Internat. J. Math. & Math. Sci.
Vol. 8 No. 2 (1985) 413-415

A NOTE ON PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

S. H. CHANG

Department of Mathematics
Cleveland State University
Cleveland, Ohio 44115

(Received September 15, 1983)

ABSTRACT. The existence of periodic solution for a certain functional differential equation with quasibounded nonlinearity is established.

KEY WORDS AND PHRASES. *Quasibounded nonlinearity, periodic solution.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 34K15.

1. INTRODUCTION.

Let C_r denote the Banach space of continuous R^n - valued functions on $[-r,0]$ with the supremum norm, i.e., for each $\phi \in C_r$, $\|\phi\| = \max_{-r\le\theta\le 0} |\phi(\theta)|$. Also for a given continuous R^n - valued function x defined on $[-r,b)$ with $b > 0$ and for $0 \le t < b$, let x_t be the function in C_r defined by $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-r,0]$.

Consider the following functional differential equation

$$x'(t) = L(t,x_t) + f(t,x_t), \tag{1.1}$$

where L and f are continuous mappings from $[0,\infty) \times C_r$ into R^n, $L(t+T,\phi) = L(t,\phi)$ and $f(t+T,\phi) = f(t,\phi)$ for all $(t,\phi) \in [0,\infty) \times C_r$ and for some $T > 0$, $L(t,\phi)$ is linear in ϕ for fixed t, and f maps closed and bounded sets into bounded sets. Assume that the equation

$$x'(t) = L(t,x_t) \tag{1.2}$$

has no nontrivial T-periodic solutions. Also, without loss of generality we assume $T \ge r$.

Fennell [2] has established the existence of T-periodic solution for the equation (1.1) by assuming

$$\lim_{\|\phi\|\to\infty} \frac{|f(t,\phi)|}{\|\phi\|} = 0 \tag{1.3}$$

uniformly in t. It is the purpose of this note to generalize Fennell's result by relaxing this requirement. We shall see that the limit in (1.3) can be allowed to be positive.

Using the mapping f in (1.1), we give the following definition. The function

f is said to be quasibounded with respect to ϕ if the number

$$|f| = \min_{0<\rho<\infty} \left(\max_{\substack{\|\phi\| \geq \rho \\ 0 \leq t \leq T}} \frac{|f(t,\phi)|}{\|\phi\|} \right) \tag{1.4}$$

is finite; in this case, $|f|$ is called the quasinorm of f. In recent years, equations with quasibounded nonlinearities have been studied extensively. We shall show that if f is quasibounded and has a quasinorm smaller than a certain positive number then Eq. (1.1) has at least one T-periodic solution. Our proof uses a technique generalizing that used in [2].

2. THE RESULTS.

Under the assumption for (1.2), the functional differential equation

$$x'(t) = L(t,x_t) + h(t), \tag{2.1}$$

where L is the same as in (1.1) and $h: [0,\infty) \to R^n$ is continuous and T-periodic, has a unique T-periodic solution. Let $x(\psi,h): [-r,\infty) \to R^n$ denote the solution of (2.1) with initial value $\psi \in C_r$. Let $U: C_r \to C_r$ be the operator defined by $U\phi = x_T(\phi,0)$. Then U is completely continuous and the T-periodic solution of (2.1) is determined by the initial function $\psi = (I-U)^{-1}x_T(0,h)$. Let $\ell(t)$ be the norm of the operator $L(t,\phi)$,

$$E = \exp\left(\int_0^T \ell(s)ds\right),$$

and

$$K = TE^2\|(I-U)^{-1}\| + TE. \tag{2.2}$$

THEOREM. If, in addition to the given assumptions for the equation (1.1), f is quasibounded with respect to ϕ and has a quasinorm $|f| < 1/K$, where K is given by (2.2), then (1.1) has at least one T-periodic solution.

PROOF. The following inequality

$$\|x_t(\phi,h)\| \leq \{\|\phi\| + \int_0^t |h(s)|ds\}\exp\left(\int_0^t \ell(s)ds\right), \quad t \geq 0, \tag{2.3}$$

which follows from (2.1) and Gronwall's lemma, will be needed.

Let X be the Banach space of continuous T-periodic functions from $[-r,\infty)$ into R^n with the supremum norm. For each $\phi \in X$, let $\hat{f}(\phi)(t) = f(t,\phi_t)$. Then $\hat{f}(\phi): [0,\infty) \to R^n$ is continuous and T-periodic. Let $\psi = (I-U)^{-1}x_T(0,\hat{f}(\phi))$. Then $\psi \in C_r$. Now, define a mapping $P: X \to X$ by $P\phi = x(\psi,\hat{f}(\phi))$, i.e., $P\phi$ is the unique T-periodic solution of

$$x'(t) = L(t,x_t) + f(t,\phi_t).$$

Then P is a continuous mapping.

Since $|f| < 1/K$, there exists $\varepsilon > 0$ such that $|f| + \varepsilon < 1/K$. Then by the definition of quasiboundedness (1.4) there exists $\rho(\varepsilon) > 0$ such that

$$\frac{|f(t,\phi)|}{\|\phi\|} < \frac{1}{K} \quad \text{whenever} \quad \|\phi\| \geq \rho(\varepsilon) \text{ and } 0 \leq t \leq T.$$

Let

$$N = \max\{|f(t,\phi)| : \phi \in C_r,\ \|\phi\| \le \rho(\varepsilon),\ 0 \le t \le T\}.$$

Then let $M = \max\{KN, \rho(\varepsilon)\}$ and

$$D = \{\phi \in X : \|\phi\| \le M\}.$$

We claim that (i) $P(D) \subset D$ and (ii) $P(D)$ is relatively compact.

Using the inequality (2.3), we obtain that

$$\|P\phi\| = \max_{0\le t\le T} |P\phi(t)| \le K \max_{0\le s\le T} |f(s,\phi_s)|.$$

Now for $\phi \in D$ and $0 \le s \le T$, if $\|\phi_s\| \le \rho(\varepsilon)$ then $K|f(s,\phi_s)| \le KN \le M$ and if $\|\phi_s\| > \rho(\varepsilon)$ then $K|f(s,\phi_s)| < \|\phi_s\| \le \|\phi\| \le M$. Thus $\|P\phi\| \le M$ whenever $\phi \in D$. This proves (i). (ii) can be established by using an argument similar to that used in [2].

By Schauder's fixed point theorem ([3], or see [1, p. 131]) there exists $\phi \in D$ such that $P\phi = \phi$, which completes the proof of the theorem.

COROLLARY (FENNELL [2]). If, in addition to the given assumptions for the equation (1.1), f satisfies the condition (1.3), then (1.1) has at least one T-periodic solution.

PROOF. The condition (1.3) implies that $|f| = 0$.

REFERENCES

1. CRONIN, J., "Fixed points and topological degree in nonlinear analysis", Mathematical Surveys, No. 11, American Mathematical Society, Providence, 1964.

2. FENNELL, R.E., Periodic solutions of functional differential equations, J. Math. Anal. Appl. 39 (1972), 198-201.

3. SCHAUDER, J., Der Fixpunktsatz in Funktionalraumen, Studia Math. 2 (1930), 171-180.

Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São Josè dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk