

NOTES ON WHITEHEAD SPACE OF AN ALGEBRA

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Let R be a ring, and denote by $[R, R]$ the group generated additively by the additive commutators of R . When $R_n = M_n(R)$ (the ring of $n \times n$ matrices over R), it is shown that $[R_n, R_n]$ is the kernel of the regular trace function modulo $[R, R]$. Then considering R as a simple left Artinian F -central algebra which is algebraic over F with $\text{Char } F = 0$, it is shown that R can decompose over $[R, R]$, as $R = Fx + [R, R]$, for a fixed element $x \in R$. The space $R/[R, R]$ over F is known as the Whitehead space of R . When R is a semisimple central F -algebra, the dimension of its Whitehead space reveals the number of simple components of R . More precisely, we show that when R is algebraic over F and $\text{Char } F = 0$, then the number of simple components of R is greater than or equal to $\dim_F R/[R, R]$, and when R is finite dimensional over F or is locally finite over F in the case of $\text{Char } F = 0$, then the number of simple components of R is equal to $\dim_F R/[R, R]$.

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1. Introduction. Additive commutator elements of a ring R and the groups and structures they make have a great role in the general specification of a ring, and their study is one of the approaches to recognize rings in noncommutative ring theory [2, 3, 4, 5]. The reason is clear, they have covered the secrets of noncommutative behaviour of the structure. In recent years, these elements are returned once again under a full consideration, and a lot of wonderful works has been done on them [1, 10, 11, 12, 13]. Our study here is also among these studies, and it reveals some of bilateral relations between substructure given by additive commutators (the additive commutator group $[R, R]$, the additive Whitehead group, and the space $R/[R, R]$) and some characteristics of the ring. In what follows let R be a ring. By $[R, R]$ we denote the group generated additively by the additive commutators of R . Following [2], the additive group $R/[R, R]$ is called the additive Whitehead group of R . This group is an F -vector space when R is a central F -algebra, and is called the Whitehead space of R .

2. Results. Our first result is about the additive commutator subgroup of a matrix ring over a given ring.

PROPOSITION 2.1. *Let R be a unitary ring and let $R_n = M_n(R)$ be the ring of $n \times n$ matrices over R . Consider the regular trace function on R_n , as $\text{tr}: R_n \rightarrow R$, then*

$$[R_n, R_n] = \{A \in R_n \mid \text{tr}(A) \in [R, R]\}. \quad (2.1)$$

PROOF. The inclusion “ \subseteq ” follows by the fact that $\text{tr}(AB - BA) \in [R, R]$. In order to show the reverse inclusion, let $\{E_{ij}\}$ be the matrix units and note that if $i \neq j$, we have $E_{ij} = E_{ii}E_{ij} - E_{ij}E_{ii} \in [R_n, R_n]$ and $E_{ii} - E_{jj} = E_{ij}E_{ji} - E_{ji}E_{ij} \in [R_n, R_n]$. For any

$A = (a_{ij}) \in R_n$, we have the following congruence:

$$A = \sum a_{ij}E_{ij} \equiv \sum a_{ii}E_{ii} \equiv \sum a_{ii}E_{11} \pmod{[R_n, R_n]}. \quad (2.2)$$

In particular, if $\text{tr}(A) \in [R, R]$, then $A \in [R_n, R_n]$. \square

COROLLARY 2.2. *Consider the trace function on R_n module of $[R, R]$. Clearly the group isomorphism $R_n/[R_n, R_n] \cong R/[R, R]$ can be derived.*

THEOREM 2.3. *Let R be a left Artinian central simple F -algebra which is algebraic over F with $\text{Char}F = 0$. Then R decomposes over $[R, R]$ as $R = Rx + [R, R]$, for a fixed $x \in R$.*

PROOF. By Wedderburn-Artin theorem, $R = M_n(D)$ for a division ring D and suitable $n \in \mathbb{N}$ [6, 14]. We divide our proof into two parts.

(i) Let $n = 1$, in other words let $R = M_1(D) = D$ be a division ring. Let $a \in R$ and let $f(t) = t^r + b_1t^{r-1} + \cdots + b_r$ be the minimal polynomial of a over F , where $b_i \in F$, $i = 1, 2, \dots, r$ and $r = \dim_F F(a)$. By the Wedderburn theorem [9, page 265], $f(t)$ splits completely in $R[t]$, this means that there exists $c_i \in R^* = D - \{0\}$, $i = 1, 2, \dots, r-1$, such that $f(t) = (t-a)(t-c_1ac_1^{-1}) \cdots (t-c_{r-1}ac_{r-1}^{-1})$. Then we have

$$\begin{aligned} \text{Tr}_{F(a)/F}(a) &= a + c_1ac_1^{-1} + c_2ac_2^{-1} + \cdots + c_{r-1}ac_{r-1}^{-1} \\ &= ra + (c_1ac_1^{-1} - a) + \cdots + (c_{r-1}ac_{r-1}^{-1} - a) \\ &= ra + (c_1(ac_1^{-1}) - (ac_1^{-1})c_1) + \cdots + (c_{r-1}(ac_{r-1}^{-1}) - (ac_{r-1}^{-1})c_{r-1}) \\ &= ra + d_1 + d_2 + \cdots + d_{r-1} = ra + d, \end{aligned} \quad (2.3)$$

where $d_1, \dots, d_{r-1}, d \in [R, R]$. This simply yields $a \in F + [R, R]$ which imply that $R = F + [R, R]$, $x = 1$.

(ii) Let $n \in \mathbb{N}$ be an arbitrary positive integer. We have $R = M_n(D)$, where D is a division ring. By (i), $D = F + [D, D]$, so

$$R = M_n(D) = M_n(F + [D, D]) = M_n(F) + M_n([D, D]) \subseteq M_n(F) + [R, R] \subseteq R. \quad (2.4)$$

This implies that $R = M_n(F) + [R, R]$. By this formula, given $A \in R$, there exist $B \in M_n(F)$ and $C \in [R, R]$ such that $A = B + C$, hence $A = (B - (\text{tr}B/n)I) + (\text{tr}B/n)I + C$, where I is the identity matrix of size n . By [Proposition 2.1](#), $(B - (\text{tr}B/n)I) \in [R, R]$, and $A = (\text{tr}B/n)I + ((B - (\text{tr}B/n)I) + C)$, consequently

$$R = FI + [R, R], \quad x = I. \quad (2.5)$$

\square

To see a different statements and initial ideas of these theorems we refer the reader to [1, 2]. Also a multiplicative version of [Theorem 2.3](#) could be found in [11].

Now, we are going to state our main result, which is about the Whitehead space of a semisimple ring. This theorem is a generalization of a nice theorem due to R. Brauer [8, page 130].

THEOREM 2.4. *Let R be a left Artinian semisimple central F -algebra and let k be the number of left simple components of R . Then,*

- (i) *if R is algebraic over F and $\text{Char } F = 0$, then $k \geq \dim_F R/[R, R]$;*
- (ii) *if R is finite dimensional over F , or is locally finite over F , and $\text{Char } F = 0$, then $k = \dim_F R/[R, R]$.*

PROOF. Consider the following chain of functions:

$$R \xrightarrow{f_1} M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k) \xrightarrow{f_2} D_1/[D_1, D_1] \times \cdots \times D_k/[D_k, D_k], \quad (2.6)$$

where f_1 is the isomorphism given by the Wedderburn-Artin theorem for the decomposition of a semisimple left Artinian ring into a direct product of simple ring [6, 14], and f_2 is the F -algebra homomorphisms, by considering component-wise the trace function on $M_{n_i}(D_i) \text{ mod } [D_i, D_i]$, $i = 1, \dots, k$.

By [Proposition 2.1](#) we have, $\ker(f_2 \circ f_1) = [R, R]$, noting that $[R, R] \cong [R_1, R_1] \times \cdots \times [R_k, R_k]$, where $R_{n_i} = M_{n_i}(D_i)$, $i = 1, \dots, k$. Therefore the following F -isomorphism holds:

$$R/[R, R] \cong D_1/[D_1, D_1] \times \cdots \times D_k/[D_k, D_k]. \quad (2.7)$$

It remains to compute the dimension of Whitehead space of a division ring in the two cases (i) and (ii) above.

First let D be algebraic over F and $\text{Char } F = 0$. We show that any two elements $\bar{a}, \bar{b} \in D/[D, D]$ are linearly dependent. By [Theorem 2.3](#), there exist elements $\alpha, \beta \in F$ and $d_1, d_2 \in [D, D]$, such that $a = \alpha + d_1$ and $b = \beta + d_2$. In other words, $\beta\bar{a} - \alpha\bar{b} = \bar{0}$ in $D/[D, D]$. Hence in this case $\dim_F D/[D, D] \leq 1$.

Now let D be finite dimensional F -central algebra. Let $\text{RT}_{D/F} : D \rightarrow F$ be the reduced trace function which is surjective by [7, page 148]. Furthermore, by a theorem of Amitsur and Rowen [5, page 171] its kernel is equal to $[D, D]$ and so it is a hyperplane over F , in this case $\dim_F D/[D, D] = 1$.

As a latter case let D be a locally finite division ring over its center F and $\text{Char } F = 0$. Now consider the function $\text{TR} : D \rightarrow F$ defined by

$$\text{TR}(x) = \frac{1}{\deg_F(x)} \text{Tr}_{F(x)/F}(x), \quad (2.8)$$

we show that this function is an F -linear surjective map, whose kernel is $[D, D]$. The claim then is clear.

First note that in this case $1 \notin [D, D]$, for if $1 \in [D, D]$, then there exist some x_i 's and y_i 's in D , such that $1 = \sum(x_i y_i - y_i x_i)$. Let D_1 be the division ring generated by F together with x_i 's and y_i 's. Taking the reduced trace of D_1 over its centre of both sides of $1 = \sum(x_i y_i - y_i x_i)$, we get a contradicting result. Therefore $[D, D] \cap F = \{0\}$. Now, by considering the trace formula (given in the proof of [Theorem 2.3](#)) for elements a , b and $\lambda a + b$ ($\lambda \in F$) in D , it is readily verified that

$$\frac{1}{r} \text{Tr}(\lambda a + b) = \frac{\lambda}{n} \text{Tr}(a) + \frac{1}{m} \text{Tr}(b), \quad (2.9)$$

where r , n , and m are degrees of $\lambda a + b$, a and b . So TR is F -linear. The surjectivity is clear. In order to specify the kernel of TR , consider the trace formula for elements of $[D, D]$. Suppose that $a \in [D, D]$. Now, we have $\text{Tr}_{F(a)/F}(a) = na + d \in [D, D] \cap F$, where n is the degree of a over F and $d \in [D, D]$. Therefore $\text{TR}(a) = 0$. By the same argument we can see that if $\text{TR}(a) = 0$, then $a \in [D, D]$. \square

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