order α for $n \geq 1$. We study a class of integral operators defined on $R_n(\alpha)$. Finally an argument theorem is proved.

KEY WORDS AND PHRASES: Univalent, convolution, starlike, convex

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1 INTRODUCTION.

Let A denote the family of functions f which are analytic in the unit disc $\Delta = \{z: |z| < 1\}$ and normalised such that $f(0) = 0 = f'(0) - 1$. The Hadamard product or convolution of two functions $f, g \in A$ is denoted by $f * g$. Let $D^n f = (z/(1-z))^{n+1} * f$, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ which implies that

$$D^n f = z(z^{n-1} f)^{(n)} / n!, \quad n \in \mathbb{N}_0.$$

Denote by $S^*(\alpha)$ and $K(\alpha)$ the subfamilies of A whose members are, respectively, starlike of order α and convex of order α , $0 \leq \alpha < 1$. Then

$$f \in S^*(\alpha) \iff \operatorname{Re}(D^1 f / D^0 f) > \alpha, \quad z \in \Delta,$$

$$f \in K(\alpha) \iff \operatorname{Re}(D^2 f / D^1 f) > (1+\alpha)/2, \quad z \in \Delta$$

Ruscheweyh [16] introduced the classes $\{K_n\}$ of functions $f \in A$ which satisfy the condition

$$\operatorname{Re}(D^{n+1} f / D^n f) > \frac{1}{2}, \quad z \in \Delta \quad (1.1)$$

so that the definition of K_n is a natural extension of $S^*(1/2)$, and $K(0)$

He proved that $K_{n+1} \subset K_n$ for each $n \in \mathbb{N}_0$. Since $K_0 = S^*(1/2)$, the elements of K_n are univalent and starlike of order $1/2$.

In this paper, we consider the classes of functions $f \in A$ which

satisfy the condition

$$\operatorname{Re}(z(D^n f)' / D^n f) > \alpha, \quad z \in \Delta \quad (1.2)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote these classes by $R_n(\alpha)$. We have $R_0(\alpha) = S^*(\alpha)$ and $R_1(\alpha) = K(\alpha)$ for $0 \leq \alpha < 1$. The classes $R_n = R_n(0)$ were considered earlier by Singh and Singh [17]. It is readily seen that for each $n \geq 0$, $R_n(\alpha) \subset R_n(0)$ and for each $n \geq 1$, $R_n(\alpha) \subset K_n$. We note that in definition (1.2), restriction $\alpha \geq 0$ can be replaced by $\alpha \geq (1-n)/2$ for each $n \geq 1$ and, further, that the negative choices of α permit us fully to partition K_n into classes $R_n(\alpha) \subset K_n$ ($n \geq 1$) such that

$$\bigcup R_n(\alpha) = K_n, \\ \frac{1-n}{2} \leq \alpha < 1$$

It can be easily seen that $R_{n+1}(\alpha) \subset R_n(\alpha)$ for each $n \in N_0$ and for all α . These inclusion relations establish that $R_n(\alpha) \subset S^*(\alpha)$ for each $n \geq 0$ and $R_n(\alpha) \subset K(\alpha)$ for each $n \geq 1$.

An important problem in univalent functions is the following: Given a compact family F and an operator J defined on F , is $J(f) \in F$ for every $f \in F$? Libera [11] established that the operator

$$J(f) = \frac{2}{z} \int_0^z f(t) dt \quad (1.3)$$

preserves convexity, starlikeness, and close-to-convexity. Bernardi [5] greatly generalised Libera's results. Many authors [1,2,7,8,12,15,17] studied operators of the form

$$J(f) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt, \quad (1.4)$$

where γ is a real (or complex) constant and f belongs to some favoured class of univalent functions from A . Recently, operators (1.4) have been studied in more general form by Causey and White [6], Miller, Mocanu and Reade [14], Barnard and Kellogg [3], and Bajpai [2].

In this paper, we study a class of integral operators of the form (1.4) defined on our family $R_n(\alpha)$. We also obtain an argument theorem for the class $R_n(\alpha)$.

2. INTEGRAL OPERATORS.

Let γ be a complex number with $\operatorname{Re} \gamma \neq -1$. We define h_γ by

$$h_{\gamma}(z) = \sum_{j=1}^{\infty} \frac{\gamma+1}{\gamma+j} z^j, \quad z \in \Delta. \quad (2.1)$$

Let the operator $J: A \rightarrow A$ be defined by $F = J(f)$, where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t) t^{\gamma-1} dt \quad (2.2)$$

Then the function F can also be written in the form

$$F(z) = f(z) * h_{\gamma}(z).$$

We need the following result of Jack [9] which is also due to Suffridge [18]

LEMMA. Let w be nonconstant and analytic in $|z| < r < 1, w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r$ at z_0 , then $z_0 w'(z_0) = k w(z_0)$, where k is a real number and $k \geq 1$.

We first give a condition on $f \in A$ for which the function $J(f)$ belongs to $R_n(\alpha)$.

THEOREM 1. Let $0 \leq \alpha < 1$, and $\gamma \neq -1$ be a complex constant such that $\operatorname{Re} \gamma \geq -\alpha$, $\operatorname{Im} \gamma \geq 0$, and $|\gamma|^2 + 2\alpha(1 + \operatorname{Re} \gamma) \geq 1$. If for a given $n \in \mathbb{N}$, $f \in A$ satisfies the condition

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \alpha - \frac{(1-\alpha)(\alpha + \operatorname{Re} \gamma)}{2\{|\gamma|^2 + 2\alpha \operatorname{Re} \gamma + \alpha^2 + (1-\alpha)\operatorname{Im} \gamma\}} \quad (2.3)$$

for all $z \in \Delta$, then $F(z)$ given by (2.2) belongs to $R_n(\alpha)$.

PROOF From (2.2), we obtain

$$z(D^n F(z))' + \gamma D^n F(z) = (\gamma+1)D^n f(z). \quad (2.4)$$

Define w in Δ by

$$\frac{z(D^n F(z))'}{D^n F(z)} = \frac{1+(2\alpha-1)w(z)}{1+w(z)}. \quad (2.5)$$

Here $w(z)$ is analytic in Δ with $w(0) = 0$ and $w(z) \neq -1$, $z \in \Delta$.

We need to show that $|w(z)| < 1$ for all $z \in \Delta$. In view of (2.4),

(2.5) yields

$$\frac{D^n f(z)}{D^n F(z)} = \frac{(1+\gamma) + (2\alpha-1+\gamma)w(z)}{(1+\gamma)(1+w(z))} \quad (2.6)$$

Differentiating (2.6) logarithmically and simplifying, we obtain

$$\frac{z(D^n f(z))'}{D^n f(z)} = \alpha + (1-\alpha) \frac{1-w(z)}{1+w(z)} - \frac{2(1-\alpha)zw'(z)}{(1+w(z))(1+\gamma+(2\alpha-1+\gamma)w(z))} \quad (2.7)$$

Now (2.7) should yield $|w(z)| < 1$ for all $z \in \Delta$ for otherwise, there exists a point $z_0 \in \Delta$ at which $|w(z_0)| = 1$ and by Lemma, we have $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. For this value of $z = z_0$, we find that (2.7) yields

$$\operatorname{Re} \frac{z_0(D^n f(z_0))'}{D^n f(z_0)} = \alpha - \frac{2k(1-\alpha)(\alpha+\operatorname{Re} \gamma)}{|(1+\gamma)+(2\alpha-1+\gamma)w(z_0)|^2} \quad (2.8)$$

$$\leq \alpha - \frac{(1-\alpha)(\alpha+\operatorname{Re} \gamma)}{2\{|\gamma|^2 + 2\alpha\operatorname{Re} \gamma + \alpha^2 + (1-\alpha)\operatorname{Im} \gamma\}}$$

which contradicts (2.3). Hence $|w(z)| < 1$ for all $z \in \Delta$ and by (2.5), it follows that $F(z) \in R_n(\alpha)$.

COROLLARY. If for a given $n \in \mathbb{N}_0$, $f \in A$ satisfies the condition

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \frac{2\alpha(\gamma+\alpha)-(1-\alpha)}{2(\gamma+\alpha)}, \quad z \in \Delta, \quad (2.9)$$

where (α, γ) is any point in the set

$$D = \{(\alpha, \gamma) : \gamma + 2\alpha \geq 1, 0 \leq \alpha < 1, \gamma > -1\},$$

then $F(z)$ given by (2.2) belongs to $R_n(\alpha)$.

PROOF. If $\gamma \neq -1$ is a real constant such that $\gamma + \alpha \geq 0$, then

$$|\gamma|^2 + 2\alpha(1+\operatorname{Re} \gamma) \geq 1 \text{ implies } (\gamma+1)(\gamma+2\alpha-1) \geq 0. \text{ The result follows}$$

from Theorem 1

It is easy to show that if $f \in R_n(\alpha)$, then f satisfies the condition

(2.3). Thus it follows from Theorem 1 that $J(R_n(\alpha)) \subset R_n(\alpha)$. More precisely,

we state the result in

THEOREM 2. If $f \in R_n(\alpha)$, then the function

$$J(f) = \frac{\gamma+1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt$$

is again an element of $R_n(\alpha)$, where $\gamma \neq -1$ is a complex constant with restrictions as stated in Theorem 1.

REMARK 1. Letting $n = 0 = \gamma - 1$ and $n = 1 = \gamma$, in Theorem 1, we get $L(S^*(\beta)) \subset S^*(\alpha)$ and $L(K(\beta)) \subset K(\alpha)$ respectively, where L is the Libera transform defined in (1.3), and

$$\beta = ((2\alpha^2 + 3\alpha - 1)/2(1+\alpha)) < \alpha.$$

These results improve the earlier results due to Libera [11] and Bernardi [5]

in the sense that their results hold under much weaker conditions

In [2], Bajpai has established that $J(S^*) \subset S^*(\alpha)$ for some α . We generalize this result in

THEOREM 3. Let $J:A \rightarrow A$ be defined as in (2.2), where γ is a complex constant. If $f \in R_n$, then $J(f) \in R_n(\alpha)$, where α satisfies the inequality

$$\alpha(|1+\gamma|+|2\alpha-1+\gamma|)^2 \leq 2(1-\alpha)(\alpha+\operatorname{Re}\gamma), \quad \text{and } 0 \leq \alpha < 1$$

PROOF. Proceeding as in Theorem 1 and applying Lemma, we have

$$\begin{aligned} \operatorname{Re} \frac{z_0(D^n f(z_0))'}{D^n f(z_0)} &\leq \alpha - \frac{2(1-\alpha)(\alpha+\operatorname{Re}\gamma)}{|(1+\gamma)+(2\alpha-1+\gamma)w(z_0)|^2} \\ &\leq \alpha - \frac{2(1-\alpha)(\alpha+\operatorname{Re}\gamma)}{(|1+\gamma|+|2\alpha-1+\gamma|)^2}, \end{aligned}$$

where $\operatorname{Re}\gamma \geq -\alpha$. Since the right hand side is ≤ 0 , we have a contradiction for $f \in R_n \equiv R_n(0)$. Thus we must have $|w(z)| < 1$ for all z in Δ and by (2.5), it follows that $J(f) \in R_n(\alpha)$.

REMARK 1. If we let $n=0=\gamma-1$ in the above theorem, then

$L(S^*) \subset S^*(\frac{\sqrt{17}-3}{4})$, where $L(f) = (2/z) \int_0^z f(t)dt$. Thus we have recovered a result of Miller, Mocanu and Reade ([14], pp 162-163).

REMARK 2. If $n=1$, γ is a real constant such that $\gamma+\alpha \geq 0$, and $f \in K$, then it follows from Theorem 3 that the function $F(z)$ in (2.2) is an element of $K(\alpha)$, where

$$\alpha = \frac{-(2\gamma+1) + \sqrt{(2\gamma-1)^2+8(1+\gamma)}}{4}.$$

This result was proved by Miller, Mocanu and Reade ([14], pp 165).

Further, this is an improvement of an earlier result due to Bernardi [5],

who proved that $f \in K$ implies $F \in K$.

For $\gamma = n$, where $n \in N_0$, we have an improvement over Theorem 2

THEOREM 4. Let

$$F(z) = f(z) * h_n(z) = \frac{n+1}{z^n} \int_0^z f(t)t^{n-1}dt \quad (2.10)$$

If $f \in R_n(\alpha)$, then $F \in R_{n+1}(\alpha)$

PROOF. From (2.10), we obtain

$$z(D^{n+1}F(z))' + nD^{n+1}F(z) = (n+1)D^{n+1}f(z) \quad (2.11)$$

and

$$z(D^n F(z))' + nD^n F(z) = (n+1)D^n f(z) \quad (2.12)$$

Using the identity

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z) \quad (2.13)$$

in (2.11) and (2.12), we obtain

$$(n+1)D^{n+1} f(z) = (n+2)D^{n+2} F(z) - D^{n+1} F(z) \quad (2.14)$$

and

$$D^n f(z) = D^{n+1} F(z) \quad (2.15)$$

In view of the identity (2.13) and the relations (2.14) and (2.15),

$f \in R_n(\alpha)$ yields

$$\operatorname{Re} \left\{ \frac{(n+2)D^{n+2} F(z) - (n+1)D^{n+1} F(z)}{D^{n+1} F(z)} \right\} > \alpha$$

which implies that

$$\operatorname{Re} \left\{ \frac{z(D^{n+1} F(z))'}{D^{n+1} F(z)} \right\} > \alpha, \quad z \in \Delta$$

This proves that $F \in R_{n+1}(\alpha)$.

REMARK For $n = 0$, Theorem 4 gives the well known result:

$$J(S^*(\alpha)) \subset K(\alpha), \text{ where } J(f) = \int_0^z (f(t)/t) dt$$

We now investigate the converse of Theorem 2. In fact, we find the sharp radius of the disc in which $f \in R_n(\beta)$ when F , defined in (2.2), is in $R_n(\alpha)$ for $0 \leq \alpha < 1$, $0 < \beta \leq 1$: In [12], Libera and Livingston have solved this converse problem for the case $n = 0$, $\gamma = 1$ when $\alpha \leq \beta < 1$. These authors were not able to obtain suitable results for the complementary case when $\beta < \alpha$. However, the method used in the next theorem gives results that are more general and also covers both $\beta \geq \alpha$ and $\beta < \alpha$.

THEOREM 5. If F is an element of $R_n(\alpha)$ for $n \geq 0$ and $0 \leq \alpha < 1$,

$$F(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \quad (2.16)$$

with $z \in \Delta$, $\operatorname{Re} \gamma \geq -\alpha$, and $0 \leq \beta < 1$, then the function f is an element of $R_n(\beta)$ for $|z| < r_0$, where r_0 is the smallest positive root in $(0, 1)$ of the equation

$$(\gamma+2\alpha-1)(2\alpha-\beta-1)r^2 + 2((\gamma+\alpha)(\alpha-\beta) - (1-\alpha)(2-\alpha))r + (\gamma+1)(1-\beta) = 0 \quad (2.17)$$

The result is sharp

PROOF Since $F \in R_n(\alpha)$, we can write

$$\frac{z(D^n F(z))'}{D^n F(z)} = \alpha + (1-\alpha)P_n(z), \quad (2.18)$$

where $P_n(z)$ is analytic in Δ and satisfies the conditions $P_n(0) = 1$

$\operatorname{Re} P_n(z) > 0$ for $z \in \Delta$. Using the identity

$$z(D^n F(z))' = (n+1)D^{n+1}F(z) - nD^n F(z) \quad (2.19)$$

in (2.18) and then taking logarithmic derivative, we obtain

$$z(D^{n+1}F(z))' = D^{n+1}F(z) \left[\alpha + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{n+\alpha+(1-\alpha)P_n(z)} \right] \quad (2.20)$$

From (2.16) we obtain

$$z(D^{n+1}F(z))' + \gamma D^{n+1}F(z) = (\gamma+1)D^{n+1}f(z). \quad (2.21)$$

From (2.20) and (2.21) we have

$$(\gamma+1)D^{n+1}f(z) = D^{n+1}F(z) \left[\alpha + \gamma + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{n+\alpha+(1-\alpha)P_n(z)} \right] \quad (2.22)$$

Also (2.18) together with the identity (2.4) yields

$$(1+\gamma)D^n f(z) = D^n F(z)(\alpha + \gamma + (1-\alpha)P_n(z)). \quad (2.23)$$

Now from the relations (2.22), (2.23), and (2.18) we conclude that

$$\frac{z(D^n f(z))'}{D^n f(z)} - \beta = \alpha - \beta + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{\alpha + \gamma + (1-\alpha)P_n(z)}. \quad (2.24)$$

Using the well known estimates

$$|zP_n'(z)| \leq (2r/(1-r^2))\operatorname{Re} P_n(z)$$

and

$$\operatorname{Re} P_n(z) \geq (1-r)/(1+r), \quad |z| = r$$

in (2.24), we obtain

$$\operatorname{Re} \left[\frac{z(D^n f(z))'}{D^n f(z)} - \beta \right] \geq (\alpha - \beta) + \frac{(1-\alpha)((1-r)(\gamma+1+(\gamma+2\alpha-1)r)-2r)}{(1-r)((\gamma+2\alpha-1)r+\gamma+1)} \quad (2.25)$$

where $\operatorname{Re} \gamma \geq -\alpha$. Therefore,

$$\operatorname{Re} \left\{ \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \beta$$

if the right side of (2.25) is positive, which is satisfied provided that

$r < r_0$, where r_0 is the smallest positive root in $(0,1)$ of (2.17).

The result in the theorem is sharp with the function f defined by

$$f(z) = (1/(1+c))z^{1-c}(z^c F(z))', \quad (2.26)$$

where $c = \operatorname{Re} \gamma \geq -\alpha$, and F is given by

$$z \frac{(D^n F(z))'}{D^n F(z)} = \frac{1-(2\alpha-1)z}{1-z} \quad (2.27)$$

REMARK. By specializing choices of α, β, γ , and n , theorem 5 gives rise to the corresponding results obtained earlier in [3,4,8,12,13,15] and by many others

3 AN ARGUMENT THEOREM.

THEOREM 6 If $f \in R_n(\alpha)$, then

$$\left| \arg \frac{D^k f(z)}{z} \right| \leq 2(1-\alpha) \sin^{-1} r + \sum_{m=0}^{k-1} \sin^{-1} \left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r^2} \right)$$

for each $k (0 \leq k \leq n+1)$.

PROOF We may write

$$\frac{D^k f(z)}{z} = \frac{f(z)}{z} \prod_{m=0}^{k-1} \frac{D^{m+1} f(z)}{D^m f(z)}, \quad 0 \leq k \leq n+1,$$

which yields

$$\left| \arg \frac{D^k f(z)}{z} \right| \leq \left| \arg \frac{f(z)}{z} \right| + \sum_{m=0}^{k-1} \left| \arg \frac{D^{m+1} f(z)}{D^m f(z)} \right|. \quad (3.1)$$

Since $R_{n+1}(\alpha) \subset R_n(\alpha) \quad \forall n \in N_0$, it follows that $f \in R_m(\alpha)$ for each

$m (0 \leq m \leq n)$ Setting

$$\frac{D^{m+1} f(z)}{D^m f(z)} = q_m(z), \quad (0 \leq m \leq n), \quad (3.2)$$

we note that $\operatorname{Re}(q_m(z)) \geq (m+\alpha)/(m+1)$

Therefore, the function

$$\begin{aligned} w(z) &= \frac{(q_m(z) - \frac{m+\alpha}{m+1}) - (1 - \frac{m+\alpha}{m+1})}{(q_m(z) - \frac{m+\alpha}{m+1}) + (1 - \frac{m+\alpha}{m+1})} \\ &= \frac{q_m(z) - 1}{q_m(z) - (\frac{2(m+\alpha)}{m+1} - 1)} \end{aligned}$$

is analytic with $w(0) = 0$ and $|w(z)| < 1$ in Δ Hence by Schwarz's

Lemma,

$$\left| \frac{q_m(z) - 1}{q_m(z) + 1 - 2(m+\alpha)/(m+1)} \right| < |z|$$

for z in Δ . Now it is easy to see that the values of $q_m(z)$ are contained in the circle of Apollonius whose centre is at the point $(m+1-(m+2\alpha-1)r^2)/((1+m)(1-r^2))$ and has radius $2(1-\alpha)r/((m+1)(1-r^2))$.

Thus $\max_{z \in \Delta} |\arg q_m(z)|$ is attained at the points where

$$\arg q_m(z) = \pm \sin^{-1} \left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r} \right)$$

which gives

$$\left| \arg \frac{D^{m+1}f(z)}{D^m f(z)} \right| \leq \sin^{-1} \left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r} \right), \quad (3.3)$$

for $0 \leq m \leq n$, and $z \in \Delta$.

Next, note that $R_n(\alpha) \subset S^*(\alpha)$, $n \geq 0$, and $f \in S^*(\alpha)$ if and only if $F(z) = \int (f(z)/z) dz$ is in $K(\alpha)$. But for $F \in K(\alpha)$, we have

$$|\arg F'(z)| \leq 2(1-\alpha)\sin^{-1} r \quad (|z| = r)$$

Thus $f \in R_n(\alpha)$ implies

$$\left| \arg \frac{f(z)}{z} \right| \leq 2(1-\alpha)\sin^{-1} r \quad (3.4)$$

Applying (3.3) and (3.4) to (3.1) we obtain the result.

For $n = 0$, we obtain

COROLLARY If $f \in S^*(\alpha)$, then (3.4)

and

$$|\arg f'(z)| \leq 2(1-\alpha)\sin^{-1} r + \sin^{-1} \left(\frac{2(1-\alpha)r}{1-(2\alpha-1)r^2} \right)$$

REMARK The case $n = 0$, $\alpha = 0$ was proved by Krzyz [10].

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Manuscript Due	December 1, 2008
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