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HOLOMORPHIC EXTENSION OF GENERALIZATIONS OF H^p FUNCTIONS

RICHARD D. CARMICHAEL

Department of Mathematical Sciences
New Mexico State University
Las Cruces, New Mexico 88003, U.S.A.

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ABSTRACT. In recent analysis we have defined and studied holomorphic functions in tubes in $\mathbb{C}^n$ which generalize the Hardy H^p functions in tubes. In this paper we consider functions $f(z)$, $z = x+iy$, which are holomorphic in the tube $T^C = \mathbb{R}^n + iC$, where C is the finite union of open convex cones C_j, $j = 1, \dots, m$, and which satisfy the norm growth of our new functions. We prove a holomorphic extension theorem in which $f(z)$, $z \in T^C$, is shown to be extendable to a function which is holomorphic in $T^{O(C)} = \mathbb{R}^n + iO(C)$, where $O(C)$ is the convex hull of C, if the distributional boundary values in $\mathcal{S}'$ of $f(z)$ from each connected component T^{C_j} of T^C are equal.

KEY WORDS AND PHRASES. Generalization of H^p Functions in Tube Domains, Holomorphic Extension, Fourier-Laplace Transform, Edge of the Wedge Theorem.
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1. INTRODUCTION.

The purpose of this paper is to prove a holomorphic extension theorem (edge of the wedge theorem) for functions which are holomorphic in a tube in $\mathbb{C}^n$ and which satisfy a norm growth condition that generalizes the norm growth for H^p functions in tubes. The basis for the analysis presented here is the analysis in our papers Carmichael [1-2].

We begin by stating some needed definitions. A set $C \subset \mathbb{R}^n$ is a cone (with vertex at the origin $\bar{0} = (0,0,\dots,0)$ in $\mathbb{R}^n$) if $y \in C$ implies $\lambda y \in C$ for all positive scalars λ. A regular cone is an open convex cone C such that $\bar{C}$ does not contain any entire straight line. The dual cone C^* of a cone C is defined as $C^* = \{t \in \mathbb{R}^n: \langle t,y\rangle \geq 0$ for all $y \in C\}$; C^* is always closed and convex (Vladimirov [3, p. 218]). The intersection of the cone C with the unit sphere in $\mathbb{R}^n$ is called the projection of C and is denoted $pr(C)$. The function

$$u_C(t) = \sup_{y \in pr(C)} (-\langle t,y\rangle)$$

is the indicatrix of the cone C, and we note that $C^* = \{t \in \mathbb{R}^n: u_C(t) \leq 0\}$. The set $T^C = \mathbb{R}^n + iC$ is a tube in $\mathbb{C}^n$. The convex hull (convex envelope) of a cone C will be denoted by $O(C)$, and $O(C)$ is also a cone. Put $C_* = \mathbb{R}^n \setminus C^*$; the number

$$\rho_C = \sup_{t \in C_*} \frac{u_{O(C)}(t)}{u_C(t)}$$

characterizes the nonconvexity of the cone C (Vladimirov [3, p. . .]. Following Vladimirov [4, p. 930] we say that a cone $C \subset \mathbb{R}^n$ with interior points has an admissible set of vectors if there are vectors $e_k \in C$, $|e_k| = 1$, $k = 1,2, \ldots ,n$, which form a basis for $\mathbb{R}^n$; equivalently we say that such a set of n vectors in C is admissible for the cone C.

Let B denote a proper open subset of $\mathbb{R}^n$. Let $0 < p < \infty$ and $A \geq 0$. Let d(y) denote the distance from $y \in B$ to the complement of B in $\mathbb{R}^n$. The space $S_A^p(T^B)$ (Carmichael [1, pp. 80-81]), $T^B = \mathbb{R}^n + iB$, is the set of all functions f(z), $z = x + iy \in T^B$, which are holomorphic in T^B and which satisfy

$$||f(x+iy)||_{L^p} = \left(\int_{\mathbb{R}^n} |f(x+iy)|^p \, dx\right)^{1/p} \leq \tag{1.1}$$

$$\leq M\,(1+(d(y))^{-r})^s \exp(2\pi A|y|), \quad y \in B,$$

for some constants $r \geq 0$ and $s \geq 0$ which can depend on f, p, and A but not on $y \in B$ and for some constant $M = M(f,p,A,r,s)$ which can depend on f, p, A, r, and s but not on $y \in B$. We defined and studied the functions $S_A^p(T^B)$ in Carmichael [1-2]. The spaces $S_A^p(T^B)$ were defined to generalize the H^p functions in tubes (Stein and Weiss [5, Chapter III]) and to contain the previous generalizations of the H^p functions of Vladimirov [6] and Carmichael and Hayashi [7].

We proved in Carmichael [1, Theorem 4.1, p. 92] that if B is a proper open connected subset of $\mathbb{R}^n$ then any element $f(z) \in S_A^p(T^B)$, $1 < p \leq 2$, $A \geq 0$, has a Fourier-Laplace integral representation for $z \in T^B$ in terms of a function g(t) which satisfies certain norm growth properties. In addition we proved in Carmichael [1, Corollary 4.1, p. 93] that if B = C, an open convex cone in $\mathbb{R}^n$, then f(x+iy) has a unique boundary value as $y \to \bar{0}$, $y \in C$, in the strong topology of $\mathcal{S}'$, the space of tempered distributions.

In this paper we prove a holomorphic extension theorem (edge of the wedge theorem) for holomorphic functions in T^C which satisfy (1.1) for $y \in C$ where C is a finite union of open convex cones in $\mathbb{R}^n$; the extended function is holomorphic in $T^{O(C)}$ where O(C) is the convex hull of C. To obtain our extension theorem we use the information from Carmichael [1] which is contained in the preceding paragraph.

We proceed to the result of this paper after making the following definition; the subspace $\mathcal{S}'_p$ of $\mathcal{S}'$, $1 \leq p < \infty$, is defined to be the set of all measurable functions g(t), $t \in \mathbb{R}^n$, such that there exists a real number $b \geq 0$ for which $((1 + |t|^p)^{-b}\, g(t)) \in L^p$ (Carmichael [1, p. 83]).

All subsequent notation and terminology in this paper are that of Carmichael [1-2].

2. HOLOMORPHIC EXTENSION.

Let C be an open cone in $\mathbb{R}^n$ such that $C = \bigcup_{j=1}^{m} C_j$ where the C_j, $j = 1, \ldots ,m$, are open convex cones in $\mathbb{R}^n$ and m is a positive integer. Let f(z) be holomorphic in the tubular cone $T^C = \mathbb{R}^n + iC$ and satisfy (1.1) for $y \in C$ and for $1 < p \leq 2$. For any $y \in C_j$, $j = 1, \ldots ,m$, the distance from y to the boundary of C is larger than or equal to the distance from y to the boundary of C_j from which it follows that $f(z) \in S_A^p(T^{C_j})$, $1 < p \leq 2$, $j = 1, \ldots ,m$. Thus by Carmichael [1, Corollary 4.1, p. 93] there exist measurable functions $g_j(t) \in \mathcal{S}'_q$, $(1/p) + (1/q) = 1$, with $\mathrm{supp}(g_j) \subseteq \{t: u_{C_j}(t) \leq A\}$

almost everywhere such that

$$f(z) = \int_{R^n} g_j(t)\exp(2\pi i\langle z,t\rangle)\,dt, \quad z \in T^{C_j},\ j = 1, \ldots, m, \tag{2.1}$$

pointwise and

$$\lim_{\substack{y\to\bar{0}\\ y\in C_j}} f(x+iy) = \mathcal{F}[g_j] \in \mathcal{S}',\ j = 1, \ldots, m, \tag{2.2}$$

in the strong topology of $\mathcal{S}'$ with $\mathcal{F}[g_j]$ being the $\mathcal{S}'$ Fourier transform of $g_j \in \mathcal{S}'_q \subset \mathcal{S}'$.

We now state and prove the main result of this paper.

THEOREM. Let C be an open cone in R^n which is the union of a finite number of open convex cones, $C = \bigcup_{j=1}^{m} C_j$, such that $(O(C))^*$ contains interior points and has an admissible set of vectors. Let $f(z)$, $z = x+iy$, be holomorphic in the tubular cone T^C and satisfy (1.1) for $y \in C$ and $1 < p \le 2$. Let the boundary values of $f(x+iy)$ in the strong topology of $\mathcal{S}'$ corresponding to each connected component C_j, $j = 1, \ldots, m$, of C given in (2.2) be equal in $\mathcal{S}'$. Then there is a function $F(z)$ which is holomorphic in $T^{O(C)}$ and which satisfies $F(z) = f(z)$, $z \in T^C$, where $F(z)$ is of the form

$$F(z) = P(z)\,H(z), \quad z \in T^{O(C)},$$

with $P(z)$ being a polynomial in z and $H(z) \in S^2_{A\,\rho_C}(T^{O(C)}) \cap S^q_{A\,\rho_C}(T^{O(C)})$, $(1/p) + (1/q) = 1$.

PROOF. By hypothesis the boundary values in (2.2) above are equal in $\mathcal{S}'$. Since the Fourier transform is a topological isomorphism of $\mathcal{S}'$ onto $\mathcal{S}'$ we have that the elements $g_j(t) \in \mathcal{S}'_q \subset \mathcal{S}'$, $(1/p) + (1/q) = 1$, $j = 1, \ldots, m$, obtained in the first paragraph of this section satisfy

$$g_1(t) = g_2(t) = \ldots = g_m(t) \tag{2.3}$$

in $\mathcal{S}'$. We call this common value $g(t)$ and have $g(t) \in \mathcal{S}'_q$, $(1/p) + (1/q) = 1$. Now

$$u_C(t) = \max_{j=1,\ldots,m} u_{C_j}(t),\ t \in R^n. \tag{2.4}$$

We have $u_C(t) = u_{O(C)}(t)$, $t \in C^*$, (Vladimirov [3, p. 219, (54)]); and from the definition of ρ_C we have $u_{O(C)}(t) \le \rho_C\, u_C(t)$, $t \in C_* = R^n \setminus C^*$. Since $1 \le \rho_C < \infty$ (Vladimirov [3, p. 220]) here we have $u_{O(C)}(t) \le \rho_C\, u_C(t)$, $t \in R^n$. From (2.4) we now obtain

$$u_{O(C)}(t) \le \rho_C \max_{j=1,\ldots,m} u_{C_j}(t), \quad t \in R^n. \tag{2.5}$$

From (2.3) and the fact that $\mathrm{supp}(g_j) \subseteq \{t: u_{C_j}(t) \le A\}$ almost everywhere, $j = 1, \ldots, m$, we have that $g \in \mathcal{S}'_q \subset \mathcal{S}'$ vanishes on $\bigcup_{j=1}^{m} \{t: u_{C_j}(t) > A\}$ as a distribution. Now let $t \in \{t: u_{O(C)}(t) > A\,\rho_C\}$; for such a point t we have by (2.5) that

$$A\,\rho_C < u_{O(C)}(t) \le \rho_C \max_{j=1,\ldots,m} u_{C_j}(t)$$

and hence

$$\max_{j=1,\ldots,m} u_{C_j}(t) > A.$$

Thus if $t \in \{t: u_{0(C)}(t) > A\rho_C\}$ then $t \in \bigcup_{j=1}^{m} \{t: u_{C_j}(t) > A\}$ and on this latter set g vanishes. Since $\{t: u_{0(C)}(t) \le A\rho_C\}$ is a closed set in $\mathbb{R}^n$ we thus have

$$\text{supp}(g) \subseteq \{t: u_{0(C)}(t) \le A\rho_C\} \tag{2.6}$$

in $\mathcal{S}'$ and $\{t: u_{0(C)}(t) \le A\rho_C\} = (0(C))^* + \overline{N(\bar{0};A\rho_C)}$ (Vladimirov [4, Lemma 1, p. 936]) with $\overline{N(\bar{0};A\rho_C)}$ being the closure of the open ball in $\mathbb{R}^n$ centered at $\bar{0}$ and with radius $A\rho_C$. Recall from section 1 that the dual cone $(0(C))^*$ is closed and convex and by hypothesis in this Theorem $(0(C))^*$ contains interior points and has an admissible set of vectors. Since $g \in \mathcal{S}'_q \subset \mathcal{S}'$ has order 0 then by Vladimirov [4, Theorem 1, p. 930]

$$g(t) = \prod_{k=1}^{n} \langle e_k, \text{gradient}\rangle^2 \ G(t) \tag{2.7}$$

where $\{e_k\}_{k=1}^n$ is an admissible set of vectors for the cone $(0(C))^*$, $G(t)$ is a continuous function of $t \in \mathbb{R}^n$ which is unique corresponding to $\{e_k\}_{k=1}^n$ and the order 0 of $g \in \mathcal{S}'_q \subset \mathcal{S}'$, $\text{supp}(G) \subseteq \{t: u_{0(C)}(t) \le A\rho_C\} = (0(C))^* + \overline{N(\bar{0};A\rho_C)}$, and

$$|G(t)| \le K\ (1 + |t|), \quad t \in \mathbb{R}^n, \tag{2.8}$$

where the constant K is independent of $t \in \mathbb{R}^n$. (In Vladimirov [4, Theorem 1, p. 930] the term "acute" in our present situation means that $((0(C))^*)^* = \overline{0(C)}$ (Vladimirov [3, p. 218]) should have non-empty interior (Vladimirov [4, p. 930]) which is certainly the case in this Theorem.) Since $G(t)$ is continuous on $\mathbb{R}^n$, then $\text{supp}(G) \subseteq \{t: u_{0(C)}(t) \le A\rho_C\}$ as a function (Schwartz [8, Chapter 1, sections 1 and 3]). (This fact is also obtained in the proof of Vladimirov [4, Theorem 1], and the containment $\text{supp}(G) \subseteq \{t: u_{0(C)}(t) \le A\rho_C\}$ which is stated preceding to (2.8) gives the support of $G(t)$ as a function.) We now choose a function $\lambda(t) \in C^\infty$, $t \in \mathbb{R}^n$, such that for any n-tuple α of nonnegative integers $|D^\alpha\lambda(t)| \le M_\alpha$, $t \in \mathbb{R}^n$, where M_α is a constant which depends only on α; and for $\epsilon > 0$, $\lambda(t) = 1$ for t on an ϵ neighborhood of $\{t: u_{0(C)}(t) \le A\rho_C\}$ and $\lambda(t) = 0$ for $t \in \mathbb{R}^n$ but not on a 2ϵ neighborhood of $\{t: u_{0(C)}(t) \le A\rho_C\}$ (Carmichael [1, p. 94]). We have that $(\lambda(t)\exp(2\pi i\langle z,t\rangle)) \in \mathcal{S}$ as a function of $t \in \mathbb{R}^n$ for $z \in T^{0(C)}$. Recalling (2.6) we now put

$$F(z) = \int_{\mathbb{R}^n} g(t)\exp(2\pi i\langle z,t\rangle)\,dt = \int_{\mathbb{R}^n} g(t)\,\lambda(t)\exp(2\pi i\langle z,t\rangle)\,dt, \quad z \in T^{0(C)}. \tag{2.9}$$

From (2.7) and $\text{supp}(G) \subseteq \{t: u_{0(C)}(t) \le A\rho_C\}$ as a function we have (Vladimirov [4, (3.1), p. 931])

$$F(z) = \left(\prod_{k=1}^{n} \langle e_k, -2\pi iz\rangle^2\right) H(z), \quad z \in T^{0(C)}, \tag{2.10}$$

where

$$H(z) = \int_{\{t: u_{0(C)}(t) \le A\rho_C\}} G(t)\exp(2\pi i\langle z,t\rangle)\,dt, \quad z \in T^{0(C)}. \tag{2.11}$$

From the continuity of $G(t)$ and (2.8) we easily have $G(t) \in \mathcal{S}'_p$ for all p, $1 \le p < \infty$; this combined with the support of $G(t)$ as a function and Carmichael [1, Theorem 6.1, p. 98] yield

$$(\exp(-2\pi\langle y,t\rangle)\,G(t)) \in L^p, \quad y \in 0(C), \tag{2.12}$$

and

$$||\exp(-2\pi\langle y,t\rangle)\ G(t)||_{L^p} \le M\ (1 + (d(y))^{-r})^s\ \exp(2\pi A\ \rho_C\ |y|),\quad y \in O(C)\ , \qquad (2.13)$$

for constants $r = r(G,p,A) \ge 0$, $s = s(G,p,A) \ge 0$, and $M = M(G,p,A,r,s) > 0$, which are independent of $y \in O(C)$, and for all p, $1 \le p < \infty$. Then (2.12), (2.13), and Carmichael [1, Theorem 5.1, p. 97] prove $H(z) \in S^q_{A\rho_C}(T^{O(C)})$, $(1/p) + (1/q) = 1$, for all p, $1 < p \le 2$, and in particular $H(z) \in S^2_{A\rho_C}(T^{O(C)})$. Then by (2.10), F(z) defined in (2.9) is holomorphic in $T^{O(C)}$, and of course (2.10) is the desired representation of F(z) in the statement of the Theorem where the polynomial P(z) is

$$P(z) = \prod_{k=1}^{n} \langle e_k, -2\pi i z\rangle^2$$

and $H(z) \in S^2_{A\rho_C}(T^{O(C)}) \cap S^q_{A\rho_C}(T^{O(C)})$, $(1/p) + (1/q) = 1$, is given in (2.11). By (2.3), (2.6), and the definition of $\lambda(t)$ preceding (2.9), we see that (2.1) can be rewritten as

$$f(z) = \int_{\mathbb{R}^n} g(t)\ \lambda(t)\ \exp(2\pi i\langle z,t\rangle)\ dt =$$

$$= \int_{\mathbb{R}^n} g(t)\ \exp(2\pi i\langle z,t\rangle)\ dt,\quad z \in T^{C_j},\quad j = 1,\dots,m .$$

These identities and (2.9) show that F(z) is the desired holomorphic extension of f(z) to $T^{O(C)}$ and $F(z) = f(z)$, $z \in T^C$. The proof of the Theorem is complete.

We emphasize that cones C exist for which the hypotheses of the Theorem are satisfied corresponding to C and $(O(C))^*$, and examples are easily constructed. If O(C) in the Theorem is regular (i.e. if $\overline{O(C)}$ does not contain an entire straight line in this case since O(C) is open and convex) then the interior of $(O(C))^*$ is not empty; the Theorem applies in this case if $(O(C))^*$ has an admissible set of vectors.

In the Theorem we have desired to obtain a result in which the holomorphic extension function could be represented in terms of an $S^p_{A\rho_C}(T^{O(C)})$ space; this happens under the assumptions on $(O(C))^*$ in the Theorem. Under these assumptions we were able to conclude that the continuous function G(t) in the representation (2.7) had pointwise support in $\{t: u_{O(C)}(t) \le A\rho_C\}$. From this fact we were able to use Carmichael [1, Theorem 6.1] and then Carmichael [1, Theorem 5.1] to obtain that H(z) in (2.11) belongs to $S^q_{A\rho_C}(T^{O(C)})$, $(1/p) + (1/q) = 1$, for all p, $1 < p \le 2$; and hence the desired representation of the holomorphic extension function F(z) was obtained in (2.10).

From the proof of the Theorem the common value $g(t) \in \mathcal{D}'_q$, $(1/p) + (1/q) = 1$, $1 < p \le 2$, in (2.3) has $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \le A\rho_C\}$ in $\mathcal{S}'$ (recall (2.6)). If supp(g) is contained in this set almost everywhere as a function as well then the restrictions on $(O(C))^*$ in the Theorem can be deleted in obtaining a holomorphic extension result as we show in the following corollary.

COROLLARY 1. Let C be an open cone in $\mathbb{R}^n$ which is the union of a finite number of open convex cones, $C = \bigcup_{j=1}^{m} C_j$. Let f(z), $z = x + iy$, be holomorphic in the tubular cone T^C and satisfy (1.1) for $y \in C$ and $1 < p \le 2$. Let the boundary values of $f(x+iy)$ in the strong topology of $\mathcal{S}'$ corresponding to each connected component C_j,

$j = 1, \ldots, m$, of C given in (2.2) be equal in $\mathcal{S}'$ and let this common value g(t) have support in $\{t: u_{O(C)}(t) \le A\rho_C\}$ almost everywhere (as well as in $\mathcal{S}'$). Then there is a function F(z) which is holomorphic in $T^{O(C)}$ and which satisfies $F(z) = f(z)$, $z \in T^C$; and if $p = 2$, $F(z) \in S^2_{A\rho_C}(T^{O(C)})$.

PROOF. Proceeding as in the proof of the Theorem we obtain the common value $g(t) \in \mathcal{S}'_q$, $(1/p) + (1/q) = 1$, from (2.3) and $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \le A\rho_C\}$ in $\mathcal{S}'$. By our assumption $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \le A\rho_C\}$ almost everywhere; thus by Carmichael [1, Theorem 6.1, p. 98], g(t) satisfies

$$(\exp(-2\pi\langle y,t\rangle)\, g(t)) \in L^q, \quad y \in O(C), \tag{2.14}$$

and

$$\|\exp(-2\pi\langle y,t\rangle)\, g(t)\|_{L^q} \le M\,(1 + (d(y))^{-r})^s \exp(2\pi A\rho_C|y|), \quad y \in O(C), \tag{2.15}$$

for constants $r = r(g,q,A) \ge 0$, $s = s(g,q,A) \ge 0$, and $M = M(g,q,A,r,s) > 0$ which are independent of $y \in O(C)$. Then by Carmichael [1, Theorem 3.1, pp. 84-85] the function

$$F(z) = \int_{\mathbb{R}^n} g(t)\exp(2\pi i\langle z,t\rangle)\,dt = \int_{\mathbb{R}^n} g(t)\,\lambda(t)\exp(2\pi i\langle z,t\rangle)\,dt, \quad z \in T^{O(C)}, \tag{2.16}$$

is holomorphic in $T^{O(C)}$ where $\lambda(t) \in C^\infty$ is the function defined in the proof of the Theorem. As in the proof of the Theorem F(z) is the desired holomorphic extension of f(z) to $T^{O(C)}$. If $p = 2$ then $q = 2$; in this case (2.14), (2.15), and Carmichael [1, Theorem 5.1, p. 97] yield that $F(z) \in S^2_{A\rho_C}(T^{O(C)})$. The proof is complete.

We have a more general holomorphic extension theorem than either the Theorem or Corollary 1. Here O(C) is as general as possible and we make no assumption on the constructed g(t) in (2.3). We lose the explicit information on F(z) being in an $S^p_{A\rho_C}(T^{O(C)})$ space however.

COROLLARY 2. Let the open cone C and the function f(z) be as in the hypothesis of Corollary 1 with $1 < p \le 2$. Let the boundary values of $f(x+iy)$ in the strong topology of $\mathcal{S}'$ corresponding to each connected component C_j, $j = 1, \ldots, m$, of C given in (2.2) be equal in $\mathcal{S}'$. Then there is a holomorphic function F(z) in $T^{O(C)}$ such that $F(z) = f(z)$, $z \in T^C$.

PROOF. Define F(z), $z \in T^{O(C)}$, as in (2.16) where $g \in \mathcal{S}'_q \subset \mathcal{S}'$, $(1/p) + (1/q) = 1$, is the common value in (2.3) in $\mathcal{S}'$ and $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \le A\rho_C\}$ in $\mathcal{S}'$ from the proof of the Theorem. Then F(z) is holomorphic in $T^{O(C)}$ by the necessity of Vladimirov [3, Theorem 2, p. 239] and is the desired holomorphic extension of f(z) to $T^{O(C)}$ because of (2.3) and (2.1). (Recall the proof of the Theorem.) The proof is complete.

Notice from Vladimirov [3, Theorem 2, p. 239] that F(z) in Corollary 2 does satisfy a pointwise growth estimate; but we cannot conclude that F(z) is in an $S^p_{A\rho_C}(T^{O(C)})$ space for any p in Corollary 2.

In the Theorem and Corollaries 1 and 2 the holomorphic extension function F(z), $z \in T^{O(C)}$, is defined by (2.9) (i.e. (2.16)) where $g(t) \in \mathcal{S}'_q \subset \mathcal{S}'$, $(1/p) + (1/q) = 1$, and $\text{supp}(g) \subseteq \{t: u_{O(C)}(t) \le A\rho_C\}$ in $\mathcal{S}'$. Since O(C) is an open convex cone then in each of the results we can also conclude that

$$\lim_{\substack{y\to\bar{0}\\ y\in O(C)}} F(x+iy) = \mathcal{F}[g] \in \mathcal{S}' \tag{2.17}$$

in the strong topology of $\mathcal{S}'$ by the boundary value proof in Carmichael [1, Corollary 4.1, p. 93]; here $\mathcal{F}[g]$ is the $\mathcal{S}'$ Fourier transform. Further, if $O(C)$ is a regular cone, $A = 0$, and $p = 2$, in Corollary 1 then we can conclude in Corollary 1 that

$$F(z) = \langle \mathcal{F}[g], K(z-t)\rangle = \langle \mathcal{F}[g], Q(z;t)\rangle, \quad z \in T^{O(C)}, \tag{2.18}$$

in $\mathcal{S}'$ by Carmichael [1, Corollary 4.2, p. 94] where $\mathcal{F}[g]$ is the boundary value in (2.17) and $K(z-t)$ and $Q(z;t)$ are the Cauchy and Poisson kernels (Carmichael [1, p. 83]), respectively, corresponding to the tube $T^{O(C)}$. (Recall from the sentence preceding the statement of Carmichael [1, Corollary 4.2, p. 94] that $g \in \mathcal{S}'_2$ implies $\mathcal{F}[g] \in \mathcal{D}'_{L^2} \subset \mathcal{S}'$.)

If the cone C is $(0,\infty)$ or $(-\infty,0)$ or $(-\infty,0) \cup (0,\infty)$ in 1 dimension then of course $d(y) = |y|$, $y \in C$, in (1.1). We have the following interesting result in 1 dimension for $C = (-\infty,0) \cup (0,\infty)$. Note that $(O(C))^* = \{0\}$ here which does not have interior points; so the following result is like Corollary 2.

COROLLARY 3. Let $f(z)$ be holomorphic in $\mathbb{R}^1 + iC$, $C = (-\infty,0) \cup (0,\infty)$, and satisfy (1.1) for $1 < p \leq 2$. Let the boundary values of $f(x+iy)$ in the strong topology of $\mathcal{S}'$ from the upper and lower half planes given in (2.2) be equal in $\mathcal{S}'$. Then there is an entire holomorphic function $F(z)$ such that $F(z) = f(z)$, $z \in \mathbb{R}^1 + iC$.

PROOF. First note that $O(C) = (-\infty,\infty)$. Obtain $g(t) \in \mathcal{S}'_q \subset \mathcal{S}'$, $(1/p) + (1/q) = 1$, $1 < p \leq 2$, as in Corollary 2 and define

$$F(z) = \int_{\mathbb{R}^1} g(t)\exp(2\pi i\langle z,t\rangle)\,dt = \int_{\mathbb{R}^1} g(t)\,\lambda(t)\exp(2\pi i\langle z,t\rangle)\,dt, \quad z \in \mathbb{C}^1, \tag{2.19}$$

as in (2.16). Here $(O(C))^* = \{0\}$ and $\operatorname{supp}(g) \subseteq \{t: u_{O(C)}(t) \leq A\rho_C\} = (O(C))^* + \overline{N(0;A\rho_C)} = [-A\rho_C, A\rho_C]$. Thus $g \in \mathcal{S}'_q$ has compact support here, and hence $g \in \mathcal{E}'$. $F(z)$ in (2.19) is the Fourier-Laplace transform of a distribution of compact support and hence is an entire holomorphic function in $\mathbb{C}^1$ (Hörmander [9, Theorem 1.7.5, p. 20]). $F(z) = f(z)$, $z \in \mathbb{R}^1 + iC$, as before.

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The author's permanent address is Department of Mathematics and Computer Science, Wake Forest University, Winston-Salem, North Carolina 27109, U.S.A.

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