

ON THE HADAMARD PRODUCTS OF SCHLICHT FUNCTIONS AND APPLICATIONS

H.S. AL-AMIRI

Department of Mathematics
Bowling Green State University
Bowling Green, Ohio 43403

(Received April 25, 1984)

ABSTRACT. We show that each of the schlicht classes of starlike, convex, close-to-convex and strongly starlike with respect to symmetric points is invariant under the Hadamard product with the class of convex functions. The influence of certain operators over these classes is also investigated.

KEY WORDS AND PHRASES. Hadamard products, schlicht functions, and starlike, convex, close-to convex and strongly starlike with respect to symmetric points.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. Primary 30C45

1. INTRODUCTION.

Let A be the class of analytic functions on the unit disk $U = \{z: |z| < 1\}$. By S we denote those functions in A which are univalent and normalized by $f(0) = f'(0) - 1 = 0$. Let C , S^* and K be the usual subclasses of S which are consisting of the convex, starlike and close-to-convex functions, respectively.

The Hadamard product (convolution) of two power series $f(z) = \sum_0^{\infty} a_n z^n$, $g(z) = \sum_0^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_0^{\infty} a_n b_n z^n$. Pólya and Schoenberg [9] have conjectured that $f * g$ is in C whenever both f and g are in C . This conjecture was finally shown to be true by Ruscheweyh and Sheil-Small [12].

THEOREM A (Ruscheweyh and Sheil-Small): Let h be in C . If f is in C, S^* or K then $h * f$ is in C, S^* or K respectively.

The proof of Theorem A requires a key lemma that turned out to be beautiful in its simplicity and powerful in its applicability to various problems. We state this key lemma of Ruscheweyh and Sheil-Small in the form stated and proved by Barnard and Kellogg in [1].

LEMMA A. Let ϕ and g be analytic in $|z| < 1$ with $\phi(0) = g(0) = 0$ and $\phi'(0)g'(0) \neq 0$. Suppose that for each $\alpha (|\alpha| = 1)$ and $\sigma (|\sigma| = 1)$ we have

$$[\phi * \left(\frac{1 + \alpha \sigma z}{1 - \sigma z} \right) g](z) \neq 0 \quad (1.1)$$

on $0 < |z| < r < 1$. Then for each F in A the image of $|z| < r$ under

$(\phi * Fg)/(\phi * g)$ is a subset of the convex hull of $F(U)$.

REMARK 1. In [12], it was shown that condition [1] is satisfied for all z in U whenever ϕ is in C and g is in S^* .

A function $f : A$ is said to be starlike with respect to symmetric points in U , denoted $f \in S_s^*$, if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0, \text{ for all } z \text{ in } U. \quad (1.2)$$

This class was introduced by Sakaguchi [13].

REMARK 2. Sakaguchi has shown that $C_s \subset S_s^*$. Further, if $f \in S_s^*$ then $(f(z) - f(-z))/2$ is in S^* .

We say that f is strongly starlike of order λ , with respect to symmetric points in U , denoted $f \in S_s^{*(\lambda)}$, if and only if

$$|\arg \{ \frac{zf'(z)}{f(z) - f(-z)} \}| < \frac{\lambda\pi}{2}, \quad 0 < \lambda < 1, \quad z \in U. \quad (1.3)$$

This class was introduced by Padmanabhan and Thangamani [8].

Das and Singh [4] introduced the classes of function f which are convex with respect to symmetric points in U , denoted $f \in C_s$, and those which are close-to-convex with respect to symmetric points, denoted $f \in K_s$ as follows: $f \in C_s$ if and only if

$$\operatorname{Re} \frac{z(zf'(z))'}{zf'(z) + zf'(-z)} > 0, \quad z \in U, \quad (1.4)$$

and $f \in K_s$ if and only if there exists $K \in S_s^*$ such that

$$\operatorname{Re} \frac{zf'(z)}{K(z) - K(-z)} > 0, \quad z \in U. \quad (1.5)$$

REMARK 3. In [4], it was shown that both C_s and K_s are subsets of S and furthermore, $f \in C_s$ if and only if $zf' \in S_s^*$.

The purpose of this note is to investigate these subclasses of S which are symmetric with respect to points in U via the Hadamard products. In section 2, we state our main theorem which shows that these classes are invariant under the Hadamard product with convex functions. In section 3, we use the results of section 2 and an observation made in [1] to obtain results concerning the radii of these classes under the influence of certain well known operators.

2. MAIN RESULTS

The elegant technique and results of Ruscheweyh and Sheil-Small allow us to produce a theorem similar to Theorem A for the classes S_s^* , C_s , K_s and $S_s^{*(\lambda)}$, $0 < \lambda < 1$.

THEOREM 1. Let $h \in C$. If f is in S_s^* , C_s , K_s or $S_s^{*(\lambda)}$, $0 < \lambda \leq 1$, then $h * f$ is in S_s^* , C_s , K_s or $S_s^{*(\lambda)}$ respectively.

PROOF. The proof of the four cases requires an application of Lemma A. In order to apply this lemma, we need to identify the convex function ϕ and the starlike function g for the different cases.

Case 1. Let $f \in S_s^*$ and $h \in C$. Then $g(z) = \frac{f(z) - f(-z)}{2} \in S^*$ by Remark 2. Thus

condition (1.1) is satisfied according to Remark 1, i.e.,

$$[h * \left(\frac{1+\alpha\sigma z}{1-\sigma z}\right)g](z) \neq 0$$

for all z in U . Now let

$$F(z) = \frac{z f'(z)}{f(z) - f(-z)} . \quad (1.6)$$

Applying Lemma A with F as given in (2.1) we obtain

$$\begin{aligned} \frac{h * Fg}{h * g}(z) &= \frac{h(z) * \frac{z f'(z)}{(f(z) - f(-z))} \cdot \frac{(f(z) - f(-z))}{2}}{h(z) * \frac{(f(z) - f(-z))}{2}} \\ &= \frac{h(z) * z f'(z)}{(h * f)(z) - (h * f)(-z)} = \frac{z(h * f)'(z)}{(h * f)(z) - (h * f)(-z)} \end{aligned}$$

which is contained in the convex hull of F in U . This implies

$$\operatorname{Re} \frac{z(h * f)'(z)}{(h * f)(z) - (h * f)(-z)} > 0, \quad z \in U,$$

or $h * f \in S_s^*$ according to (1.2).

CASE 2. Let $f \in C_s$ and $h \in C$. This case follows immediately from Remark 3 and Case 1 above.

CASE 3. Let $f \in K_s$ and $h \in C$. Let $K(z)$ be the associated function appearing in (1.5). Then according to Remark 2 $g(z) = \frac{K(z) - K(-z)}{2} \in S^*$. Thus condition (1.1) is satisfied for h and g of this case. Applying Lemma A with

$$F(z) = \frac{z f'(z)}{K(z) - K(-z)}$$

we easily get after some simplification as in Case 1 that

$$\frac{z(h * f)'(z)}{(h * K)(z) - (h * K)(-z)}$$

is contained in the convex hull of F . That is

$$\operatorname{Re} \frac{z(h * f)'(z)}{(h * K)(z) - (h * K)(-z)} > 0, \quad z \in U.$$

Thus $h * f \in K_s$ provided $h * K \in S_s^*$. However, the latter is true as seen by Case 1.

CASE 4. Let $f \in S_s^*(\lambda)$, $0 < \lambda \leq 1$, and $h \in C$. It is clear from (1.3) and (1.2) that $S_s^*(\lambda) \subset S_s^*$ for all $0 < \lambda \leq 1$. Let g and F as defined in Case 1, then, as in the proof of Case 1,

$$\frac{z(h * f)'(z)}{(h * f)(z) - (h * f)(-z)}$$

lies in the convex hull of F . This means

$$\left| \operatorname{arg} \frac{z(h * f)'(z)}{(h * f)(z) - (h * f)(-z)} \right| < \lambda\pi/2, \quad 0 < \lambda \leq 1.$$

Thus $h * f \in S_s^*(\lambda)$ by (1.3). This completes the proof of Theorem 1.

3. APPLICATIONS

For $X \subset A$, let $r_c(X)$ denote the largest positive number so that every $f \in X$

is convex in $|z| < r_c(X)$. Consequently, the conclusion of Theorem 1 remains valid for $h * f$ in the disk $|z| < r_c(X)$ for each $h \in X$ and every f in the given four classes. This observation was first used effectively by Barnard and Kellogg in [1] to unify and reduce the amount of work usually associated with such problems. Let

$$h_\gamma(z) = \sum_{n=1}^{\infty} \frac{\gamma + n}{\gamma + 1} z^n = \frac{\gamma}{1 + \gamma} \frac{z}{1 - z} + \frac{1}{1 + \gamma} \frac{z}{(1 - z)^2},$$

$$H(z) = \sum_{n=1}^{\infty} \frac{\gamma + 1}{\gamma + n} z^n, \text{ and}$$

$$P(z) = \sum_{n=1}^{\infty} \frac{1 - x^n}{(1 - x)^n} z^n = \frac{1}{1 - x} \log \frac{1 - xz}{1 - z},$$

where $\operatorname{Re}\gamma \geq 0$, $|x| \leq 1$, $x \neq 1$ and for all z in U . The operators B_j , $j = 1, 2, 3$, defined below have been studied by many authors. Let

$$B_1 f(z) = \frac{\gamma f(z) + z f'(z)}{1 + \gamma} \in (h_\gamma * f)(z), \operatorname{Re}\gamma > 0,$$

$$B_2 f(z) = \frac{1 + \gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt \in (H_\gamma * f)(z), \operatorname{Re}\gamma \geq 0,$$

$$B_3 f(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt} dt \in (P * f)(z), |x| \leq 1, x \neq 1.$$

For example, the operator B_1 is due to Livingston [7] when $\gamma = 1$, B_2 is due to Libera [5] when $\gamma = 1$ while B_3 was first used by Pommerenke in [10], see also [2], [3], [6], [11]. Consequently, the operators B_j , $j = 1, 2, 3$, preserve the classes S_s^* , C_s , K_s or $S_s^*(\lambda)$ up to $r_c(h_\gamma)$, $r_c(H_\gamma)$ or $r_c(P)$ as mentioned at the start of this section. However, $r_c(H_\gamma) = 1$, $\operatorname{Re}\gamma \geq 0$ was shown by Ruscheweyh [11], and $r_c(P) = 1$ was shown by Pommerenke [10]. We need only to calculate $r_c(h_\gamma)$. From the definition of h_γ we get

$$h_\gamma'(z) = \frac{1 + \frac{1 - \gamma}{1 + \gamma} z}{(1 - z)^3} = \frac{1 + cz}{(1 - z)^3}$$

where $c = \frac{1 - \gamma}{1 + \gamma}$. Thus

$$1 + \frac{z h_\gamma''(z)}{h_\gamma'(z)} = \frac{2 + z}{1 - z} - \frac{1}{1 + cz}.$$

This implies

$$\operatorname{Re}(1 + \frac{z h_\gamma''(z)}{h_\gamma'(z)}) \geq \frac{2 - r}{1 + r} - \frac{1}{1 - |c|r}, |z| = r.$$

This shows that $r_c(h_\gamma)$ is the smallest positive number r satisfying

$$|c|r^2 - 2(1 + |c|)r + 1 = 0$$

or

$$r_c(h_\gamma) = \frac{1}{1 + |c| + \sqrt{1 + |c| + |c|^2}}.$$

Finally, we have the following theorem.

THEOREM 2. If f is in S_s^* , C_s , K_s or $S_s^*(\lambda)$, $0 < \lambda \leq 1$, then

i) $B_j f$ is in S_s^* , C_s , K_s or $S_s^*(\lambda)$, respectively for each $j = 2, 3$, and

ii) $B_1 f$ is in S_s^* , C_s , K_s or $S_s^*(\lambda)$ respectively for

$$|z| < r_c(h_\gamma) = 1/(1 + |c| + \sqrt{1 + |c| + |c|^2}), \quad c = \frac{1 - \gamma}{1 + \gamma},$$

$$R \cdot \gamma \geq 0.$$

REMARK 4. Theorem 2 part (i) was shown in [8] for the class $S_s^*(\lambda)$ only. Part (i) with $\gamma = 1$ and $x = -1$ was shown in [4] for the classes S_s^* , C_s and K_s .

REFERENCES

1. BARNARD, R. W., KELLOGG, C., Applications of convolution operators to problems in univalent function theory, Michigan Math. J., 27 (1980), 81-94.
2. BARNARDI, S. D., Convex and Starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969), 429-446.
3. CAUSEY, W. M., The close to convexity and univalence of an integral, Math. Z. 99 (1967), 207-212.
4. DAS, R. N. and SINGH, P., On subclasses of schlicht mappings, Indian J. Pure Appl. Math. 8 (1977), 864-872.
5. LIBERA, R. J., Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1965), 775-758.
6. LIBERA, R. J. and LIVINGSTON, A. E., On the univalence of some classes of regular functions, Proc. Amer. Math. Soc. 30 (1971), 327-336.
7. LIVINGSTON, A. E., On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966), 352-357.
8. PADMANABHAN, K. S. and THANGAMANI, J., The effect of certain integral operators on some classes of starlike functions with respect to symmetric points, Bull. Math. de la Soc. Sci. Math. de la R.S. de Rawmanie 25 (1982), 355-360.
9. PÓLYA, G. and SCHOENBERG, I. J., Remarks on de la Vallée Poussin means and convex conformal maps of the circle, Pacific J. Math. 8 (1958), 295-334.
10. POMMERENKE, C., On close-to-convex analytic functions, Trans. Amer. Math. Soc. 114 (1965), 176-186.
11. RUSCHEWEYH, S., New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.
12. RUSCHEWEYH, S. and SHEIL-SMALL, T., Hadamard products of schlicht functions and the Polya-Schoenberg conjecture, Comment. Math. Helv. 48 (1973), 119-135.
13. SAKAGUCHI, K., On a certain univalent mapping, J. Math. Soc. Japan 11 (1959), 72-80.

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods:** artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields:** asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects:** decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site <http://www.hindawi.com/journals/jamds/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk