

ON THE ORDER OF EXPONENTIAL GROWTH OF THE SOLUTION OF THE LINEAR DIFFERENCE EQUATION WITH PERIODIC COEFFICIENT IN BANACH SPACE

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ABSTRACT. An equation of the form $y' - A(t)y = f(t)$ is considered, where $\Delta y = \frac{y(t+\delta) - y(t)}{\delta}$, and the necessary and sufficient criteria for the exponential growth of the solution of this equation is obtained.

KEY WORDS AND PHRASES. *Difference equations, solution of exponential growth.*

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1. INTRODUCTION.

Let E be a complex Banach space. Denote by $\{A(t) : t \geq 0\}$ a family of linear bounded operators from E into itself. We assume that $A(t)$ is periodic and strongly continuous in $t \in [0, \infty)$.

Let $\|\cdot\|$ be the norm in E . Denote by E_α the set of all elements $f(t) \in E$ such that

$$\sup \|f(t)\| \exp(-\alpha t) < \infty.$$

2. RESULTS.

Let $\Delta y = \frac{y(t+\delta) - y(t)}{\delta}$, $\delta > 0$, $y(t)$ be a solution of the difference equation

$$\Delta y - A(t)y = f(t), \quad t \geq \delta \tag{2.1}$$

such that

$$y(t) = 0, \quad 0 \leq t < \delta \tag{2.2}$$

where 0 is the zero of E .

Let us assume that $f \in E_\alpha$. The solution of equation (2.1) can be written in the

form

$$y(t) = \delta \sum_{i=0}^{t-\delta} A(i) y(i) + \delta \sum_{i=0}^{t-\delta} f(i) \quad (2.3)$$

where $t = [n\delta]$, $[a]$ denotes the greatest positive integer $\leq a$ and δ is a positive integer.

Without loss of generality we suppose that $\delta = 1$.

Putting $t = 1, 2, \dots, n$ in (2.3), one obtains

$$y(t) = \sum_{j=1}^{n-1} \prod_{i=n-1}^j (I + A(i)) f(j-1) + f(t-1) \quad (2.4)$$

where I is the unit operator. Let w be the period of $A(t)$.

$$\left[\prod_{i=n-1}^j B(i) = B(n-1) B(n-2) \dots B(j), j \leq n-1 \right]$$

Substituting $t = [S w]$ into equation (2.4), we obtain

$$y(t) = \sum_{r=1}^S \left[\prod_{k=w-1}^0 [I + A(k)] \left\{ \sum_{j=1}^{s-r} f_j((r-1)w+j-1) + f((r-1)w + w-1) \right\} \right] \quad (2.5)$$

where

$$\begin{aligned} f_1(\xi w) &= A(w-1) A(w-2) \dots A(1) f(\xi w) \\ f_2(\xi w+1) &= A(w-1) A(w-2) \dots A(2) f(\xi w+1) \\ &\dots \\ &\dots \\ f_{w-1}(\xi w+w-2) &= A(w-1) f(\xi w + w-2). \end{aligned}$$

Setting $B = \prod_{k=w-1}^0 [I + A(k)]$ in (2.5) we get

$$y(t) = \sum_{r=1}^{\frac{t}{w}} B^{\frac{t-rw}{w}} \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f((r-1)w+w-1) \right\}.$$

The last equation can be written in the form

$$y(t) = -\frac{1}{2\pi i} \oint_{\gamma} \sum_{r=1}^{\frac{t}{w}} \lambda^{\frac{t-rw}{w}} (B-\lambda I)^{-1} \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f((r-1)w+w-1) \right\} \quad (2.6)$$

where γ is a contour which circumscribes all the specter of the operator B , [1].

It can be seen that if $f \in E_{\alpha}$, then $(B-\lambda I)^{-1} f \in E_{\alpha}$ for every $\lambda \in \gamma$. From equation (2.6) we obtain a necessary and sufficient criterion for the exponential growth of the solution with an index β . Let σ_B denote the specter of the operator B . Assume that $\lambda_0 \in \sigma_B$. Set $\alpha_0 = \frac{1}{w} \ln |\lambda_0|$. The following theorem holds:

THEOREM. If $f \in E_{\alpha}$, then the solution y of equation (2.1) belongs to E_{β} such that

$$\beta = \alpha, \text{ when } \alpha > \alpha_0$$

$$\beta > \alpha, \text{ when } \alpha = \alpha_0$$

$$\beta = \alpha_0, \text{ when } \alpha < \alpha_0.$$

PROOF. To prove the sufficiency, we consider the following three cases:

(1) If $\alpha > \frac{1}{w} \ln |\lambda|$ then $y(t)$ defined by (2.6) belongs to E_α .

(2) If $\alpha > \frac{1}{w} \ln |\lambda|$ then from (2.6) we obtain

$$\begin{aligned} \|y\| &\leq D \sum_{r=1}^{\frac{t}{w}} \exp\left(\frac{1}{w} \ln |\lambda| (t-rw)\right) \left\{ \sum_{j=1}^{w-1} \|f_j((r-1)w+j-1)\| \right. \\ &\quad \left. + \|f((r-1)w + w-1)\| \right\} \\ &< D_1 \exp(\alpha t) \cdot \frac{t}{w} + D_2 \exp(\alpha t) \cdot (w-2) \frac{t}{w} \\ &< D' \exp(\alpha t) \cdot t \end{aligned}$$

(where D, D_1, D_2 and D' are constants).

This means that $y \in E_\beta$ where $\beta > \alpha$.

(3) If $\alpha < \frac{1}{w} \ln |\lambda|$ and $\|f\| \leq c \exp(\alpha t)$, then from (2.6) we have

$$\begin{aligned} \|y\| &\leq C_1 \exp\left(\frac{1}{w} \ln |\lambda| \cdot t\right) \\ \text{and } y &\in E_{\frac{1}{w} \ln |\lambda|} \quad (\alpha < \frac{1}{w} \ln |\lambda|). \end{aligned}$$

We now prove the necessity:

If λ_0 is an eigenvalue and x_0 is an eigenvector for the operator B such that

$$Bx_0 = \lambda_0 x_0,$$

where x_0 is an element of Banach space such that $\|x_0\| = 1$, by taking

$f(t) = \exp(\alpha t) \cdot x_0$ equation (2.6) with

$$(B - \lambda I)^{-1} x_0 = \frac{x_0}{\lambda_0 - \lambda} \text{ becomes}$$

$$y(t) = \sum_{r=1}^{\frac{t}{w}} \exp\left(\frac{1}{w} \ln |\lambda_0| (t-rw)\right) \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f((r-1)w + w-1) \right\}. \quad (2.7)$$

Multiplying the last equation by $\exp(-\alpha_0 t)$, where $\alpha_0 = \frac{1}{w} \ln |\lambda_0|$, we have

$$y(t) \exp(-\alpha_0 t) = \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \exp(-\alpha_0 wr) \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f((r-1)w + w-1) \right\}$$

where $\theta = \arg \lambda$.

$$\begin{aligned} y(t) \exp(-\alpha_0 t) &= \exp\left(\frac{i\theta t}{w} + w(1 - \alpha_0) - 1\right) \sum_{r=1}^{\frac{t}{w}} \exp((\alpha - \alpha_0)wr) \cdot x_0 \\ &\quad + \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \sum_{j=1}^{w-1} \exp(-\alpha_0 wr) f_j((r-1)w + j-1) \end{aligned}$$

$$\begin{aligned}
&= \frac{\exp\left(\frac{i\theta t}{w} + (\alpha - \alpha_0)\right)}{\exp(\alpha - \alpha_0) - 1} [\exp(\alpha - \alpha_0)t - 1] x_0 \\
&+ \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \sum_{j=1}^{w-1} \exp(-\alpha_0 wr) \cdot f_j((r-1)w + j-1)
\end{aligned} \quad (2.8)$$

Now for the last relation we have the following cases:

1) If $\alpha > \alpha_0$ then by using formula (2.8) we get

$$\lim_{t \rightarrow \infty} y(t) \exp(-\alpha_0 t) = \infty.$$

This means that $y \notin E_{\alpha_0}$ but $y \in E_{\alpha}$ ($\alpha > \alpha_0$).

2) If $\alpha = \alpha_0$ then from (2.8)

$$\begin{aligned}
y(t) \exp(-\alpha_0 t) &= \exp(w(1 - \alpha) - 1 + \frac{i\theta t}{w}) \left(\frac{t}{w} - 1\right) x_0 \\
&+ \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \sum_{j=1}^{w-1} \exp(-\alpha_0 wr) f_j((r-1)w + j-1).
\end{aligned}$$

Using the last equation we get

$$\lim_{t \rightarrow \infty} y(t) \exp(-\alpha_0 t) = \infty.$$

This means that $y \in E_{\alpha}$ but $y \notin E_{\beta}$ ($\beta > \alpha$).

3) If $\alpha = \alpha_0$ then from (2.8) we have $y \notin E_{\alpha}$ but $y \in E_{\alpha_0}$.

This completes the proof.

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