

ON A WEAK FORM OF WEAK QUASI-CONTINUITY

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A weak form of weak quasi-continuity, which we call subweak quasi-continuity, is introduced. It is shown that subweak quasi-continuity is strictly weaker than weak quasi-continuity. Subweak quasi-continuity is used to strengthen several results in the literature concerning weak quasi-continuity. Specifically, results concerning the graph, graph function, and restriction of a weakly quasi-continuous function are extended slightly. Also, a result concerning weakly quasi-continuous retractions is strengthened.

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1. Introduction. Weakly quasi-continuous functions were introduced by Popa and Stan [9]. Recently, weak quasi-continuity has been developed further by Noiri [5, 6] and Park and Ha [8]. Due to a result by Noiri [5], weak quasi-continuity is equivalent to the weak semicontinuity developed by Arya and Bhamini [1]. The purpose of this note is to introduce the concept of subweak quasi-continuity, which we define in terms of a base for the topology on the codomain. We establish that this condition is strictly weaker than weak quasi-continuity and we use it to strengthen some of the results in the literature concerning weak quasi-continuity. For example, we show that the graph of a subweakly quasi-continuous function with a Hausdorff codomain is semiclosed. We also show that, if the graph function is subweakly quasi-continuous with respect to the usual base for the product space, then the function itself is weakly quasi-continuous, and that, if a function is subweakly quasi-continuous with respect to the base \mathcal{B} , then the restriction to a preopen set is subweakly quasi-continuous with respect to the same base. These results strengthen slightly the comparable results for weakly quasi-continuous functions. Finally, we extend a result concerning weakly quasi-continuous retractions and investigate some of the basic properties of subweakly quasi-continuous functions.

2. Preliminaries. The symbols X and Y denote topological spaces with no separation axioms assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A set A is semiopen (preopen, α -open) provided that $A \subseteq \text{Cl}(\text{Int}(A))$ ($A \subseteq \text{Int}(\text{Cl}(A))$, $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$). A set is semiclosed (preclosed, α -closed) provided that its complement is semiopen (preopen, α -open). The collection of all semiopen sets in a space X is denoted by $\text{SO}(X)$ and the collection of all semiopen sets in X containing a fixed point x is denoted by $\text{SO}(X, x)$. The intersection of all semiclosed sets containing a set A is called the semiclosure of A and denoted by $\text{sCl}(A)$. The

semi-interior of a set A , denoted by $\text{sInt}(A)$, is the union of all semiopen sets contained in A . The preclosure of A , denoted by $\text{pCl}(A)$, is the intersection of all preclosed sets containing A . Finally, if an operator is used with respect to a proper subspace, a subscript is added to the operator. Otherwise, it is assumed that the operator refers to the entire space.

DEFINITION 2.1 (Popa and Stan [9]). A function $f : X \rightarrow Y$ is said to be weakly quasi-continuous if for every $x \in X$, every open set U in X containing x , and every open set V in Y containing $f(x)$, there exists a nonempty open set W in X such that $W \subseteq U$ and $f(W) \subseteq \text{Cl}(V)$.

DEFINITION 2.2 (Arya and Bhamini [1]). A function $f : X \rightarrow Y$ is said to be weakly semicontinuous if for every $x \in X$ and every open set V in Y containing $f(x)$, there exists $U \in \text{SO}(X, x)$ for which $f(U) \subseteq \text{Cl}(V)$.

The following result by Noiri [5] shows that weak quasi-continuity and weak semicontinuity are equivalent.

THEOREM 2.3 (Noiri [5, Theorem 4.1]). A function $f : X \rightarrow Y$ is weakly quasi-continuous if and only if for every $x \in X$ and every open set V containing $f(x)$, there exists $U \in \text{SO}(X, x)$ for which $f(U) \subseteq \text{Cl}(V)$.

DEFINITION 2.4. A function $f : X \rightarrow Y$ is said to be subweakly continuous (Rose [10]) (subalmost weakly continuous (Baker [2])) if there is an open base \mathcal{B} for the topology on Y such that $\text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ ($\text{pCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$) for every $V \in \mathcal{B}$.

3. Subweakly quasi-continuous functions. The following characterization of weak quasi-continuity is due to Noiri [5].

THEOREM 3.1 (Noiri [5, Theorem 4.3(d)]). A function $f : X \rightarrow Y$ is weakly quasi-continuous if and only if $\text{sCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for every open set V in Y .

We define a function $f : X \rightarrow Y$ to be subweakly quasi-continuous provided that there is an open base \mathcal{B} for the topology on Y for which $\text{sCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for every $V \in \mathcal{B}$. Obviously, weak quasi-continuity implies subweak quasi-continuity. The following example shows that these concepts are not equivalent.

EXAMPLE 3.2. Let $X = \mathbb{R}$ have the usual topology and $Y = X$ have the discrete topology. The identity mapping $f : X \rightarrow Y$ is subweakly quasi-continuous with respect to the base consisting of the singleton sets in Y . However, f is not weakly quasi-continuous because for $V = (0, 1) \cup (1, 2)$, $\text{sCl}(f^{-1}(V)) \not\subseteq f^{-1}(\text{Cl}(V))$.

Since $\text{sCl}(A) = A \cup \text{Int}(\text{Cl}(A))$ for every set A , we have the following characterization of subweak quasi-continuity.

THEOREM 3.3. A function $f : X \rightarrow Y$ is subweakly quasi-continuous if and only if there is an open base \mathcal{B} for the topology on Y for which $\text{Int}(\text{Cl}(f^{-1}(V))) \subseteq f^{-1}(\text{Cl}(V))$ for every $V \in \mathcal{B}$.

Since $sCl(A) \subseteq Cl(A)$ for every set A , obviously, subweak continuity implies subweak quasi-continuity. The following example shows that the converse implication does not hold.

EXAMPLE 3.4. Let $X = [1/2, 3/2]$ have the usual relative topology, $Y = \{0, 1\}$ have the discrete topology, and let $f : X \rightarrow Y$ be the greatest integer function. Kar and Bhattacharya [3] showed that f is weakly quasi-continuous (their term is weakly semicontinuous) but not weakly continuous. Obviously, the function f is also not subweakly continuous.

The following two examples establish that subweak quasi-continuity is independent of subalmost weak continuity.

EXAMPLE 3.5. Let X be an indiscrete space with at least two elements and let $Y = X$ have the discrete topology. Since $pCl(\{x\}) = \{x\}$ for every $x \in X$, the identity mapping $f : X \rightarrow Y$ is subalmost weakly continuous with respect to the base consisting of the singleton sets in Y . However, since singleton sets in X are dense, f is not subweakly quasi-continuous.

EXAMPLE 3.6. Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $Y = X$ have the discrete topology. Let $f : X \rightarrow Y$ be the identity mapping. The function f is not subalmost weakly continuous, since any base for Y must include $V = \{a\}$ and $pCl(f^{-1}(V)) \not\subseteq f^{-1}(Cl(V))$. However, f is subweakly quasi-continuous with respect to the base of singleton subsets of Y .

4. Graph related properties. Recall that the graph of a function $f : X \rightarrow Y$ is the subspace $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

Park and Ha [8] proved that the graph of a weakly quasi-continuous function with a Hausdorff codomain is semiclosed. We show that weak quasi-continuity can be replaced by subweak quasi-continuity.

THEOREM 4.1. *If $f : X \rightarrow Y$ is subweakly quasi-continuous and Y is Hausdorff, then the graph of f , $G(f)$, is semiclosed.*

PROOF. Let \mathcal{B} be an open base for Y such that $sCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Let $(x, y) \in X \times Y - G(f)$. Since $y \neq f(x)$, there exists disjoint open sets V and W with $f(x) \in W$, $y \in V$, and $V \in \mathcal{B}$. Then $x \notin f^{-1}(Cl(V))$, and, since $sCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$, $x \notin sCl(f^{-1}(V))$. Therefore $(x, y) \in (X - sCl(f^{-1}(V))) \times V \subseteq X \times Y - G(f)$. Since $sCl(f^{-1}(V))$ is semiclosed, $X - sCl(f^{-1}(V))$ is semiopen. Since finite products of semiopen sets are semiopen, $(X - sCl(f^{-1}(V))) \times V$ is semiopen. Finally, since unions of semiopen sets are semiopen, it follows that $X \times Y - G(f)$ is semiopen and that $G(f)$ is semiclosed. \square

COROLLARY 4.2 (Park and Ha [8, Corollary 4.2]). *If $f : X \rightarrow Y$ is weakly quasi-continuous and Y is Hausdorff, then the graph of f , $G(f)$, is semiclosed.*

By the graph function of a function $f : X \rightarrow Y$ we mean the function $g : X \rightarrow X \times Y$ given by $g(x) = (x, f(x))$ for every $x \in X$.

THEOREM 4.3. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and let \mathcal{B} be an open base for σ . Let $\mathcal{C} = \{U \times V : U \in \tau, V \in \mathcal{B}\}$. The function f is subweakly quasi-continuous with respect to the base \mathcal{B} if and only if the graph function of f , $g : X \rightarrow X \times Y$, is subweakly quasi-continuous with respect to the base \mathcal{C} .*

PROOF. Assume that $f : (X, \tau) \rightarrow (Y, \sigma)$ is subweakly quasi-continuous with respect to the base \mathcal{B} for σ . Let $U \times V \in \mathcal{C}$, where $U \in \tau$ and $V \in \mathcal{B}$. Then $\text{sCl}(g^{-1}(U \times V)) = \text{sCl}(U \cap f^{-1}(V)) \subseteq \text{sCl}(U) \cap \text{sCl}(f^{-1}(V)) \subseteq \text{Cl}(U) \cap f^{-1}(\text{Cl}(V)) = g^{-1}(\text{Cl}(U) \times \text{Cl}(V)) = g^{-1}(\text{Cl}(U \times V))$. Thus g is subweakly quasi-continuous with respect to the base \mathcal{C} .

Assume that $g : (X, \tau) \rightarrow X \times Y$ is subweakly quasi-continuous with respect to the base \mathcal{C} for $X \times Y$. If $V \in \mathcal{B}$, then $\text{sCl}(f^{-1}(V)) = \text{sCl}(g^{-1}(X \times V)) \subseteq g^{-1}(\text{Cl}(X \times V)) = g^{-1}(X \times \text{Cl}(V)) = f^{-1}(\text{Cl}(V))$. Therefore, f is subweakly quasi-continuous with respect to the base \mathcal{B} . \square

In Theorem 4.3, if we take \mathcal{B} to be σ , the topology on Y , then we have the following result.

COROLLARY 4.4. *If the graph function $g : X \rightarrow X \times Y$ of a function f is subweakly quasi-continuous with respect to the usual base for the product space, then the function f is weakly quasi-continuous.*

COROLLARY 4.5 (Noiri [5, The “only if” part of Theorem 6.3.4]). *If the graph function $g : X \rightarrow X \times Y$ of a function f is weakly quasi-continuous, then the function f is weakly quasi-continuous.*

5. Additional properties

DEFINITION 5.1 (Kar and Bhattacharya [4]). A space X is said to be semi- T_1 provided that for every pair of distinct points x and y in X there exist sets $U \in \text{SO}(X, x)$ and $V \in \text{SO}(X, y)$ such that $y \notin U$ and $x \notin V$.

THEOREM 5.2. *If Y is Hausdorff and $f : X \rightarrow Y$ is a subweakly quasi-continuous injection, then X is semi- T_1 .*

PROOF. Let x_1 and x_2 be distinct points in X and let \mathcal{B} be an open base for Y such that $\text{sCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for every $V \in \mathcal{B}$. Since Y is Hausdorff and $f(x_1) \neq f(x_2)$, there exist disjoint open sets U and V in Y with $f(x_1) \in U$ and $f(x_2) \in V$, and $V \in \mathcal{B}$. Then, since $f(x_1) \notin \text{Cl}(V)$, we have $x_1 \in X - f^{-1}(\text{Cl}(V)) \subseteq X - \text{sCl}(f^{-1}(V))$ which is semiopen and does not contain x_2 . Therefore X is semi- T_1 . \square

The function in Example 3.6 is a subweakly quasi-continuous injection with a Hausdorff codomain and a non- T_1 -domain. Therefore, the conclusion that X is semi- T_1 in Theorem 5.2 cannot be strengthened to T_1 .

Since the restriction of the function f in Example 3.6 to the set $A = \{a, c\}$ is not subweakly quasi-continuous, we see that the restriction of a subweakly quasi-continuous function can fail to be subweakly quasi-continuous. Noiri [5] proved that the restriction of weakly quasi-continuous function to an open set is weakly quasi-continuous and Arya and Bhamini [1] extended this result to α -open sets. Finally, Park and Ha [8]

extended the result further to preopen sets. In what follows, we establish the analogous result for subweakly quasi-continuous functions.

THEOREM 5.3. *If $f : X \rightarrow Y$ is subweakly quasi-continuous with respect to the base \mathcal{B} for Y and A is a preopen set in X , then $f|_A : A \rightarrow Y$ is subweakly quasi-continuous with respect to the base \mathcal{B} .*

PROOF. Let $V \in \mathcal{B}$, then using (Noiri [7, Lemma 3.3]) we see that $sCl_A(f|_A^{-1}(V)) = A \cap sCl(f|_A^{-1}(V)) = A \cap sCl(f^{-1}(V) \cap A) \subseteq A \cap sCl(f^{-1}(V)) \subseteq A \cap f^{-1}(Cl(V)) = f|_A^{-1}(Cl(V))$. Therefore, $f|_A : A \rightarrow Y$ is subweakly quasi-continuous with respect to the base \mathcal{B} . \square

In Theorem 5.3, if we let \mathcal{B} be the topology, then we have the following result.

COROLLARY 5.4 (Park and Ha [8, Theorem 3.8]). *If $f : X \rightarrow Y$ is weakly quasi-continuous and A is an preopen set in X , then $f|_A : A \rightarrow Y$ is weakly quasi-continuous.*

THEOREM 5.5. *If $f : X \rightarrow Y$ is subweakly quasi-continuous and A is an open set in Y containing $f(X)$, then $f : X \rightarrow A$ is subweakly quasi-continuous.*

PROOF. Let \mathcal{B} be an open base for Y for which $sCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Then $\mathcal{C} = \{V \cap A : V \in \mathcal{B}\}$ is an open base for the relative topology on A . Let $V \cap A \in \mathcal{C}$, where $V \in \mathcal{B}$. Then $sCl(f^{-1}(V \cap A)) = sCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V)) = f^{-1}(Cl(V) \cap A)$. The proof is completed by establishing that $Cl(V) \cap A \subseteq Cl_A(V \cap A)$.

Let $y \in Cl(V) \cap A$ and let $W \subseteq A$ be open in A with $y \in W$. Since A is open in Y , W is open in Y . Because $y \in Cl(V)$, $W \cap V \neq \emptyset$. Therefore $W \cap (V \cap A) \neq \emptyset$, which proves that $y \in Cl_A(V \cap A)$. Thus $Cl(V) \cap A \subseteq Cl_A(V \cap A)$.

Now, it follows that $f : X \rightarrow A$ is subweakly quasi-continuous. \square

Park and Ha [8] defined a function $f : X \rightarrow A$, where $A \subseteq X$, to be a weakly quasi-continuous retraction provided that f is weakly quasi-continuous and $f|_A$ is the identity on A . It is then proved (Park and Ha [8, Theorem 3.15]) that, if $f : X \rightarrow A$ is a weakly quasi-continuous retraction and X is Hausdorff, then A is semiclosed in X . We prove the following comparable result for subweakly quasi-continuous functions.

THEOREM 5.6. *Let $A \subseteq X$ and let $f : X \rightarrow X$ be a subweakly quasi-continuous function such that $f(X) = A$ and $f|_A$ is the identity on A . If X is Hausdorff, then A is semiclosed.*

PROOF. Assume A is not semiclosed. Let $x \in sCl(A) - A$. Let \mathcal{B} be an open base for the topology on X such that $sCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Since $x \notin A$, $x \neq f(x)$. Because X is Hausdorff, there exist disjoint open sets V and W such that $x \in V$, $f(x) \in W$, and $V \in \mathcal{B}$. Let $U \in SO(X, x)$. Then $x \in U \cap V$, which is semiopen in X (Noiri [7]). Since $x \in sCl(A)$, $(U \cap V) \cap A \neq \emptyset$. So there exists $y \in (U \cap V) \cap A$. Since $y \in A$, $f(y) = y$ and therefore $y \in f^{-1}(V)$. Thus $U \cap f^{-1}(V) \neq \emptyset$ and we see that $x \in sCl(f^{-1}(V))$. However, $f(x) \in W$, which is open and disjoint from V . Hence $f(x) \notin Cl(V)$ or $x \notin f^{-1}(Cl(V))$, which contradicts the assumption that f is subweakly quasi-continuous. \square

LEMMA 5.7. *If $A \subseteq Y$ and $f : X \rightarrow A$ is weakly quasi-continuous, then $f : X \rightarrow Y$ is weakly quasi-continuous.*

PROOF. If V is an open set in Y , then $sCl(f^{-1}(V)) = sCl(f^{-1}(V \cap A)) \subseteq f^{-1}(Cl_A(V \cap A)) = f^{-1}(A \cap Cl(V \cap A)) = f^{-1}(Cl(V \cap A)) \subseteq f^{-1}(Cl(V))$. \square

Thus a weakly quasi-continuous retraction satisfies the hypothesis of [Theorem 5.6](#) and we have the following corollary.

COROLLARY 5.8 (Park and Ha [8, Theorem 3.15]). *If $f : X \rightarrow A$, where $A \subseteq X$, is a weakly quasi-continuous retraction and X is Hausdorff, then A is semiclosed.*

THEOREM 5.9. *Let Y be a Hausdorff space, $f_1 : X \rightarrow Y$ continuous, and $f_2 : X \rightarrow Y$ subweakly quasi-continuous. Then $\{x \in X : f_1(x) = f_2(x)\}$ is semiclosed.*

PROOF. Let $A = \{x \in X : f_1(x) = f_2(x)\}$ and let $x \in X - A$. Let \mathcal{B} be an open base for the topology on Y for which $sCl(f_2^{-1}(V)) \subseteq f_2^{-1}(Cl(V))$ for every $V \in \mathcal{B}$. Since Y is Hausdorff and $f_1(x) \neq f_2(x)$, there exist disjoint open sets V and W in Y for which $f_1(x) \in V$, $f_2(x) \in W$, and $V \in \mathcal{B}$. Since $f_2(x) \notin Cl(V)$, we have $x \in X - f_2^{-1}(Cl(V)) \subseteq X - sCl(f_2^{-1}(V))$. Therefore $x \in f_1^{-1}(V) \cap (X - sCl(f_2^{-1}(V))) \subseteq X - A$. Since $f_1^{-1}(V)$ is open, $X - sCl(f_2^{-1}(V))$ is semiopen, and the intersection of an open set and a semiopen set is semiopen (Noiri [7]), we see that $X - A$ is semiopen and that A is semiclosed. \square

COROLLARY 5.10. *Let Y be Hausdorff, $f_1 : X \rightarrow Y$ continuous, and $f_2 : X \rightarrow Y$ subweakly quasi-continuous. If f_1 and f_2 agree on a dense subset of X , then $f_1 = f_2$.*

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