

ANALYTIC REPRESENTATION OF THE DISTRIBUTIONAL FINITE HANKEL TRANSFORM

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ABSTRACT. Various representations of finite Hankel transforms of generalized functions are obtained. One of the representations is shown to be the limit of a certain family of regular generalized functions and this limit is interpreted as a process of truncation for the generalized functions (distributions). An inversion theorem for the generalized finite Hankel transform is established (in the distributional sense) which gives a Fourier-Bessel series representation of generalized functions.

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1. INTRODUCTION.

Zemanian [1] extended Hankel transformations to the distribution space H'_μ . H'_μ is the dual of the space of testing functions H_μ defined as follows: for each real number μ , let

$$H_\mu = \{\phi : (0, \infty) \rightarrow \mathbb{C} \mid \phi \text{ is smooth on } (0, \infty) \text{ and } \phi \text{ satisfies (1.1)}\}$$

$$y_{m,k}^\mu(\phi) = \sup_{0 < x < \infty} |x^m (x^{-1} D)^k [x^{-\mu - \frac{1}{2}} \phi(x)]| < \infty, \text{ for each } m, k = 0, 1, 2, \dots \quad (1.1)$$

H'_μ consists of certain distributions of slow growth. Then later [2] he obtained a more general result by removing the restriction on the slow growth of the distributions. He defined the Hankel transformation of a distribution of rapid growth in the space B'_μ . B'_μ is the dual of B_μ , the strict inductive limit of the testing

function spaces $B_{\mu,b}$ (defined in section 2) as b tends to infinity through a monotonically increasing sequence of positive numbers.

We take advantage of the fact that functions in $B_{\mu,b}$ are identically zero after b , to define the finite Hankel transformation for the generalized functions in its dual $B'_{\mu,b}$. This is done by generalizing Parseval's equation. We find that for $\mu \geq -\frac{1}{2}$, the finite Hankel transform h_{μ} maps $B'_{\mu,b}$ isomorphically onto the generalized function space $Y'_{\mu,b}$ (defined in section 3). The aim of the present paper is to obtain various representations of the generalized functions in $Y'_{\mu,b}$ and to find an inversion formula for the generalized finite Hankel transform which also gives another representation of the members of $B'_{\mu,b}$ as a Fourier-Bessel series.

We follow the notation and terminology of Schwartz [3] and Zemanian [4,5]. Here I denotes the open interval $(0, \infty)$. The letters x, y, t and w are used as real variables on I . The k^{th} derivative of an ordinary or generalized function $f(x)$ is usually denoted by $D^k f(x)$ (though the symbol $D_x^k f(x)$ is also used). $D(I)$ denotes the space of smooth functions that have compact support on I . The topology of $D(I)$ is that which makes its dual the space $D'(I)$ of Schwartz's distributions [3, vol. I, p.65].

2. TESTING FUNCTION SPACES $B_{\mu,b}$ AND $Y_{\mu,b}$.

Let $b > 0$ be a fixed arbitrary real number. Then for $\mu \in \mathbb{R}$, where \mathbb{R} is the set of real numbers, we define

$$B_{\mu,b} = \{\phi : I \rightarrow \mathbb{C} \mid z(x) \text{ is smooth, } \phi(x) \equiv 0 \text{ for } x > b \text{ and } \phi \text{ satisfies (2.1)}\}$$

$$r_k^{\mu}(\phi) = \sup_{0 < x < \infty} |(x^{-1} D)^k [x^{-\mu-\frac{1}{2}} \phi(x)]| < \infty, \text{ for each } k = 0, 1, 2, \dots \quad (2.1)$$

$B_{\mu,b}$ is a linear space to which we assign the topology generated by the countable set of seminorms r_k^{μ} . $B_{\mu,b}$ is a sequentially complete countably multi-normed space [2].

Classically, for $\mu + \frac{1}{2} \geq 0$, the finite Hankel transform of a testing function ϕ in $B_{\mu,b}$ is defined as

$$\phi(\lambda_n) = \int_0^b \phi(x) \sqrt{\lambda_n} x J_{\mu}(\lambda_n x) dx, \quad n=1,2,3,\dots, \quad (2.2a)$$

where as usual J_{μ} denotes the Bessel function of the first kind of order μ and λ_n ($n = 1, 2, 3, \dots$) are positive roots of $J_{\mu}(bz) = 0$ (arranged in ascending order of magnitude). However, $\phi(\lambda_n)$ can be extended to the analytic function of the complex variable $z = y+iw$ by

$$\phi(z) = \int_0^b \phi(x) \sqrt{zx} J_{\mu}(xz) dx. \quad (2.2)$$

Note that $\phi(z)$ is an analytic function on the finite z -plane except for a branch point at $z = 0$ [4, p. 145]. Henceforth, the finite Hankel transform of a testing function ϕ in $B_{\mu,b}$ shall be defined as the analytic function $\phi(z)$ given in (2.2) and denoted by $h_{\mu}\phi = \phi$.

For a given real number $b > 0$, $Y_{\mu,b}$ is the space of functions $\phi(z)$ which satisfy:

$z^{-\mu-\frac{1}{2}}\phi(z)$ is an even entire function of z and for each non-negative integer k , the quantity

$$a_{b,k}^{\mu}(\phi) = \sup_z |e^{-b|z|} z^{2k-(\mu+\frac{1}{2})} \phi(z)| \quad (2.3)$$

is finite. The topology of $Y_{\mu,b}$ is the one generated by using the $a_{b,k}^{\mu}$, $k = 0,1,2,\dots$, as seminorms. $Y_{\mu,b}$ is a sequentially complete countably normed space. For further properties of these spaces one can look into Zemanian [4, 2].

For a given testing function ϕ in $Y_{\mu,b}$, consider the function

$$\phi(x) = \int_0^\infty \phi(y) \sqrt{xy} J_{\mu}(xy) dy = h_{\mu}^{-1}[\phi]. \quad (2.4)$$

Then for $\mu \geq -\frac{1}{2}$, Zemanian [2, Theorem 1] has proved:

Theorem 2.1. For $\mu \geq -\frac{1}{2}$, h_{μ} is an isomorphism from $B_{\mu,b}$ onto $Y_{\mu,b}$.

Here isomorphism means topological isomorphism. Henceforth, the symbol ϕ shall be used to denote a testing function in $Y_{\mu,b}$ whose pre-image is a testing function ϕ in $B_{\mu,b}$.

For a given ϕ in $Y_{\mu,b}$, the classical inverse of the finite Hankel transform (2.2a) is a Fourier-Bessel series of the form, [6,7],

$$\frac{2}{b^2} \sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}} \frac{J_{\mu}(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} \phi(\lambda_n). \quad (2.5)$$

Since ϕ satisfies (2.3), we have

$$|\phi(\lambda_n)| \leq \frac{A_{k\mu}}{\lambda_n^{2k-(\mu+\frac{1}{2})}}, \quad k = 0,1,2,\dots, \quad n = 1,2,3,\dots,$$

where $A_{k\mu}$ is constant. Also $(x/\lambda_n)^{\frac{1}{2}} [J_{\mu}(x\lambda_n)/J_{\mu+1}^2(b\lambda_n)]$ is smooth and bounded on $0 < \lambda_n x < \infty$, for $\mu \geq -\frac{1}{2}$. Consequently, the Fourier-Bessel series (2.5) converges absolutely and uniformly in x for all $x > 0$. Let us write

$$\psi(x) = \frac{2}{b^2} \sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}} \frac{J_{\mu}(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} \phi(\lambda_n),$$

then

$$|(x^{-1}D)^k x^{-\mu-\frac{1}{2}} \psi(x)| = \frac{2}{b^2} \left| \sum_{n=1}^{\infty} (x/\lambda_n)^{-\mu-k} \frac{J_{\mu+k}(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} \lambda_n^{2k+\mu+\frac{1}{2}} \phi(\lambda_n) \right|. \quad (2.6)$$

Since $\phi(\lambda_n)$ is of rapid descent as $\lambda_n \rightarrow \infty$ and $(x\lambda_n)^{-\mu-k} J_{\mu+k}(x\lambda_n)$ is smooth and bounded on $(0,\infty)$ for $\mu \geq -\frac{1}{2}$, it follows that the right-hand side of (2.6) converges absolutely and uniformly for all $x > 0$ and for every $k = 0,1,2,\dots$. Hence the left-hand side is continuous and bounded on $0 < x < \infty$ for each $k = 0,1,2,\dots$. Hence

$$\gamma_k^{\mu}(\psi) < \infty, \quad k = 0,1,2,\dots$$

$$\text{Moreover, } (x^{-1}D)^k (x^{-\mu-\frac{1}{2}} \psi(x)) = x^{-\mu-\frac{1}{2}} [a_{k0} \frac{\psi}{x^{2k}} + a_{k1} \frac{D\psi}{x^{2k-1}} + \dots + a_{kk} \frac{D^k \psi}{x^k}],$$

where the a_{kj} denote constants and $D = \frac{d}{dx}$. So we see that the Fourier-Bessel series defines an infinitely differentiable function $\psi(x)$ satisfying $\gamma_k^{\mu}(\psi) < \infty$ for each $k = 0,1,2,\dots$. But ψ may not be in $B_{\mu,b}$ as ψ may not be zero for $x > b$. But

$$\phi(x) = \lim_{\epsilon \rightarrow 0^+} \lambda_{\epsilon}(x) \psi(x) \in B_{\mu,b},$$

where $\lambda_\varepsilon(x)$ is defined as:

$$\lambda_\varepsilon(x) = \begin{cases} E(x/2\varepsilon), & 0 < x < 2\varepsilon \\ 1, & 2\varepsilon \leq x \leq b-2\varepsilon \\ 1-E(\frac{x-b+2\varepsilon}{2\varepsilon}), & b-2\varepsilon < x < b, \\ 0, & x \geq b, \end{cases} \quad (2.7)$$

for $0 < \varepsilon < b/4$, and

$$E(u) = \frac{\int_0^u \exp(1/x(x-1)) dx}{\int_0^1 \exp(1/x(x-1)) dx}. \quad (2.8)$$

Note that $\lambda_\varepsilon(x)$ is a multiplier in $B_{\mu,b}$ for each $0 < \varepsilon < b/4$ since, for any $\phi \in B_{\mu,b}$, we have

$$\gamma_k^u(\lambda_\varepsilon \phi) \leq \sum_{n=0}^k \binom{k}{n} \gamma_n^u(\phi) \sup_{0 < x < b} |(x^{-1}D)^{k-n} \lambda_\varepsilon(x)|.$$

Now pick X such that $0 < X < 2\varepsilon$. Then

$$\sup_{X < x < b} |(x^{-1}D)^m \lambda_\varepsilon(x)| < \infty,$$

and

$$\sup_{0 < x < X} |(x^{-1}D)^m \lambda_\varepsilon(x)| \leq A \sup_{0 < x < X} |(x^{-1}D)^{m-1} [x^{-1} \exp(\frac{4\varepsilon^2}{x(2\varepsilon-x)})]|,$$

where A is a constant.

So we see that $\gamma_k^u(\lambda_\varepsilon \phi) < \infty$. Also $\lambda_\varepsilon(x)$ is smooth on $(0, \infty)$. Hence $\lambda_\varepsilon \phi \in B_{\mu,b}$. It is easy to see that

$$\lim_{\varepsilon \rightarrow 0^+} \lambda_\varepsilon(x) \phi(x) = \phi(x), \text{ for any } \phi \text{ in } B_{\mu,b}.$$

3. GENERALIZED FUNCTION SPACES $B'_{\mu,b}$ AND $Y'_{\mu,b}$.

The spaces $B'_{\mu,b}$ and $Y'_{\mu,b}$ are the dual spaces of $B_{\mu,b}$ and $Y_{\mu,b}$ respectively. We shall use only the weak topology of $B'_{\mu,b}$, that is, the topology assigned to it by the seminorms

$$\rho_\phi(f) = |\langle f, \phi \rangle|, \quad \phi \in B_{\mu,b}, \quad f \in B'_{\mu,b}.$$

Since $B_{\mu,b}$ is a sequentially complete countably normed space, $B'_{\mu,b}$ is also sequentially complete [4, Theorem 1.83]. Similarly, we equip $Y'_{\mu,b}$ with the weak topology generated by the seminorms $\zeta_\phi(F) = |\langle F, \phi \rangle|$, $\phi \in Y_{\mu,b}$, $F \in Y'_{\mu,b}$.

$Y'_{\mu,b}$ is a sequentially complete space.

We now construct a generalized function in $B'_{\mu,b}$ which is not in $D'(I)$. Let $\{\tau_n\}$ be a monotone increasing sequence of positive numbers with limit $b+1$. For every $\phi \in B_{\mu,b}$, the formula

$$\langle f, \phi \rangle = \sum_{n=1}^{\infty} \phi(\tau_n) \quad (3.1)$$

is easily seen to define a generalized function f in $B'_{\mu,b}$. On the other hand, if ϕ is an arbitrary testing function in $D(I)$, then $\sum_n \phi(\tau_n)$ is in general an

infinite sum and it need not be convergent.

Note that

(i) $B'_{\mu,b}$ contains every regular distribution that corresponds to a function which is Lebesgue integrable on $0 < x < b$. In this case we have

$$\langle f, \phi \rangle = \int_0^b f(x) \phi(x) dx, \phi \in B'_{\mu,b}.$$

(ii) If f is a tempered distribution whose support is contained in $[X, \infty)$ for some $X > 0$, then $f \in B'_{\mu,b}$.

(iii) Similarly, every regular distribution F , which can be defined by a locally integrable function $F(y)$ through the equation

$$\langle F, \phi \rangle = \int_0^\infty F(y) \phi(y) dy$$

for every ϕ in $Y'_{\mu,b}$, belongs to $Y'_{\mu,b}$. Note that $F(y)$ need not be integrable over $0 < y < \infty$, though typically it would be of slow growth, i.e., for some integer $N > 0$, $y^{-N} F(y) \rightarrow 0$ as $y \rightarrow \infty$.

4. FINITE HANKEL TRANSFORMATION OF $B'_{\mu,b}$.

Henceforth, we assume that $\mu \geq -\frac{1}{2}$. For $f \in B'_{\mu,b}$, $\phi \in B'_{\mu,b}$ and $\phi = h_\mu \phi \in Y'_{\mu,b}$, we define the finite Hankel transform $F = h_\mu f$ by

$$\langle F, \phi \rangle = \langle f, \phi \rangle. \quad (4.1)$$

The above equation also defines the inverse Hankel transform $f = h_\mu^{-1} F$. From Theorem 2.1 one readily obtains:

Theorem 4.1. h_μ is an isomorphism from $B'_{\mu,b}$ onto $Y'_{\mu,b}$.

Example 1. The finite Hankel transform of the delta function $\delta(x-k)$ is given by the equation (4.1):

$$\begin{aligned} \langle h_\mu \delta(x-k), \phi(z) \rangle &= \langle \delta(x-k), \phi(x) \rangle, \text{ for } 0 < k < b, \\ &= \langle \delta(x-k), \int_0^\infty \phi(\sqrt{xy}) J_\mu(xy) dy \rangle, \text{ (using (2.4))} \\ &= \int_0^\infty \phi(y) \sqrt{ky} J_\mu(ky) dy < \infty. \end{aligned}$$

This defines a regular distribution $F(z) = (kz)^{\frac{1}{2}} J_\mu(kz)$ in $Y'_{\mu,b}$. Consequently, $h_\mu \delta(x-k) = (kz)^{\frac{1}{2}} J_\mu(kz)$, for $0 < k < b$.

Example 2. The finite Hankel transform of $\delta(x-k)$ for $k > b$ is the "Zero" generalized function in $Y'_{\mu,b}$, since $\langle \delta(x-k), \phi(x) \rangle = \phi(k) = 0$ for all ϕ in $B'_{\mu,b}$.

Example 3. The finite Hankel transform of the generalized function in $B'_{\mu,b}$ defined by (3.1) is the generalized function defined by the series

$$F = \sum_{n=1}^{\infty} \sqrt{y\tau_n} J_\mu(y\tau_n)$$

since, for $\phi \in Y'_{\mu,b}$,

$$\begin{aligned} \langle F, \phi \rangle &= \sum_{n=1}^{\infty} \int_0^\infty \phi(y) \sqrt{\tau_n y} J_\mu(\tau_n y) dy \\ &= \sum_{n=1}^{\infty} \phi(\tau_n) = \langle f, \phi \rangle \end{aligned}$$

Example 4. Suppose f is a regular generalized function, corresponding to a Lebesgue integrable function over $(0, b)$, in $B'_{\mu,b}$. Then the ordinary finite Hankel transform of f is given by

$$\int_0^b f(x) \sqrt{\lambda_n x} J_\mu(\lambda_n x) dx; n = 1, 2, 3, \dots.$$

Since f is integrable over $(0, b)$, its finite Hankel transform may be extended to the analytic function

$$F(z) = \int_0^b f(x) \sqrt{zx} J_\mu(zx) dx.$$

We show that $F = h_\mu(f)$. Since f is a regular generalized function,

$$\langle h_\mu f, \phi \rangle = \langle f, \phi \rangle$$

$$= \int_0^b f(x) \phi(x) dx$$

$$= \int_0^b f(x) \left(\int_0^\infty \phi(y) \sqrt{xy} J_\mu(xy) dy \right) dx \quad (\text{using (2.4)}).$$

Since the integrand $f(x)\phi(y) (xy)^{\frac{1}{2}} J_\mu(xy)$ is absolutely integrable over the domain $0 < x < b$, $0 < y < \infty$, the order of integration may be changed, and we obtain

$$\begin{aligned} \langle h_\mu f, \phi \rangle &= \int_0^\infty dy \phi(y) \int_0^b dx f(x) (xy)^{\frac{1}{2}} J_\mu(xy) \\ &= \langle F, \phi \rangle. \end{aligned}$$

Note that $F(\lambda_n) = \int_0^b f(x) (\lambda_n x)^{\frac{1}{2}} J_\mu(\lambda_n x) dx$, gives that $|F(\lambda_n)|$ is bounded.

Hence, for any ψ in $Y_{\mu, b}$, equation (2.3) ensures that the series $\sum_{n=1}^{\mu} F(\lambda_n) \phi(\lambda_n)$ converges. Furthermore, if a sequence $\{\phi_m\}$ converges in $Y_{\mu, b}$ then the sequence of numbers $\sum_{n=1}^{\infty} F(\lambda_n) \phi_m(\lambda_n)$ also converges. Hence, the sum $\sum_{n=1}^{\infty} F(\lambda_n) \phi(\lambda_n)$ defines a continuous linear functional on $Y_{\mu, b}'$.

Next we investigate a representation for the finite Hankel transform of a generalized function in $B_{\mu, b}'$. Let $D(0, b)$ be the space of infinitely differentiable functions on $(0, b)$ with compact support contained in $(0, b)$. The topology of $D(0, b)$ is that which makes its dual $D'(0, b)$ of Schwartz's distribution. Then $D(0, b) \subset B_{\mu, b}$ and h_μ maps $D(0, b)$ into a subspace of $Y_{\mu, b}$. Let W be the subspace of $Y_{\mu, b}$ onto which $D(0, b)$ is mapped. Then we have

Theorem 4.2. For any generalized function f in $B_{\mu, b}'$, there exists a continuous function $F(y)$ of slow growth such that the finite Hankel transform $h_\mu f$ of f restricted to W is equivalent, in the functional sense, to the regular generalized function F in $Y_{\mu, b}'$.

Proof. For a given generalized function f , there exists an integer $r \geq 0$ and a continuous function $h(x)$ [5, Theorem 3.4.2] such that

$$\langle f, \phi \rangle = \langle D^r h, \phi \rangle, \text{ for every } \phi \text{ in } D(0, b).$$

We take $h = 0$ outside $(0, b)$. Then using (2.4), we have, for $\phi \in D(0, b)$,

$$\langle F, \phi \rangle = \langle h_\mu f, h_\mu \phi \rangle = \langle f, \phi \rangle = \langle D^r h, \phi \rangle$$

$$= (-1)^r \langle h(x), D^r \phi(x) \rangle$$

$$\begin{aligned}
 &= (-1)^r \int_0^b dx h(x) D^r \phi(x) \\
 &= (-1)^r \int_0^b dx h(x) \int_0^\infty dy \phi(y) \frac{\partial^r}{\partial x^r} (\sqrt{xy} J_\mu(xy)) \quad (4.2)
 \end{aligned}$$

$$\begin{aligned}
 &= (-1)^r \int_0^b dx h(x) \int_0^\infty dy \phi(y) \sum_{i=0}^r [(-1)^i a_i(\mu) y^i x^{i-r} \sqrt{xy} J_{\mu+i}(xy)] \\
 &= \sum_{i=0}^r (-1)^{i+r} a_i(\mu) \int_0^b dx x^{i-r} h(x) [\int_0^\infty dy \phi(y) y^i \sqrt{xy} J_{\mu+i}(xy)] \quad (4.3)
 \end{aligned}$$

where $a_i(\mu)$ is a constant depending on μ , for each i . If $g_{i-r}(x) = x^{i-r} h(x)$, then $g_{i-r}(x)$ is continuous on $(0, \infty)$ and $g_{i-r}(x) = 0$ outside $(0, b)$. Since $\phi(y)$ is of rapid descent and $(xy)^{\frac{1}{2}} J_{\mu+i}(xy)$ is bounded on $0 < xy < \infty$, the order of integration in (4.3) may be interchanged. Therefore, (4.3) becomes

$$\langle F, \phi \rangle = \sum_{i=0}^r (-1)^{i+r} a_i(\mu) \int_0^\infty dy \phi(y) y^i h_{\mu+i}[g_{i-r}(x)].$$

Denote $h_{\mu+i}[g_{i-r}(x)]$ by $G_{i-r}^{\mu+i}(y)$, then for $\phi \in W$,

$$\langle F, \phi \rangle = \langle \sum_{i=0}^r (-1)^{i+r} a_i(\mu) y^i G_{i-r}^{\mu+i}(y), \phi(y) \rangle. \quad (4.4)$$

Clearly, the continuous function

$$G_{i-r}^{\mu+i}(y) = \int_0^b g_{i-r}(x) (xy)^{\frac{1}{2}} J_{\mu+i}(xy) dx,$$

may be extended to an analytic function. Equation (4.4) gives

$$h_\mu f \mid W = \sum_{i=0}^r (-1)^{i+r} a_i(\mu) y^i G_{i-r}^{\mu+i}(y) = F(y).$$

Finally since, $|G_{i-r}^{\mu+i}(y)| < \infty$, it is obvious that $F(y)$ is of slow growth.

Example 5. From Example 1, we know that $h_\mu \delta(x-k) = (kz)^{\frac{1}{2}} J_\mu(kz)$, for $0 < k < b$.

On the other hand, if we define

$$h(x) = \begin{cases} 0, & \text{for } x \geq k, \\ x-k, & \text{for } x < k, \end{cases}$$

we obtain from Theorem 4.2

$$h_\mu \delta(x-k) = F(y) = a_0(\mu) G_{-2}^\mu(y) - a_1(\mu) y G_{-1}^{\mu+1}(y) + a_2(\mu) y^2 G_0^{\mu+2}(y).$$

It is easily seen that

$$a_0(\mu) = \mu^2 - \frac{1}{4}, \quad a_1(\mu) = 2(\mu+1), \quad \text{and} \quad a_2(\mu) = 1.$$

Also

$$G_{-2}^\mu(y) = \int_k^b \left(\frac{1}{x} - \frac{k}{x^2}\right) (xy)^{\frac{1}{2}} J_\mu(xy) dx,$$

$$G_{-1}^{\mu+1}(y) = \int_k^b \left(1 - \frac{k}{x}\right) (xy)^{\frac{1}{2}} J_{\mu+1}(xy) dx,$$

and

$$G_0^{\mu+2}(y) = \int_k^b (x-k) (xy)^{\frac{1}{2}} J_{\mu+2}(xy) dx.$$

Hence (4.4) gives

$$F(y) = (yk)^{\frac{1}{2}} J_{\mu}(yk) - \frac{1}{2}(1 + \frac{k}{b})(yb)^{\frac{1}{2}} J_{\mu}(yb) + (1 - \frac{k}{b})y(yb)^{\frac{1}{2}} J'_{\mu}(yb).$$

This is another representation of $h_{\mu} \delta(x-k)$. It can be shown that this representation is equivalent to the one given in example 1, since

$$\begin{aligned} < -\frac{1}{2}(1 + \frac{k}{b})(yb)^{\frac{1}{2}} J_{\mu}(yb), \phi > &= -\frac{1}{2}(1 + \frac{k}{b}) \int_0^{\infty} \phi(y) (yb)^{\frac{1}{2}} J_{\mu}(yb) dy, \phi \in Y_{\mu,b} \\ &= -\frac{1}{2}(1 + \frac{k}{b}) \phi(b) = 0. \end{aligned}$$

Note. $\phi(b) = 0$ follows from the continuity property.

$$\text{Thus } (1 - \frac{k}{b}) < y(yb)^{\frac{1}{2}} J'_{\mu}(yb), \phi > = ((1 - \frac{k}{b}) \int_0^{\infty} y \phi(y) (yb)^{\frac{1}{2}} J'_{\mu}(yb) dy).$$

From (2.4) we have

$$\phi'(x) = \frac{1}{2x} \int_0^{\infty} \phi(y) (xy)^{\frac{1}{2}} J_{\mu}(xy) dy + \int_0^{\infty} y \phi(y) (xy)^{\frac{1}{2}} J'_{\mu}(xy) dy.$$

Therefore,

$$\phi'(b) = \frac{1}{2b} \int_0^{\infty} \phi(y) (by)^{\frac{1}{2}} J_{\mu}(by) dy + \int_0^{\infty} y \phi(y) (by)^{\frac{1}{2}} J'_{\mu}(by) dy.$$

So we have

$$(1 - \frac{k}{b}) < y(by)^{\frac{1}{2}} J'_{\mu}(by), \phi > = (1 - \frac{k}{b}) [\phi'(b) - \frac{1}{2b} \phi(b)] = 0.$$

Hence

$$< F, \phi > = < (yk)^{\frac{1}{2}} J_{\mu}(yk), \phi(y) >.$$

Thus we get the same result as derived in Example 1.

Example 6. We have shown in Example 2 that $h_{\mu} \delta(x-k) = 0$ for $k > b$. This also follows from Theorem 4.2. Take $r = 2$ and define $h(x) = 0$ for $x \leq k$ and $h(x) = x-k$ for $x > k$. It is easily seen that $G_{-2}^{\mu}(y) = G_{-1}^{\mu+1}(y) = G_0^{\mu+2}(y) = 0$, hence $F(y) = 0$.

Corollary 4.3. For any generalized function f in $B'_{\mu,b}$ with r , $h(x)$ and $F(y)$ defined as in the proof of Theorem 4.2, we have

$$F(y) = < D^r h(x), \lambda_{\epsilon}(x) (xy)^{\frac{1}{2}} J_{\mu}(xy) >, \text{ as } \epsilon \rightarrow 0^+. \quad (4.5)$$

Proof. Theorem 4.2 gives

$$\begin{aligned} h_{\mu} f|_W &= F(y) = \sum_{i=0}^r (-1)^{i+r} a_i(\mu) y^i G_{i-r}^{\mu}(y), \text{ for } f \in B'_{\mu,b} \\ &= (-1)^r \int_0^b dx \sum_{i=0}^r (-1)^i a_i(\mu) x^{-r} (xy)^i (xy)^{\frac{1}{2}} J_{\mu+i}(xy) \\ &= (-1)^r \int_0^b \frac{\partial^r}{\partial x^r} [\sqrt{xy} J_{\mu}(xy)] h(x) dx \\ &= (-1)^r \int_0^b \lambda_{\epsilon}(x) \frac{\partial^r}{\partial x^r} [\sqrt{xy} J_{\mu}(xy)] h(x) dx \text{ as } \epsilon \rightarrow 0^+ \\ &= (-1)^r < h(x), \lambda_{\epsilon} \frac{\partial^r}{\partial x^r} [\sqrt{xy} J_{\mu}(xy)] > \text{ as } \epsilon \rightarrow 0^+. \end{aligned}$$

But since $\lambda_{\epsilon}(x) = 1$ on $(0, b)$ as $\epsilon \rightarrow 0^+$ (and $h(x) = 0$ outside $(0, b)$), the order of differentiation may be interchanged in the preceding equation to give

$$\begin{aligned} F(y) &= (-1)^r < h(x), D_x^r [\lambda_{\epsilon}(x) (xy)^{\frac{1}{2}} J_{\mu}(xy)] > \text{ as } \epsilon \rightarrow 0^+, \\ &= < D_x^r h(x), \lambda_{\epsilon}(x) (xy)^{\frac{1}{2}} J_{\mu}(xy) >, \text{ as } \epsilon \rightarrow 0^+, \end{aligned}$$

(from the definition of distributional differentiation).

Example 7. While calculating the finite Hankel transform of $\delta(x-k)$, $0 < k < b$ (Example 5) using the method of Theorem 4.2, it was necessary to evaluate certain integrals to find $F(y)$. This may be avoided by using the above Corollary. From the definition of $h(x)$, we see that

$$D h(x) = \begin{cases} 0, & 0 < x \leq k, \\ 1, & x > k, \end{cases}$$

and

$$D^2 h(x) = \delta(x-k).$$

Hence, (4.7) gives

$$\begin{aligned} F(y) &= \lim_{\epsilon \rightarrow 0^+} \langle \delta(x-k), \lambda_\epsilon(x) (xy)^{\frac{1}{2}} J_\mu(xy) \rangle \\ &= \lim_{\epsilon \rightarrow 0^+} \lambda_\epsilon(k) (ky)^{\frac{1}{2}} J_\mu(ky) \\ &= (ky)^{\frac{1}{2}} J_\mu(ky), \text{ since } \lambda_\epsilon(k) = 1 \text{ as } \epsilon \rightarrow 0^+. \end{aligned}$$

5. SOME STRUCTURE THEOREMS

In this section we shall obtain representations for members of $B_{\mu,b}$, $B'_{\mu,b}$ and $Y'_{\mu,b}$ under suitable conditions. Note that the structure formula given by Theorem 4.2 is valid only when $F \in Y'_{\mu,b}$ is restricted to W , a subspace of $Y_{\mu,b}$. Here we will obtain a more general result, viz., a structure formula shall be established for $F \in Y'_{\mu,b}$ restricted to a larger subspace than W of $Y_{\mu,b}$. This section is very similar to section 3.4 of Zemanian [5, p. 86-93], consequently the corresponding results will be stated without proof or, perhaps, with only an indication of the proof. To begin with, we define certain spaces associated with $B_{\mu,b}$.

Definition 5.1. We define the spaces $B_{\mu,b}^0$, $C_{\mu,b}^0$ and $B_{\mu,b}^{(1)}$ by

$$B_{\mu,b}^0 = \{ \phi \in B_{\mu,b} : \phi = o(x^{\mu+\frac{1}{2}}) \text{ as } x \rightarrow 0^+ \}, \quad (5.1)$$

$$\begin{aligned} C_{\mu,b}^0 &= \{ g : (0, \infty) \rightarrow \mathbb{C} \mid g \text{ is continuous on } (0, \infty), g = 0 \text{ for } x > b \\ g(x) &= o(x^{\mu+\frac{1}{2}}) \text{ as } x \rightarrow 0^+ \}, \end{aligned} \quad (5.2)$$

and

$$B_{\mu,b}^{(1)} = \{ \phi \in B_{\mu,b} : \phi' = o(x^{\mu+3/2}) \text{ as } x \rightarrow 0^+ \}. \quad (5.3)$$

$B_{\mu,b}^0$ carries the natural topology induced on it by $B_{\mu,b}$.

Note that

$$\phi \in B_{\mu,b} \rightarrow \phi = o(x^{\mu+\frac{1}{2}}), \text{ as } x \rightarrow 0^+.$$

This is true because $\gamma_0^\mu(\phi) = \sup_{0 < x < b} |x^{-\mu-\frac{1}{2}} \phi(x)| < \infty$.

We prove an interesting property of functions in $B_{\mu,b}^0$ in

Lemma 5.2. Let $\phi \in B_{\mu,b}^0$. Then $\phi = o(x^{\mu+5/2})$ as $x \rightarrow 0^+$. (5.4)

Proof. Let $\phi \in B_{\mu,b}^0$. Now

$$(t^{-1} D_t)[t^{-\mu-\frac{1}{2}} \phi(t)] = t^{-\mu-\frac{1}{2}} \left[\frac{\phi'}{t} - (\mu + \frac{1}{2}) \frac{\phi}{t^2} \right]. \quad (5.5)$$

Write $n(t) = \frac{\phi'}{t} - (\mu + \frac{1}{2}) \frac{\phi}{t^2}$. Clearly $n(t)$ is a smooth function on $(0, \infty)$ and $n(t) = 0$, for $t > b$. Also, $\gamma_k^\mu(r_i) = \gamma_{k+1}^\mu(\phi) < \infty$, for each $k = 0, 1, 2, \dots$.

Hence, $n(t) \in B_{\mu, b}$. Therefore, $n(t) = o(t^{\mu+\frac{1}{2}})$, as $t \rightarrow 0^+$.

Hence

$$\frac{d}{dt} (t^{-\mu-\frac{1}{2}} \phi(t)) = o(t), \quad \text{as } t \rightarrow 0^+. \quad (5.6)$$

Integrating (5.6), we obtain

$$t^{-\mu-\frac{1}{2}} \phi(t) = o(t^2), \quad \text{as } t \rightarrow 0^+, \quad (5.7)$$

proving Lemma 5.2.

We assign a norm to the space $C_{\mu, b}^0$ by

$$\|j\|_0 = \sup_{x \in (0, b)} |x^{-\mu-\frac{1}{2}} g(x)|, \quad \text{for each } g \in C_{\mu, b}^0. \quad (5.8)$$

Thus $C_{\mu, b}^0$ becomes a topological vector space. We need the following lemma which is stated without proof since the proof is identical to the proof of [5, Lemma 1, p. 88].

Lemma 5.3. $B_{\mu, b}^0$ is a dense subset of $C_{\mu, b}^0$.

The following proposition gives an integral representation for the functions in $B_{\mu, b}^0$.

Proposition 5.4. Let $\phi \in B_{\mu, b}^0$. Then ϕ satisfies the integral equation

$$\phi(x) = \int_0^b u(x, t) (t^{-1} D_t^2) (t^{-\mu-\frac{1}{2}} \phi(t)) dt, \quad (5.9)$$

where

$$u(x, t) = x^{\mu+\frac{1}{2}} u^*(x, t), \quad (5.10)$$

and

$$u^*(x, t) = \begin{cases} \frac{x^3}{2b^2} t(t^2 - b^2), & \text{for } 0 < x \leq t \leq b, \\ \frac{t^3}{2b^2} x(x^2 - b^2), & \text{for } 0 < t \leq x \leq b, \\ 0 & \text{, elsewhere,} \end{cases} \quad (5.11)$$

for $0 < x < \infty$, $0 < t < \infty$.

Proof. Trivial.

Next we prove that generalized functions in $B_{\mu, b}'$ are distributional derivatives of certain continuous functions.

We start with the following boundedness property of $f \in B_{\mu, b}'$.

For each $f \in B_{\mu, b}'$, there exists a non-negative integer r and a positive constant A such that for all $\phi \in B_{\mu, b}$,

$$|\langle f, \phi \rangle| \leq A \max_{0 \leq k \leq r} \gamma_k^\mu(\phi) = o_r^\mu(\phi) \quad (\text{say}). \quad (5.12)$$

Suppose $f \in B_{\mu, b}'$ is such that (5.12) is satisfied with $r = 0$. Then

$$|\langle f, \phi \rangle| \leq A \sup_{0 < x < b} |x^{-\mu-\frac{1}{2}} \phi(x)|. \quad (5.13)$$

We now extend f , satisfying the inequality (5.13), continuously and uniquely onto the space $C_{\mu, b}^0$. Let g in $C_{\mu, b}^0$ be arbitrary. Then by Lemma 5.3, there exists a sequence $\{\phi_n\}$ of testing functions in $B_{\mu, b}^0$ such that ϕ_n converges to g in

$C_{\mu,b}^0$. We define $\langle f, g \rangle$ by

$$\langle f, g \rangle = \lim_{n \rightarrow \infty} \langle f, \phi_n \rangle. \quad (5.14)$$

This defines a continuous linear functional on $C_{\mu,b}^0$ satisfying the inequality (5.13).

Clearly $u(x,t) \in C_{\mu,b}^0$, hence the following definition makes sense for $f \in B_{\mu,b}^1$ satisfying (5.13).

Definition 5.5. For $f \in B_{\mu,b}^1$ satisfying the inequality (5.13), define

$$h(t) = \langle f(x), u(x,t) \rangle. \quad (5.15)$$

Note: (i) $h(t) = 0$ for $t \geq b$, as $u(x,t) = 0$ for $t \geq b$

$$(ii) |h(t) - h(\tau)| \leq 3Ab^2|t^{-1}|, 0 < t \leq b, 0 < \tau \leq b. \quad (5.16)$$

Lemma 5.6. For $\phi \in B_{\mu,b}$, $(\frac{1}{t} D_t)^2 [t^{-\mu-\frac{1}{2}} \phi(t)]$ is uniformly continuous on $(0,b]$.

Proof. Let

$$n(t) = (\frac{1}{t} \frac{d}{dt})^2 [t^{-\mu-\frac{1}{2}} \phi(t)], \phi \in B_{\mu,b}^0.$$

Then, $n(t) = 0(1)$ as $t \rightarrow 0^+$, and $|n'(t)| < \infty$, proving the Lemma 5.6.

Theorem 5.7. For $f \in B_{\mu,b}^1$ satisfying the condition

$$|\langle f, \phi \rangle| \leq A \sup_{0 < x < b} |x^{-\mu-\frac{1}{2}} \phi(x)|, \forall \phi \in B_{\mu,b}^0,$$

we have

$$\langle f, \phi \rangle = \int_0^b h(t) (t^{-1} D_t)^2 [t^{-\mu-\frac{1}{2}} \phi(t)] dt \quad (5.17)$$

for every $\phi \in B_{\mu,b}^0$. Here, $h(t)$ is the continuous function defined by equation

(5.15). If $D_t [t^{-1} D_t (t^{-1} h(t))]$ is Lebesgue integrable over $(0,b)$, (5.17) can be written as

$$\langle f, g \rangle = \langle t^{-\mu-\frac{1}{2}} D_t (t^{-1} D_t (t^{-1} h(t))), \phi(t) \rangle, \quad (5.18)$$

for every $\phi \in B_{\mu,b}^0$.

Proof. The proof of (5.17) is very similar to the proof of [5, equation (9), p. 90-91] and (5.18) follows easily from (5.17).

We now generalize Theorem 5.7 for the case when $|\langle f, \phi \rangle| \leq \rho_r^\mu(\phi)$, $r > 0$. For this we need the following:

Definition 5.8. For each non-negative integer n , we define the spaces $B_{\mu,b}^{(n)}$, $H_{\mu,b}^{(n)}$, $B_{\mu,b}^\infty$, and $H_{\mu,b}^\infty$ by

$$B_{\mu,b}^{(n)} = \{\phi \in B_{\mu,b} : \phi^{(n)}(x) = o(x^{\mu+3/2}) \text{ as } x \rightarrow 0^+\}, \quad (5.19)$$

$$H_{\mu,b}^{(n)} = \{n(x) = \frac{\phi'(x)}{(\mu+\frac{1}{2})x} - \frac{\phi(x)}{x^2} : \phi \in B_{\mu,b}^{(n)}\}, \quad (5.20)$$

where $\phi^{(n)}(x) = D^n \phi(x)$ for $n \neq 0$ and $\phi^{(0)}(x) = \phi(x)$,

$$B_{\mu,b}^\infty = \{\phi \in B_{\mu,b} : \phi^{(k)}(x) = o(x^{\mu+\frac{1}{2}}), \text{ as } x \rightarrow 0^+, \text{ for each } k = 0, 1, 2, \dots\}, \quad (5.21)$$

and

$$H_{\mu,b}^\infty = \{n(x) = \frac{\phi'(x)}{(\mu+\frac{1}{2})x} - \frac{\phi(x)}{x^2} : \phi \in B_{\mu,b}^\infty\}. \quad (5.22)$$

Note that $H_{\mu,b}^{(1)} \subset B_{\mu,b}^0$ and $B_{\mu,b}^0 = B_{\mu,b}^{(0)}$. Let $\phi_0 \in B_{\mu,b}^0$ be such that

$$\int_0^b t^{-\mu+\frac{1}{2}} \phi_0(t) dt = 1. \quad (5.23)$$

In the subsequent development we shall need the following lemma, whose proof is omitted since it is similar to that of [5, Lemma 1, p.68].

Lemma 5.9. Let ϕ_0 be a fixed testing function in $B_{\mu,b}^0$ satisfying (5.23). Then any testing function ϕ in $B_{\mu,b}^0$ may be decomposed uniquely according to

$$\phi = K\phi_0 + \eta \quad (5.24)$$

where η is in $H_{\mu,b}^{(1)}$ and the constant K is given by

$$K = \int_0^b t^{-\mu+\frac{1}{2}} \phi(t) dt. \quad (5.25)$$

Suppose f is a regular generalized function in $B_{\mu,b}'$ generated by a differentiable function f such that f is Lebesgue integrable over $(0,b)$ and f' is bounded on $(0,b]$. Then for $\eta \in H_{\mu,b}^{(1)}$ we have

$$\begin{aligned} \langle f, \eta \rangle &= \int_0^b f(x) \eta(x) dx \\ &= \frac{1}{\mu+\frac{1}{2}} \int_0^b x^{\mu-\frac{1}{2}} f(x) \frac{d}{dx} [x^{-\mu-\frac{1}{2}} \phi(x)] dx \quad (\text{for some } \phi \in B_{\mu,b}^{(1)}) \\ &= -\frac{1}{\mu+\frac{1}{2}} \int_0^b \frac{(x^{\mu-\frac{1}{2}} f(x))'}{x^{\mu+\frac{1}{2}}} \phi(x) dx, \end{aligned}$$

since $\phi(b) = 0$ and $\frac{\phi(x)}{x} = o(x^{\mu+3/2})$ as $x \rightarrow 0+$.

Therefore

$$\langle f, \eta \rangle = \frac{1}{\mu+\frac{1}{2}} \langle x^{-\mu-\frac{1}{2}} D x^{\mu-\frac{1}{2}} f(x), -\phi(x) \rangle. \quad (5.26)$$

But

$$\eta = \frac{x^{\mu-\frac{1}{2}}}{\mu+\frac{1}{2}} D x^{\mu-\frac{1}{2}} \phi(x),$$

therefore,

$$\langle f(x), x^{\mu-\frac{1}{2}} D x^{\mu-\frac{1}{2}} \phi(x) \rangle = \langle x^{-\mu-\frac{1}{2}} D x^{\mu-\frac{1}{2}} f(x), -\phi(x) \rangle. \quad (5.27)$$

Let

$$L_{\mu} = x^{\mu-\frac{1}{2}} D x^{\mu-\frac{1}{2}}, \quad (5.28)$$

and

$$T_{\mu} = x^{-\mu-\frac{1}{2}} D x^{\mu-\frac{1}{2}} = I_{\mu}. \quad (5.29)$$

Then for any $\phi \in B_{\mu,b}^{(1)}$,

$$L_{\mu} \phi = x^{\mu+\frac{1}{2}} (x^{-1} D) (x^{-\mu-\frac{1}{2}} \phi(x)) \in B_{\mu,b}^0,$$

and

$$\gamma_k^{\mu} (L_{\mu} \phi) = \gamma_{k+1}^{\mu} (\phi).$$

We write the above as

Lemma 5.10. The operation $\phi \mapsto L_{\mu} \phi$ is a continuous linear mapping of $B_{\mu,b}^{(1)}$ into $B_{\mu,b}^0$.

For an arbitrary f in $B_{\mu,b}'$ and any ϕ in $B_{\mu,b}^{(1)}$, set

$$\langle T_\mu f, \phi \rangle = \langle x^{-\mu-\frac{1}{2}} D x^{\mu-\frac{1}{2}} f, \phi \rangle = \langle f, -L_\mu \phi \rangle. \quad (5.30)$$

From (5.30) we see that $T_\mu f$ is a linear functional on $B_{\mu,b}^{(1)}$. Write

$$x^{-\mu-\frac{1}{2}} D x^{\mu-\frac{1}{2}} f = g.$$

We then have formally,

$$D x^{\mu-\frac{1}{2}} f = x^{\mu+\frac{1}{2}} g.$$

Now define,

$$x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} g]^{(-1)} = f. \quad (5.31)$$

With this notation (5.26) suggests that

$$\langle x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}, n \rangle = \frac{1}{\mu+\frac{1}{2}} \langle f, -\phi \rangle. \quad (5.32)$$

Equation (5.32) defines a linear functional on $H_{\mu,b}^{(1)}$. This can be extended to all of $B_{\mu,b}^0$ by using Lemma 5.9. Assign

$$\langle x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}, \phi_0 \rangle = K_0 \text{ (arbitrary).} \quad (5.33)$$

Then for any $\phi \in B_{\mu,b}^0$, we have from Lemma 5.9,

$$\phi = K\phi_0 + n.$$

Hence the operator $x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}$ is defined for all of $B_{\mu,b}^0$.

Proposition 5.11. Suppose $f \in B_{\mu,b}'$ satisfies

$$|\langle f, \phi \rangle| \leq A \sup_{0 \leq k \leq r} \gamma_k^\mu (\phi), \quad r \geq 1 \quad (5.34)$$

for all ϕ in $B_{\mu,b}^0$. Then

$$|\langle x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}, \phi(x) \rangle| \leq A \sup_{0 \leq k \leq r-1} \gamma_k^\mu (\phi), \quad (5.35)$$

for all $\phi \in B_{\mu,b}^0$.

Proof. Using (5.32), the proof follows trivially for members of $H_{\mu,b}^{(1)}$. Now use Lemma 5.9 to complete the proof.

Proposition 5.12. For $f \in B_{\mu,b}'$ satisfying

$$|\langle f, \phi \rangle| \leq A \sup_{0 \leq k \leq 1} \gamma_k^\mu (\phi), \quad \forall \phi \in B_{\mu,b}^0,$$

we have

$$\langle f, \phi \rangle = - \int_0^b h(t) (t^{-1} D_t)^3 (t^{-\mu-\frac{1}{2}} \phi(t)) dt, \quad (5.36)$$

for each ϕ in $B_{\mu,b}^{(1)}$ and where h is a continuous function.

Proof. We know that the theorem is true for $r = 0$. For $\phi \in B_{\mu,b}^{(1)}$,

$$\langle f, \phi \rangle = -(\mu+\frac{1}{2}) \langle x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}, n \rangle, \quad n \in H_{\mu,b}^{(1)} \text{ (equation (5.32))}$$

$$= -(\mu+\frac{1}{2}) \int_0^b h(t) (t^{-1} D_t)^2 (t^{-\mu-\frac{1}{2}} \eta(t)) dt,$$

by Theorem (5.7), since

and $| \langle x^{-\mu+\frac{1}{2}} [x^{\mu+\frac{1}{2}} f]^{(-1)}, \eta \rangle | \leq A \gamma_0^\mu (\eta)$, (see Proposition (5.11))

$$\eta(x) = o(x^{\mu+\frac{1}{2}}), \text{ as } x \rightarrow 0^+.$$

Therefore,

$$\begin{aligned} \langle f, \phi \rangle &= -(\mu+\frac{1}{2}) \int_0^b h(t) (t^{-1} D_t)^2 [t^{-\mu-\frac{1}{2}} \cdot \frac{t^{\mu+\frac{1}{2}}}{\mu+\frac{1}{2}} \frac{1}{t} \frac{d}{dt} (t^{-\mu-\frac{1}{2}} \phi(t))] dt \\ &= - \int_0^b h(t) (t^{-1} D_t)^3 (t^{-\mu-\frac{1}{2}} \phi(t)) dt. \end{aligned}$$

If we replace both $B_{\mu,b}^0$ and $B_{\mu,b}^{(1)}$ by $B_{\mu,b}^\infty$ and $H_{\mu,b}^{(1)}$ by $H_{\mu,b}^\infty$, Lemmas 5.9,

5.10 and Propositions 5.11, 5.12 are still true. Hence by induction on r , we get

Theorem 5.13. For $f \in B_{\mu,b}^r$ satisfying

$$|\langle f, \phi \rangle| \leq A \sup_{0 \leq k \leq r} \gamma_k^\mu (\phi), \quad (\forall \phi \in B_{\mu,b}^0)$$

for some non-negative integer r , we have

$$\langle f, \phi \rangle = (-1)^r \int_0^b h(t) (t^{-1} D_t)^{r+2} [t^{-\mu-\frac{1}{2}} \phi(t)] dt, \quad (5.37)$$

for each $\phi \in B_{\mu,b}^0$ where h is a continuous function.

The above structure theorem helps us to say more about the finite Hankel transform of elements in $B_{\mu,b}^r$.

Let $\phi \in B_{\mu,b}^\infty$ and F, ϕ be the finite Hankel transforms of $f \in B_{\mu,b}^r$ and ϕ respectively. Then

$$\langle F, \phi \rangle = \langle f, \phi \rangle$$

$$= (-1)^{r-2} \int_0^b h(x) \int_0^\infty (y)^{\frac{1}{2}} \phi(y) \left(\frac{1}{x} \frac{d}{dx} \right)^r (x^{-\mu} J_\mu(xy)) dy dx \quad (5.38)$$

(using equations (2.4) and (5.37)).

Since

$$(-1)^r \left(\frac{1}{x} \frac{d}{dx} \right)^r (x^{-\mu} J_\mu(xy)) = y^r x^{-\mu-r} J_{\mu+r}(xy), \quad (5.39)$$

then

$$\begin{aligned} \langle F, \phi \rangle &= \int_0^b h(x) \int_0^\infty (y)^{\frac{1}{2}} \phi(y) y^r x^{-\mu-r} J_{\mu+r}(xy) dy dx \\ &= \int_0^\infty y^r \phi(y) \int_0^b \frac{h(x)}{x^{\mu+\frac{1}{2}+r}} \sqrt{xy} J_{\mu+r}(xy) dx dy. \end{aligned} \quad (5.40)$$

Now let $g_r(x) = \frac{h(x)}{x^{\mu+\frac{1}{2}+r}}$.

Since

$$\frac{\sqrt{xy} J_{\mu+r}(xy)}{x^{\mu+\frac{1}{2}+r}} \sim \frac{(xy)^{\mu+r+\frac{1}{2}}}{x^{\mu+\frac{1}{2}+r}} \text{ as } x \rightarrow 0^+,$$

so that

$$G_r^{\mu+r}(y) = \int_0^b g_r(x) (xy)^{\frac{1}{2}} J_{\mu+r}(xy) dx$$

is well defined.

Thus (5.40) becomes

$$\begin{aligned} \langle F, \phi \rangle &= \int_0^\infty y^r \phi(y) G_r^{\mu+r}(y) dy \\ &= \langle y^r G_r^{\mu+r}(y), \phi(y) \rangle \end{aligned}$$

Consequently,

$$F = y^r G_r^{\mu+r}(y) \quad (\text{in the functional sense})$$

is of slow growth since $|G_r^{\mu+r}(y)| < \infty$. We list the above as

Theorem 5.14. For any $F \in Y_{\mu,b}'$, there exists a continuous function $G(y)$ ($= y^r G_r^{\mu+r}(y)$) of slow growth such that $F|_{Y_{\mu,b}^\infty}$ is equivalent to $G(y)$, i.e., $\langle F, \phi \rangle = \langle G(y), \phi(y) \rangle$, for $\phi \in Y_{\mu,b}^\infty$, where $Y_{\mu,b}^\infty = h_\mu [B_{\mu,b}^\infty]$.

Furthermore, $G(y)$ may be extended to an analytic function.

6. FURTHER PROPERTIES OF THE FINITE HANKEL TRANSFORM

Notation: Let us write $\hat{f} = f|_{B_{\mu,b}^\infty}$, for any $f \in B_{\mu,b}'$, and likewise $\hat{F} = F|_{Y_{\mu,b}^\infty}$ for any $F \in Y_{\mu,b}'$. Also write $\hat{\phi}$ to denote the members of $B_{\mu,b}^\infty$ and $\hat{\Phi}$ for the members of $Y_{\mu,b}^\infty$.

For any \hat{f} , we have for some integer $r \geq 2$,

$$\langle \hat{f}, \hat{\phi} \rangle = (-1)^r \int_0^b h(x) \left(\frac{1}{x} \frac{d}{dx} \right)^r (x^{-\mu-\frac{1}{2}} \phi(x)) dx,$$

where $h(x)$ is a continuous function.

Definition 6.1. We define a new space $H_\mu^\infty(I)$ by

$$H_\mu^\infty(I) = \{\phi \in H_\mu : \phi^{(k)}(x) = o(x^{\mu+3/2}) \text{ as } x \rightarrow 0^+, \text{ for } k = 0, 1, 2, \dots\}, \quad (6.1)$$

where $I = (0, \infty)$.

Now define $h_b(x)$ to be the periodic extension of period b of $h(x)$ on $(0, b]$. Then for any $x \in \mathbb{R}^+$, $h_b(x) = h(x-nb)$ for some positive integer n such that $0 < x-nb \leq b$. Associated with $h_b(x)$ is the regular distribution in $H_\mu^\infty(I)$ having the value

$$\langle h_b(x), \phi(x) \rangle = \sum_{n=0}^{\infty} \int_0^b h(x) \phi(x+nb) dx, \quad (6.2)$$

for any $\phi \in H_\mu^\infty(I)$.

The right-hand side of (6.2) converges, since ϕ is of rapid descent as $x \rightarrow \infty$. Now define a functional f_b on $H_\mu^\infty(I)$ by

$$\begin{aligned}
 \langle f_b, \phi \rangle &= (-1)^r \int_0^\infty h_b(x) \left(\frac{1}{x} D_x\right)^r (x^{-\mu-\frac{1}{2}} \phi(x)) dx \\
 &= \sum_{n=0}^{\infty} (-1)^r \left[\int_0^\infty h(x-nb) \left(\frac{1}{x} D_x\right)^r (x^{-\mu-\frac{1}{2}} \phi(x)) dx \right] \\
 &= \sum_{n=0}^{\infty} (-1)^r \int_{nb}^{(n+1)b} h(x-nb) \left(\frac{1}{x} D_x\right)^r (x^{-\mu-\frac{1}{2}} \phi(x)) dx. \quad (6.3)
 \end{aligned}$$

This defines a linear continuous functional on $H_\mu^\infty(I)$. Also for $\hat{\phi}$ in $B_{\mu,b}^\infty$, we have

$$\begin{aligned}
 \langle f_b, \hat{\phi} \rangle &= (-1)^r \int_0^b h(x) \left(\frac{1}{x} D_x\right)^r (x^{-\mu-\frac{1}{2}} \phi(x)) dx, \quad (\text{since } \hat{\phi} = 0 \text{ for } x > b) \\
 &= \langle \hat{f}, \hat{\phi} \rangle.
 \end{aligned}$$

So we see that \hat{f}_b is a periodic extension of f .

Theorem 6.2. Every \hat{f} in $B_{\mu,b}^r$ may be extended to a periodic linear functional, with period b , on $H_\mu^\infty(I)$ which is continuous in the topology of H_μ .

Theorem 6.3. For every ϵ in $(0, b/4)$, and each f in $B_{\mu,b}^r$, the function

$$F_\epsilon(y) = \langle f(x), \lambda_\epsilon(x) (xy)^{\frac{1}{2}} J_\mu(xy) \rangle, \quad (6.4)$$

where $\lambda_\epsilon(x)$ is defined by (2.7), is a smooth function of slow growth and defines a regular generalized function in $Y_{\mu,b}^r$.

Proof. The proof of the above theorem is similar to the proof of [1, Lemma 12] given by Zemanian.

Theorem 6.4. The finite Hankel transform, $h_\mu f$, of a generalized function f in $B_{\mu,b}^r$ is the limit, as $\epsilon \rightarrow 0$, of the family $F_\epsilon(z)$ of regular generalized functions defined in Theorem 6.3.

Proof. Since $F_\epsilon(z)$ is a regular functional in $Y_{\mu,b}^r$, it is sufficient to show that

$$\langle h_\mu f, \phi \rangle = \langle F_\epsilon, \phi \rangle$$

for each ϕ in $Y_{\mu,b}$, as $\epsilon \rightarrow 0$. For each ϕ in $Y_{\mu,b}$ there exists a unique ϕ

in $B_{\mu,b}$ given by Equation (2.4). As $\epsilon \rightarrow 0+$, $\lambda_\epsilon(x) = 1$ on $(0, b)$.

Now we have

$$\begin{aligned}
 \langle F_\epsilon, \phi \rangle &= \langle \langle f(x), \lambda_\epsilon(x) (xy)^{\frac{1}{2}} J_\mu(xy) \rangle, \phi(y) \rangle \\
 &= \int_0^\infty \langle f(x), \lambda_\epsilon(x) (xy)^{\frac{1}{2}} J_\mu(xy) \rangle \phi(y) dy \\
 &= \langle f(x), \lambda_\epsilon(x) \int_0^\infty \phi(y) (xy)^{\frac{1}{2}} J_\mu(xy) dy \rangle \quad (\text{by [8, Corollary 5.3.2b, p. 121]}) \\
 &= \langle f(x), \lambda_\epsilon(x) \phi(x) \rangle \\
 &\rightarrow \langle f(x), \phi(x) \rangle, \quad \text{as } \epsilon \rightarrow 0+.
 \end{aligned}$$

Consequently, $\langle F_\epsilon, \phi \rangle \rightarrow \langle f(x), \phi(x) \rangle = \langle h_\mu f, \phi \rangle$, as $\epsilon \rightarrow 0+$, for each $\phi \in Y_{\mu,b}$.

Since $\lambda_\epsilon(x)(xy)^{\frac{1}{2}}J_\mu(xy) \rightarrow x(0,b)(x)(xy)^{\frac{1}{2}}J_\mu(xz)$, as $\epsilon \rightarrow 0+$, (where $x(0,b)$ is

the characteristic function of the interval $(0,b)$) and the latter is not a testing function in $B'_{\mu,b}$, it is not true, in general, that the limit of $F_\epsilon(z)$, as $\epsilon \rightarrow 0$, exists as an ordinary function. Where the limit does exist as an ordinary function, it will be denoted by $F_0(z)$.

Corollary 6.5. If f is a regular generalized function in $B'_{\mu,b}$, then the limiting function $F_0(z)$ exists and is equivalent to the finite Hankel transform of f .

Proof. If f is regular, then for each ϵ in $(0, b/4)$,

$$F_\epsilon(z) = \int_0^b f(x) \lambda_\epsilon(x)(xy)^{\frac{1}{2}} J_\mu(xz) dx.$$

As $\epsilon \rightarrow 0^+$, we obtain

$$F_0(z) = \int_0^b f(x)(xy)^{\frac{1}{2}} J_\mu(xz) dx, \quad (6.5)$$

and from Example 4, we see that $F_0(z)$ is equivalent to $h_\mu f$.

Using the function $x(0,b)(x)$, (6.5) can be written as

$$F_0(z) = \langle x(0,b)(x)f(x), \lambda(x)(xz)^{\frac{1}{2}}J_\mu(xz) \rangle,$$

where $\lambda(x)$ is a testing function in $D(R)$ such that $\lambda(x) = 1$ on $(0,b]$. In this case, to calculate the finite Hankel transform we merely truncate the regular distribution in $B'_{\mu,b}$ at $x = b$, as would be expected. In a similar way one might interpret the limit as $\epsilon \rightarrow 0+$ of the family of the functions $F_\epsilon(z)$ as a process of truncation for distributions in general, for one is replacing $f(x)$ by the distributional limit $\lambda_\epsilon(x)f(x)$ as $\epsilon \rightarrow 0+$.

Corollary 6.6. If the generalized function $f \in B'_{\mu,b}$ has support in $(0,b]$, then the function $F_0(z)$ exists and is equivalent to the finite Hankel transform of f .

Proof. Let $\lambda(x)$ be a testing function in $D(I)$ such that $\lambda(x) = 1$ on a neighborhood of the support of f . Then

$$F_\epsilon(z) = \langle f(x), \lambda_\epsilon(x)\lambda(x)(xz)^{\frac{1}{2}}J_\mu(xz) \rangle$$

such that,

$$\lim_{\epsilon \rightarrow 0^+} F_\epsilon(z) = F_0(z) = \langle f(x), \lambda(x)(xz)^{\frac{1}{2}}J_\mu(xz) \rangle,$$

since $\lambda_\epsilon(x) = 1$ on the support of f as $\epsilon \rightarrow 0+$. But Zemanian [1, Theorem 2] has proved that

$$h_\mu f = \langle f(x), \lambda(x)(xz)^{\frac{1}{2}}J_\mu(xz) \rangle,$$

for every functional in H'_μ having compact support.

Example 8. For the distribution $\delta(x-k)$, from (6.4) we obtain

$$\begin{aligned} F_\epsilon(z) &= \langle \delta(x-k), \lambda_\epsilon(x)(xz)^{\frac{1}{2}}J_\mu(xz) \rangle \\ &= \lambda_\epsilon(k)(kz)^{\frac{1}{2}}J_\mu(kz). \end{aligned}$$

But for $0 < k < b$, $\lambda_\epsilon(k) = 1$ as $\epsilon \rightarrow 0+$. And for $k \geq b$, $\lambda_\epsilon(k) = 0$.

Therefore,

$$F_0(z) = \begin{cases} \sqrt{bz} J_\mu(bz), & \text{for } 0 < k < b \\ 0, & \text{for } k \geq b, \end{cases}$$

$$= h_\mu [\delta(x-b)].$$

7. THE FOURIER-BESSEL SERIES

Classically the inverse finite Hankel transform is considered to be the Fourier-Bessel series

$$\frac{2}{b^2} \sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} \cdot \phi(\lambda_n), \quad (7.1)$$

where

$$\phi(\lambda_n) = \int_0^b \phi(x) (x\lambda_n)^{\frac{1}{2}} J_\mu(x\lambda_n) dx, \quad (7.2)$$

is the finite Hankel transform [7] of some function ϕ .

In section 2, we showed that for any $\phi \in B_{\mu,b}$ and $\phi(\lambda_n)$ given by (7.2), we have

$$\phi(x) = \frac{2}{b^2} \sum_{n=1}^{\infty} \lambda_\epsilon(x) \left(\frac{x}{\lambda_n} \right)^{\frac{1}{2}} \cdot \frac{J_\mu(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} \phi(\lambda_n), \quad \text{as } \epsilon \rightarrow 0. \quad (7.3)$$

We obtain an inversion theorem similar to (7.1) for generalized finite Hankel transforms of members in $B'_{\mu,b}$.

Theorem 7.1 (inversion). Let f be an arbitrary generalized function in $B'_{\mu,b}$, where $\mu \geq -\frac{1}{2}$. Let $F = h_\mu(f)$, be the finite Hankel transform. Then in the sense of convergence in $B'_{\mu,b}$, we have

$$f(x) = \lim_{N \rightarrow \infty} \frac{2}{b^2} \sum_{n=1}^{\infty} \left(\frac{x}{\lambda_n} \right)^{\frac{1}{2}} [J_\mu(x\lambda_n)/J_{\mu+1}^2(b\lambda_n)] F(\lambda_n). \quad (7.4)$$

Proof. Let $\lambda(x)$ be an arbitrary testing function in $B_{\mu,b}$. We wish to prove that

$$< \frac{2}{b^2} \sum_{n=1}^N \left(\frac{x}{\lambda_n} \right)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} F(\lambda_n), \phi(x) > + < f(x), \phi(x) >, \text{ as } N \rightarrow \infty.$$

Since $(x)^{\frac{1}{2}} J_\mu(x\lambda_n)$ is locally integrable over $(0,b)$,

$$\begin{aligned} & < \frac{2}{b^2} \sum_{n=1}^N \left(\frac{x}{\lambda_n} \right)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} F(\lambda_n), \phi(x) > \\ & = \int_0^b \frac{2}{b^2} \sum_{n=1}^N \frac{F(\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} \left(\frac{x}{\lambda_n} \right)^{\frac{1}{2}} \cdot J_\mu(x\lambda_n) \phi(x) dx \\ & = \sum_{n=1}^N \frac{2}{b^2} \frac{F(\lambda_n)}{\lambda_n J_{\mu+1}^2(b\lambda_n)} \phi(\lambda_n), \quad (\text{from (2.2)}) \end{aligned}$$

$$= \sum_{n=1}^N \frac{2}{b^2} \frac{\phi(\lambda_n)}{\lambda_n J_{\mu+1}^2(b\lambda_n)} \lim_{\epsilon \rightarrow 0^+} \langle f(x), \lambda_\epsilon(x)(x\lambda_n)^{\frac{1}{2}} J_\mu(x\lambda_n) \rangle$$

(from Theorem 6.4)

$$\rightarrow \langle f(x), \phi(x) \rangle, \text{ as } N \rightarrow \infty.$$

We verify this inversion Theorem by means of a numerical example.

Example 9. For $0 < k < b$, $\delta(x-k) \in E'(I) \subset H_\mu^1 \subset B_{\mu,b}^1$. The finite Hankel transform of $\delta(k-x)$ is,

$$h_\mu \delta(x-k) = (kz)^{\frac{1}{2}} J_\mu(kz) = F(z).$$

Hence,

$$F(\lambda_n) = (k\lambda_n)^{\frac{1}{2}} J_\mu(k\lambda_n), \quad n = 1, 2, 3, \dots$$

Now

$$\begin{aligned} & \langle \frac{2}{b^2} \sum_{n=1}^N \left(\frac{x}{\lambda_n}\right)^{\frac{1}{2}} \frac{J_\mu(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)} (k\lambda_n)^{\frac{1}{2}} J_\mu(k\lambda_n), \phi(x) \rangle \\ &= \frac{2}{b^2} \sum_{n=1}^N \frac{J_\mu(k\lambda_n)}{(\lambda_n)^{\frac{1}{2}} J_{\mu+1}^2(b\lambda_n)} (k)^{\frac{1}{2}} \int_0^b (x\lambda_n)^{\frac{1}{2}} J_\mu(x\lambda_n) \phi(x) dx \\ &= \frac{2}{b^2} \sum_{n=1}^N \frac{J_\mu(k\lambda_n)}{(\lambda_n)^{\frac{1}{2}} J_{\mu+1}^2(b\lambda_n)} (k)^{\frac{1}{2}} \phi(\lambda_n) \\ &= \phi(k) \\ &= \langle \delta(x-k), \phi(x) \rangle. \end{aligned}$$

This also yields

$$\delta(x-k) = \lim_{N \rightarrow \infty} \frac{2}{b^2} \sum_{n=1}^N \frac{J_\mu(k\lambda_n) J_\mu(x\lambda_n)}{J_{\mu+1}^2(b\lambda_n)}$$

in the sense of convergence in $B_{\mu,b}^1$.

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