

**ON THE ORDER OF EXPONENTIAL GROWTH OF THE SOLUTION  
OF THE LINEAR DIFFERENCE EQUATION WITH PERIODIC  
COEFFICIENT IN BANACH SPACE**

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**ABSTRACT.** An equation of the form  $y - A(t)y = f(t)$  is considered, where  $\Delta y = \frac{y(t+\delta) - y(t)}{\delta}$ , and the necessary and sufficient criteria for the exponential growth of the solution of this equation is obtained.

**KEY WORDS AND PHRASES.** *Difference equations, solution of exponential growth.*

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**1. INTRODUCTION.**

Let  $E$  be a complex Banach space. Denote by  $\{A(t) : t \geq 0\}$  a family of linear bounded operators from  $E$  into itself. We assume that  $A(t)$  is periodic and strongly continuous in  $t \in [0, \infty)$ .

Let  $\| \cdot \|$  be the norm in  $E$ . Denote by  $E_\alpha$  the set of all elements  $f(t) \in E$  such that

$$\sup \|f(t)\| \exp(-\alpha t) < \infty.$$

**2. RESULTS.**

Let  $\Delta y = \frac{y(t+\delta) - y(t)}{\delta}$ ,  $\delta > 0$ ,  $y(t)$  be a solution of the difference equation

$$\Delta y - A(t)y = f(t), \quad t \geq \delta \quad (2.1)$$

such that

$$y(t) = 0, \quad 0 \leq t < \delta \quad (2.2)$$

where  $0$  is the zero of  $E$ .

Let us assume that  $f \in E_\alpha$ . The solution of equation (2.1) can be written in the

form

$$y(t) = \delta \sum_{i=0}^{t-\delta} A(i) y(i) + \delta \sum_{i=0}^{t-\delta} f(i) \quad (2.3)$$

where  $t = [n\delta]$ ,  $[a]$  denotes the greatest positive integer  $\leq a$  and  $\delta$  is a positive integer.

Without loss of generality we suppose that  $\delta = 1$ .

Putting  $t = 1, 2, \dots, n$  in (2.3), one obtains

$$y(t) = \sum_{j=1}^{n-1} \prod_{i=n-1}^j (I + A(i)) f(j-1) + f(t-1) \quad (2.4)$$

where  $I$  is the unit operator. Let  $w$  be the period of  $A(t)$ .

$$\left[ \prod_{i=n-1}^j B(i) = B(n-1) B(n-2) \dots B(j), \quad j \leq n-1 \right]$$

Substituting  $t = [Sw]$  into equation (2.4), we obtain

$$y(t) = \sum_{r=1}^s \left[ \prod_{k=w-1}^0 [I + A(k)] \sum_{j=1}^{s-r} f_j((r-1)w+j-1) + f((r-1)w + w-1) \right] \quad (2.5)$$

where

$$\begin{aligned} f_1(\xi w) &= A(w-1) A(w-2) \dots A(1) f(\xi w) \\ f_2(\xi w+1) &= A(w-1) A(w-2) \dots A(2) f(\xi w+1) \\ &\dots \\ &\dots \\ f_{w-1}(\xi w+w-2) &= A(w-1) f(\xi w + w-2). \end{aligned}$$

Setting  $B = \prod_{k=w-1}^0 [I + A(k)]$  in (2.5) we get

$$y(t) = \sum_{r=1}^{\frac{t}{w}} B^{\frac{t-rw}{w}} \left\{ \sum_{j=1}^{w-1} f_j((r-1)w + j-1) + f((r-1)w + w-1) \right\}.$$

The last equation can be written in the form

$$y(t) = -\frac{1}{2\pi i} \oint_{\gamma} \sum_{r=1}^{\frac{t}{w}} \lambda^{\frac{t-rw}{w}} (B - \lambda I)^{-1} \left\{ \sum_{j=1}^{w-1} f_j((r-1)w + j-1) + f((r-1)w + w-1) \right\} \quad (2.6)$$

where  $\gamma$  is a contour which circumscribes all the specter of the operator  $B$ , [1].

It can be seen that if  $f \in E_\alpha$ , then  $(B - \lambda I)^{-1} f \in E_\alpha$  for every  $\lambda \in \gamma$ . From equation (2.6) we obtain a necessary and sufficient criterion for the exponential growth of the solution with an index  $\beta$ . Let  $\sigma_B$  denote the specter of the operator  $B$ . Assume that  $\lambda_0 \in \sigma_B$ . Set  $\alpha_0 = \frac{1}{w} \ln |\lambda_0|$ . The following theorem holds:

THEOREM. If  $f \in E_\alpha$ , then the solution  $y$  of equation (2.1) belongs to  $E_\beta$  such that

$$\begin{aligned}\beta &= \alpha, \text{ when } \alpha > \alpha_0 \\ \beta &> \alpha, \text{ when } \alpha = \alpha_0 \\ \beta &= \alpha_0, \text{ when } \alpha < \alpha_0.\end{aligned}$$

PROOF. To prove the sufficiency, we consider the following three cases:

(1) If  $\alpha > \frac{1}{w} \ln |\lambda|$  then  $y(t)$  defined by (2.6) belongs to  $E_\alpha$ .

(2) If  $\alpha > \frac{1}{w} \ln |\lambda|$  then from (2.6) we obtain

$$\begin{aligned}\|y\| &\leq D \sum_{r=1}^{\frac{t}{w}} \exp\left(\frac{1}{w} \ln |\lambda| (t-rw)\right) \left\{ \sum_{j=1}^{w-1} \|f_j((r-1)w+j-1)\| \right. \\ &\quad \left. + \|f((r-1)w + w-1)\| \right\} \\ &< D_1 \exp(\alpha t) \cdot \frac{t}{w} + D_2 \exp(\alpha t) \cdot (w-2) \cdot \frac{t}{w}\end{aligned}$$

$$< D' \exp(\alpha t) \cdot t$$

(where  $D$ ,  $D_1$ ,  $D_2$  and  $D'$  are constants).

This means that  $y \in E_\beta$  where  $\beta > \alpha$ .

(3) If  $\alpha < \frac{1}{w} \ln |\lambda|$  and  $\|f\| \leq c \exp(\alpha t)$ , then from (2.6) we have

$$\begin{aligned}\|y\| &\leq C_1 \exp\left(\frac{1}{w} \ln |\lambda| \cdot t\right) \\ \text{and } y &\in E_{\frac{1}{w} \ln |\lambda|} \quad (\alpha < \frac{1}{w} \ln |\lambda|).\end{aligned}$$

We now prove the necessity:

If  $\lambda_0$  is an eigenvalue and  $x_0$  is an eigenvector for the operator  $B$  such that

$$Bx_0 = \lambda_0 x_0,$$

where  $x_0$  is an element of Banach space such that  $\|x_0\| = 1$ , by taking

$f(t) = \exp(\alpha t) \cdot x_0$  equation (2.6) with

$$(B - \lambda I)^{-1} x_0 = \frac{x_0}{\lambda_0 - \lambda} \text{ becomes}$$

$$y(t) = \sum_{r=1}^{\frac{t}{w}} \exp\left(\frac{1}{w} \ln |\lambda_0| (t-rw)\right) \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f((r-1)w + w-1) \right\}. \quad (2.7)$$

Multiplying the last equation by  $\exp(-\alpha_0 t)$ , where  $\alpha_0 = \frac{1}{w} \ln |\lambda_0|$ , we have

$$y(t) \exp(-\alpha_0 t) = \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \exp(-\alpha_0 wr) \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f((r-1)w + w-1) \right\}$$

where  $\theta = \arg \lambda$ .

$$\begin{aligned}y(t) \exp(-\alpha_0 t) &= \exp\left(\frac{i\theta t}{w} + w(1 - \alpha) - 1\right) \sum_{r=1}^{\frac{t}{w}} \exp((\alpha - \alpha_0)wr) \cdot x_0 \\ &\quad + \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \sum_{j=1}^{w-1} \exp(-\alpha_0 wr) f_j((r-1)w + j-1)\end{aligned}$$

$$\begin{aligned}
 &= \frac{\exp\left(\frac{i\theta t}{w} + (\alpha - \alpha_0)\right)}{\exp(\alpha - \alpha_0) - 1} [\exp(\alpha - \alpha_0) t - 1] x_0 \\
 &+ \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \sum_{j=1}^{w-1} \exp(-\alpha_0 wr) \cdot f_j((r-1)w + j-1)
 \end{aligned} \tag{2.8}$$

Now for the last relation we have the following cases:

1) If  $\alpha > \alpha_0$  then by using formula (2.8) we get

$$\lim_{t \rightarrow \infty} y(t) \exp(-\alpha_0 t) = \infty.$$

This means that  $y \notin E_{\alpha_0}$  but  $y \in E_\alpha$  ( $\alpha > \alpha_0$ ).

2) If  $\alpha = \alpha_0$  then from (2.8)

$$\begin{aligned}
 y(t) \exp(-\alpha_0 t) &= \exp(w(1 - \alpha) - 1 + \frac{i\theta t}{w}) \left(\frac{t}{w} - 1\right) x_0 \\
 &+ \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \sum_{j=1}^{w-1} \exp(-\alpha_0 wr) f_j((r-1)w + j-1).
 \end{aligned}$$

Using the last equation we get

$$\lim_{t \rightarrow \infty} y(t) \exp(-\alpha_0 t) = \infty.$$

This means that  $y \in E_\alpha$  but  $y \in E_\beta$  ( $\beta > \alpha$ ).

3) If  $\alpha = \alpha_0$  then from (2.8) we have  $y \notin E_\alpha$  but  $y \in E_{\alpha_0}$ .

This completes the proof.

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#### REFERENCES

1. HUSSEIN, H. A. On the Bounded Solution of Certain Linear Equations in Partial Differences Equations, J. Natur. Sci. Math. 21, (1981) 165-169.
2. HUSSEIN, H. A. Estimation of the Exponential Growth of the Solution of Certain Linear Partial Differential Equations with a Highest Order Term, (Russian) Differencial'nye Uravnenija 12 (1976) 2279-2280.

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