

## INTEGRAL OPERATORS OF CERTAIN UNIVALENT FUNCTIONS

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ABSTRACT. A function  $f$ , analytic in the unit disc  $\Delta$ , is said to be in the family  $R_n(\alpha)$  if  $\operatorname{Re}\{(z^n f(z))^{(n+1)} / (z^{n-1} f(z))^{(n)}\} > (n+\alpha)/(n+1)$  for some  $\alpha (0 \leq \alpha < 1)$  and for all  $z$  in  $\Delta$ , where  $n \in \mathbb{N}_0$ ,  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ . The class  $R_n(\alpha)$  contains the starlike functions of order  $\alpha$  for  $n \geq 0$ , and the convex functions of order  $\alpha$  for  $n \geq 1$ . We study a class of integral operators defined on  $R_n(\alpha)$ . Finally an argument theorem is proved.

KEY WORDS AND PHRASES: Univalent, convolution, starlike, convex

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### I INTRODUCTION.

Let  $A$  denote the family of functions  $f$  which are analytic in the unit disc  $\Delta = \{z : |z| < 1\}$  and normalised such that  $f(0) = 0 = f'(0) - 1$ . The Hadamard product or convolution of two functions  $f, g \in A$  is denoted by  $f * g$ . Let  $D^n f = (z/(1-z))^{n+1} * f$ ,  $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$  which implies that

$$D^n f = z(z^{n-1} f)^{(n)} / n! , \quad n \in \mathbb{N}_0 .$$

Denote by  $S^*(\alpha)$  and  $K(\alpha)$  the subfamilies of  $A$  whose members are, respectively, starlike of order  $\alpha$  and convex of order  $\alpha$ ,  $0 \leq \alpha < 1$ . Then

$$f \in S^*(\alpha) \Leftrightarrow \operatorname{Re}(D^1 f / D^0 f) > \alpha, \quad z \in \Delta,$$

$$f \in K(\alpha) \Leftrightarrow \operatorname{Re}(D^2 f / D^1 f) > (1+\alpha)/2, \quad z \in \Delta$$

Ruscheweyh [16] introduced the classes  $\{K_n\}$  of functions  $f \in A$  which satisfy the condition

$$\operatorname{Re}(D^{n+1} f / D^n f) > \frac{1}{2} , \quad z \in \Delta \tag{1.1}$$

so that the definition of  $K_n$  is a natural extension of  $S^*(1/2)$ , and  $K(0)$

He proved that  $K_{n+1} \subset K_n$  for each  $n \in \mathbb{N}_0$ . Since  $K_0 = S^*(1/2)$ , the elements of  $K_n$  are univalent and starlike of order  $1/2$ .

In this paper, we consider the classes of functions  $f \in A$  which

satisfy the condition

$$\operatorname{Re}(z(D^n f)' / D^n f) > \alpha, \quad z \in \Delta \quad (1.2)$$

for some  $\alpha (0 \leq \alpha < 1)$ . We denote these classes by  $R_n(\alpha)$ . We have

$R_0(\alpha) = S^*(\alpha)$  and  $R_1(\alpha) = K(\alpha)$  for  $0 \leq \alpha < 1$ . The classes  $R_n = R_n(0)$  were considered earlier by Singh and Singh [17]. It is readily seen that for each  $n \geq 0$ ,  $R_n(\alpha) \subset R_n(0)$  and for each  $n \geq 1$ ,  $R_n(\alpha) \subset K_n$ . We note that in definition (1.2), restriction  $\alpha \geq 0$  can be replaced by  $\alpha \geq (1-n)/2$  for each  $n \geq 1$  and, further, that the negative choices of  $\alpha$  permit us fully to partition  $K_n$  into classes  $R_n(\alpha) \subset K_n$  ( $n \geq 1$ ) such that

$$\bigcup_{\frac{1-n}{2} \leq \alpha < 1} R_n(\alpha) = K_n.$$

It can be easily seen that  $R_{n+1}(\alpha) \subset R_n(\alpha)$  for each  $n \in N_0$  and for all  $\alpha$ . These inclusion relations establish that  $R_n(\alpha) \subset S^*(\alpha)$  for each  $n \geq 0$  and  $R_n(\alpha) \subset K(\alpha)$  for each  $n \geq 1$ .

An important problem in univalent functions is the following: Given a compact family  $F$  and an operator  $J$  defined on  $F$ , is  $J(f) \in F$  for every  $f \in F$ ? Libera [11] established that the operator

$$J(f) = \frac{2}{z} \int_0^z f(t) dt \quad (1.3)$$

preserves convexity, starlikeness, and close-to-convexity. Bernardi [5] greatly generalised Libera's results. Many authors [1, 2, 7, 8, 12, 15, 17] studied operators of the form

$$J(f) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt, \quad (1.4)$$

where  $\gamma$  is a real (or complex) constant and  $f$  belongs to some favoured class of univalent functions from  $A$ . Recently, operators (1.4) have been studied in more general form by Causey and White [6], Miller, Mocanu and Reade [14], Barnard and Kellogg [3], and Bajpai [2].

In this paper, we study a class of integral operators of the form (1.4) defined on our family  $R_n(\alpha)$ . We also obtain an argument theorem for the class  $R_n(\alpha)$ .

## 2. INTEGRAL OPERATORS.

Let  $\gamma$  be a complex number with  $\operatorname{Re}\gamma \neq -1$ . We define  $h_\gamma$  by

$$h_{\gamma}(z) = \sum_{j=1}^{\infty} \frac{\gamma+1}{\gamma+j} z^j, \quad z \in \Delta. \quad (2.1)$$

Let the operator  $J: A \rightarrow A$  be defined by  $F = J(f)$ , where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t) t^{\gamma-1} dt \quad (2.2)$$

Then the function  $F$  can also be written in the form

$$F(z) = f(z) * h_{\gamma}(z).$$

We need the following result of Jack [9] which is also due to Suffridge [18]

LEMMA. Let  $w$  be nonconstant and analytic in  $|z| < r < 1, w(0) = 0$

If  $|w|$  attains its maximum value on the circle  $|z| = r$  at  $z_0$ , then  $z_0 w'(z_0) = kw(z_0)$ , where  $k$  is a real number and  $k \geq 1$

We first give a condition on  $f \in A$  for which the function  $J(f)$  belongs to  $R_n(\alpha)$

THEOREM 1. Let  $0 \leq \alpha < 1$ , and  $\gamma \neq -1$  be a complex constant such that  $\operatorname{Re}\gamma \geq -\alpha$ ,  $\operatorname{Im}\gamma \geq 0$ , and  $|\gamma|^2 + 2\alpha(1 + \operatorname{Re}\gamma) \geq 1$ . If for a given  $n \in \mathbb{N}_0$ ,  $f \in A$  satisfies the condition

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \alpha - \frac{(1-\alpha)(\alpha + \operatorname{Re}\gamma)}{2(|\gamma|^2 + 2\alpha\operatorname{Re}\gamma + \alpha^2 + (1-\alpha)\operatorname{Im}\gamma)} \quad (2.3)$$

for all  $z \in \Delta$ , then  $F(z)$  given by (2.2) belongs to  $R_n(\alpha)$ .

PROOF From (2.2), we obtain

$$z(D^n F(z))' + \gamma D^n F(z) = (\gamma+1)D^n f(z). \quad (2.4)$$

Define  $w$  in  $\Delta$  by

$$\frac{z(D^n F(z))'}{D^n F(z)} = \frac{1+(2\alpha-1)w(z)}{1+w(z)}. \quad (2.5)$$

Here  $w(z)$  is analytic in  $\Delta$  with  $w(0) = 0$  and  $w(z) \neq -1$ ,  $z \in \Delta$

We need to show that  $|w(z)| < 1$  for all  $z \in \Delta$ . In view of (2.4), (2.5) yields

$$\frac{D^n f(z)}{D^n F(z)} = \frac{(1+\gamma)+(2\alpha-1+\gamma)w(z)}{(1+\gamma)(1+w(z))} \quad (2.6)$$

Differentiating (2.6) logarithmically and simplifying, we obtain

$$\frac{z(D^n f(z))'}{D^n f(z)} = \alpha + (1-\alpha) \frac{1-w(z)}{1+w(z)} - \frac{2(1-\alpha)zw'(z)}{(1+w(z))(1+\gamma+(2\alpha-1+\gamma)w(z))} \quad (2.7)$$

Now (2.7) should yield  $|w(z)| < 1$  for all  $z \in \Delta$  for otherwise, there exists a point  $z_0 \in \Delta$  at which  $|w(z_0)| = 1$  and by Lemma, we have  $z_0 w'(z_0) = kw(z_0)$ ,  $k \geq 1$ . For this value of  $z = z_0$ , we find that (2.7) yields

$$\begin{aligned} \operatorname{Re} \frac{z_0(D^n f(z_0))'}{D^n f(z_0)} &= \alpha - \frac{2k(1-\alpha)(\alpha+\operatorname{Re}\gamma)}{|(1+\gamma)+(2\alpha-1+\gamma)w(z_0)|^2} \\ &\leq \alpha - \frac{(1-\alpha)(\alpha+\operatorname{Re}\gamma)}{2\{|\gamma|^2+2\alpha\operatorname{Re}\gamma+\alpha^2+(1-\alpha)\operatorname{Im}\gamma\}} \end{aligned} \quad (2.8)$$

which contradicts (2.3). Hence  $|w(z)| < 1$  for all  $z \in \Delta$  and by (2.5), it follows that  $F(z) \in R_n(\alpha)$ .

COROLLARY. If for a given  $n \in N_0$ ,  $f \in A$  satisfies the condition

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \frac{2\alpha(\gamma+\alpha)-(1-\alpha)}{2(\gamma+\alpha)}, \quad z \in \Delta, \quad (2.9)$$

where  $(\alpha, \gamma)$  is any point in the set

$$D = \{(\alpha, \gamma) : \gamma+2\alpha \geq 1, 0 \leq \alpha < 1, \gamma > -1\},$$

then  $F(z)$  given by (2.2) belongs to  $R_n(\alpha)$ .

PROOF. If  $\gamma \neq -1$  is a real constant such that  $\gamma + \alpha \geq 0$ , then  $|\gamma|^2 + 2\alpha(1+\operatorname{Re}\gamma) \geq 1$  implies  $(\gamma+1)(\gamma+2\alpha-1) \geq 0$ . The result follows

from Theorem 1.

It is easy to show that if  $f \in R_n(\alpha)$ , then  $f$  satisfies the condition (2.3). Thus it follows from Theorem 1 that  $J(R_n(\alpha)) \subset R_n(\alpha)$ . More precisely, we state the result in

THEOREM 2 If  $f \in R_n(\alpha)$ , then the function

$$J(f) = \frac{\gamma+1}{z\gamma} \int_0^z f(t) t^{\gamma-1} dt$$

is again an element of  $R_n(\alpha)$ , where  $\gamma \neq -1$  is a complex constant with restrictions as stated in Theorem 1.

REMARK 1 Letting  $n = 0 = \gamma - 1$  and  $n = 1 = \gamma$ , in Theorem 1, we get  $L(S^*(\beta)) \subset S^*(\alpha)$  and  $L(K(\beta)) \subset K(\alpha)$  respectively, where  $L$  is the Libera transform defined in (1.3), and

$$\beta = ((2\alpha^2+3\alpha-1)/2(1+\alpha)) < \alpha.$$

These results improve the earlier results due to Libera [11] and Bernardi [5] in the sense that their results hold under much weaker conditions

In [2], Bajpai has established that  $J(S^*) \subset S^*(\alpha)$  for some  $\alpha$ . We generalize this result in

THEOREM 3. Let  $J: A \rightarrow A$  be defined as in (2.2), where  $\gamma$  is a complex constant. If  $f \in R_n$ , then  $J(f) \in R_n(\alpha)$ , where  $\alpha$  satisfies the inequality

$$\alpha(|1+\gamma| + |2\alpha-1+\gamma|)^2 \leq 2(1-\alpha)(\alpha+Re\gamma), \text{ and } 0 \leq \alpha < 1$$

PROOF Proceeding as in Theorem 1 and applying Lemma, we have

$$\begin{aligned} \operatorname{Re} \frac{z_0(D^n f(z_0))'}{D^n f(z_0)} &\leq \alpha - \frac{2(1-\alpha)(\alpha+Re\gamma)}{|(1+\gamma)+(2\alpha-1+\gamma)w(z_0)|^2} \\ &\leq \alpha - \frac{2(1-\alpha)(\alpha+Re\gamma)}{(|1+\gamma| + |2\alpha-1+\gamma|)^2}, \end{aligned}$$

where  $Re\gamma \geq -\alpha$ . Since the right hand side is  $\leq 0$ , we have a contradiction for  $f \in R_n \equiv R_n(0)$ . Thus we must have  $|w(z)| < 1$  for all  $z$  in  $\Delta$  and by (2.5), it follows that  $J(f) \in R_n(\alpha)$ .

REMARK 1 If we let  $n=0=\gamma-1$  in the above theorem, then

$L(S^*) \subset S^*(\frac{\sqrt{17}-3}{4})$ , where  $L(f) = (2/z) \int_0^z f(t)dt$ . Thus we have recovered a result of Miller, Mocanu and Reade ([14], pp 162-163).

REMARK 2 If  $n = 1$ ,  $\gamma$  is a real constant such that  $\gamma+\alpha \geq 0$ , and  $f \in K$ , then it follows from Theorem 3 that the function  $F(z)$  in (2.2) is an element of  $K(\alpha)$ , where

$$\alpha = \frac{-(2\gamma+1) + \sqrt{(2\gamma+1)^2 + 8(1+\gamma)}}{4}.$$

This result was proved by Miller, Mocanu and Reade ([14], pp 165)

Further, this is an improvement of an earlier result due to Bernardi [5], who proved that  $f \in K$  implies  $F \in K$ .

For  $\gamma = n$ , where  $n \in N_0$ , we have an improvement over Theorem 2

THEOREM 4. Let

$$F(z) = f(z) * h_n(z) = \frac{n+1}{z^n} \int_0^z f(t) t^{n-1} dt \quad (2.10)$$

If  $f \in R_n(\alpha)$ , then  $F \in R_{n+1}(\alpha)$

PROOF. From (2.10), we obtain

$$z(D^{n+1}F(z))' + nD^{n+1}F(z) = (n+1)D^{n+1}f(z) \quad (2.11)$$

and

$$z(D^n F(z))' + nD^n F(z) = (n+1)D^{n+1} F(z) \quad (2.12)$$

Using the identity

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z) \quad (2.13)$$

in (2.11) and (2.12), we obtain

$$(n+1)D^{n+1} f(z) = (n+2)D^{n+2} F(z) - D^{n+1} F(z) \quad (2.14)$$

and

$$D^n f(z) = D^{n+1} F(z) \quad (2.15)$$

In view of the identity (2.13) and the relations (2.14) and (2.15),

$f \in R_n(\alpha)$  yields

$$\operatorname{Re} \left\{ \frac{(n+2)D^{n+2} F(z) - (n+1)D^{n+1} F(z)}{D^{n+1} F(z)} \right\} > \alpha$$

which implies that

$$\operatorname{Re} \left\{ \frac{z(D^{n+1} F(z))'}{D^{n+1} F(z)} \right\} > \alpha, \quad z \in \Delta$$

This proves that  $F \in R_{n+1}(\alpha)$ .

REMARK For  $n = 0$ , Theorem 4 gives the well known result:

$$J(S^*(\alpha)) \subset K(\alpha), \text{ where } J(f) = \int_0^z (f(t)/t) dt$$

We now investigate the converse of Theorem 2. In fact, we find the sharp radius of the disc in which  $f \in R_n(\beta)$  when  $F$ , defined in (2.2), is in  $R_n(\alpha)$  for  $0 \leq \alpha < 1$ ,  $0 < \beta \leq 1$ . In [12], Libera and Livingston have solved this converse problem for the case  $n = 0$ ,  $\gamma = 1$  when  $\alpha \leq \beta < 1$ . These authors were not able to obtain suitable results for the complementary case when  $\beta < \alpha$ . However, the method used in the next theorem gives results that are more general and also covers both  $\beta \geq \alpha$  and  $\beta < \alpha$ .

THEOREM 5. If  $F$  is an element of  $R_n(\alpha)$  for  $n \geq 0$  and  $0 \leq \alpha < 1$ ,

$$F(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t)t^{\gamma-1} dt \quad (2.16)$$

with  $z \in \Delta$ ,  $\operatorname{Re} \gamma \geq -\alpha$ , and  $0 \leq \beta < 1$ , then the function  $f$  is an element of  $R_n(\beta)$  for  $|z| < r_0$ , where  $r_0$  is the smallest positive root in  $(0, 1)$  of the equation

$$(\gamma+2\alpha-1)(2\alpha-\beta-1)r^2 + 2((\gamma+\alpha)(\alpha-\beta)-(1-\alpha)(2-\alpha))r + (\gamma+1)(1-\beta) = 0 \quad (2.17)$$

The result is sharp

PROOF Since  $F \in R_n(\alpha)$ , we can write

$$\frac{z(D^n F(z))'}{D^n F(z)} = \alpha + (1-\alpha)P_n(z), \quad (2.18)$$

where  $P_n(z)$  is analytic in  $\Delta$  and satisfies the conditions  $P_n(0) = 1$

$\operatorname{Re} P_n(z) > 0$  for  $z \in \Delta$  Using the identity

$$z(D^n F(z))' = (n+1)D^{n+1}F(z) - nD^n F(z) \quad (2.19)$$

in (2.18) and then taking logarithmic derivative, we obtain

$$z(D^{n+1}F(z))' = D^{n+1}F(z)[\alpha + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{n+\alpha+(1-\alpha)P_n(z)}] \quad (2.20)$$

From (2.16) we obtain

$$z(D^{n+1}F(z))' + \gamma D^{n+1}F(z) = (\gamma+1)D^{n+1}f(z). \quad (2.21)$$

From (2.20) and (2.21) we have

$$(\gamma+1)D^{n+1}f(z) = D^{n+1}F(z)[\alpha + \gamma + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{n+\alpha+(1-\alpha)P_n(z)}] \quad (2.22)$$

Also (2.18) together with the identity (2.4) yields

$$(1+\gamma)D^n f(z) = D^n F(z)(\alpha + \gamma + (1-\alpha)P_n(z)). \quad (2.23)$$

Now from the relations (2.22), (2.23), and (2.18) we conclude that

$$\frac{z(D^n f(z))'}{D^n f(z)} - \beta = \alpha - \beta + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{\alpha + \gamma + (1-\alpha)P_n(z)}. \quad (2.24)$$

Using the well known estimates

$$|zP_n'(z)| \leq (2r/(1-r^2))\operatorname{Re} P_n(z)$$

and

$$\operatorname{Re} P_n(z) \geq (1-r)/(1+r), \quad |z| = r$$

in (2.24), we obtain

$$\operatorname{Re} \left[ \frac{z(D^n f(z))'}{D^n f(z)} - \beta \right] \geq (\alpha - \beta) + \frac{(1-\alpha)((1-r)(\gamma+1+(\gamma+2\alpha-1)r)-2r)}{(1-r)((\gamma+2\alpha-1)r+\gamma+1)} \quad (2.25)$$

where  $\operatorname{Re} \gamma \geq -\alpha$ . Therefore,

$$\operatorname{Re} \left\{ \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \beta$$

if the right side of (2.25) is positive, which is satisfied provided that

$r < r_0$ , where  $r_0$  is the smallest positive root in  $(0,1)$  of (2.17).

The result in the theorem is sharp with the function  $f$  defined by

$$f(z) = (1/(1+c))z^{1-c}(z^c F(z))', \quad (2.26)$$

where  $c = \operatorname{Re}\gamma \geq -\alpha$ , and  $F$  is given by

$$z \frac{(D^n F(z))'}{D^n F(z)} = \frac{1-(2\alpha-1)z}{1-z} \quad (2.27)$$

REMARK. By specializing choices of  $\alpha, \beta, \gamma$ , and  $n$ , theorem 5 gives rise to the corresponding results obtained earlier in [3, 4, 8, 12, 13, 15] and by many others

### 3 AN ARGUMENT THEOREM.

THEOREM 6 If  $f \in R_n(\alpha)$ , then

$$|\arg \frac{D^k f(z)}{z}| \leq 2(1-\alpha) \sin^{-1} r + \sum_{m=0}^{k-1} \sin^{-1} \left( \frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r^2} \right)$$

for each  $k(0 \leq k \leq n+1)$ .

PROOF We may write

$$\frac{D^k f(z)}{z} = \frac{f(z)}{z} \prod_{m=0}^{k-1} \frac{D^{m+1} f(z)}{D^m f(z)}, \quad 0 \leq k \leq n+1,$$

which yields

$$|\arg \frac{D^k f(z)}{z}| \leq |\arg \frac{f(z)}{z}| + \sum_{m=0}^{k-1} |\arg \frac{D^{m+1} f(z)}{D^m f(z)}|. \quad (3.1)$$

Since  $R_{n+1}(\alpha) \subset R_n(\alpha) \forall n \in N_0$ , it follows that  $f \in R_m(\alpha)$  for each

$m(0 \leq m \leq n)$  Setting

$$\frac{D^{m+1} f(z)}{D^m f(z)} = q_m(z), \quad (0 \leq m \leq n), \quad (3.2)$$

we note that  $\operatorname{Re}(q_m(z)) \geq (m+\alpha)/(m+1)$

Therefore, the function

$$\begin{aligned} w(z) &= \frac{\left( q_m(z) - \frac{m+\alpha}{m+1} \right) - \left( 1 - \frac{m+\alpha}{m+1} \right)}{\left( q_m(z) - \frac{m+\alpha}{m+1} \right) + \left( 1 - \frac{m+\alpha}{m+1} \right)} \\ &= \frac{q_m(z) - 1}{q_m(z) - \left( \frac{2(m+\alpha)}{m+1} - 1 \right)} \end{aligned}$$

is analytic with  $w(0) = 0$  and  $|w(z)| < 1$  in  $\Delta$  Hence by Schwarz's

Lemma,

$$\left| \frac{q_m(z) - 1}{q_m(z) + 1 - 2(m+\alpha)/(m+1)} \right| < |z|$$

for  $z$  in  $\Delta$ . Now it is easy to see that the values of  $q_m(z)$  are contained in the circle of Appolonius whose centre is at the point  $(m+1-(m+2\alpha-1)r^2)/((1+m)(1-r^2))$  and has radius  $2(1-\alpha)r/((m+1)(1-r^2))$ . Thus  $\max_{z \in \Delta} |\arg q_m(z)|$  is attained at the points where

$$\arg q_m(z) = \pm \sin^{-1} \left( \frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r} \right)$$

which gives

$$\left| \arg \frac{D^{m+1} f(z)}{D^m f(z)} \right| \leq \sin^{-1} \left( \frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r} \right), \quad (3.3)$$

for  $0 \leq m \leq n$ , and  $z \in \Delta$ .

Next, note that  $R_n(\alpha) \subset S^*(\alpha)$ ,  $n \geq 0$ , and  $f \in S^*(\alpha)$  if and only if  $F(z) = \int (f(z)/z) dz$  is in  $K(\alpha)$ . But for  $F \in K(\alpha)$ , we have

$$|\arg F'(z)| \leq 2(1-\alpha)\sin^{-1} r \quad (|z| = r)$$

Thus  $f \in R_n(\alpha)$  implies

$$\left| \arg \frac{f(z)}{z} \right| \leq 2(1-\alpha)\sin^{-1} r \quad (3.4)$$

Applying (3.3) and (3.4) to (3.1) we obtain the result.

For  $n = 0$ , we obtain

COROLLARY If  $f \in S^*(\alpha)$ , then (3.4)

and

$$|\arg f'(z)| \leq 2(1-\alpha)\sin^{-1} r + \sin^{-1} \left( \frac{2(1-\alpha)r}{1-(2\alpha-1)r^2} \right)$$

REMARK The case  $n = 0$ ,  $\alpha = 0$  way proved by Krzyz [10].

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