

GRADED RADICAL W TYPE LIE ALGEBRAS I

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We get a new \mathbb{Z} -graded Witt type simple Lie algebra using a generalized polynomial ring which is the radical extension of the polynomial ring $F[x]$ with the exponential function e^x .

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1. Introduction. Let F be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, \mathbb{Z}_+ and \mathbb{Z} denote the nonnegative integers and the integers, respectively. Let $F[x]$ be the polynomial ring in indeterminate x . Let $F(x) = \{f(x)/g(x) \mid f(x), g(x) \in F[x], g(x) \neq 0\}$ be the field of rational functions in one variable. We define the F -algebra $V_{\sqrt{m},e}$ spanned by

$$\left\{ e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \mid d, a_1, \dots, a_m, t \in \mathbb{Z}, f_i \neq x, \right. \\ \left. (a_1, b_1) = 1, \dots, (a_m, b_m) = 1, 1 \leq i \leq m \right\}, \quad (1.1)$$

where b_1, \dots, b_m are fixed nonnegative integers, and $(a_i, b_i) = 1, 1 \leq i \leq m$, means that a_i and b_i are relatively primes, and f_1, \dots, f_n are the fixed relatively prime polynomials in $F[x]$. The F -subalgebra $V_{\sqrt{m},e}^+$ of $V_{\sqrt{m},e}$ is spanned by

$$\left\{ e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \mid d, a_1, \dots, a_m \in \mathbb{Z}, t \in \mathbb{Z}_+, f_i \neq x, \right. \\ \left. (a_1, b_1) = 1, \dots, (a_m, b_m) = 1, 1 \leq i \leq m \right\}. \quad (1.2)$$

Let $W_{\sqrt{m},e}(\partial)$ be the vector space over F with elements $\{f\partial \mid f \in V_{\sqrt{m},e}\}$ and the standard basis $\{e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \mid e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \in W_{\sqrt{m},e}\}$. Define a Lie bracket on $W_{\sqrt{m},e}(\partial)$ as follows:

$$[f\partial, g\partial] = f(\partial(g))\partial - g(\partial(f))\partial, \quad f, g \in V_{\sqrt{m},e}. \quad (1.3)$$

It is easy to check that (1.3) defines a Lie algebra $W_{\sqrt{m},e}(\partial)$ with the underlying vector space $W_{\sqrt{m},e}(\partial)$ (see also [1, 3, 5]). Similarly, we define the Lie subalgebra $W_{\sqrt{m},e}^+(\partial)$ of $W_{\sqrt{m},e}(\partial)$ using the F -algebra $V_{\sqrt{m},e}^+$ instead of $V_{\sqrt{m},e}$.

The Lie algebra $W_{\sqrt{m},e}(\partial)$ has a natural \mathbb{Z} -gradation as follows:

$$W_{\sqrt{m},e}(\partial) = \bigoplus_{d \in \mathbb{Z}} W_{\sqrt{m},e}^d, \quad (1.4)$$

where $W_{\sqrt{m},e}^d$ is the subspace of the Lie algebra $W_{\sqrt{m},e}(\partial)$ generated by elements of the form $\{e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \mid f_1, \dots, f_n \in F[x], a_1, \dots, a_m, t \in \mathbb{Z}, m \in \mathbb{Z}_+\}$. We call the subspace $W_{\sqrt{m},e}^d$ the d -homogeneous component of $W_{\sqrt{m},e}(\partial)$.

We decompose the d -homogeneous component $W_{\sqrt{m},e}^d$ as follows:

$$W_{\sqrt{m},e}^d = \bigoplus_{s_1, \dots, s_m \in \mathbb{Z}} W_{(d,s_1,\dots,s_m)}, \quad (1.5)$$

where $W_{(d,s_1,\dots,s_m)}$ is the subspace of $W_{\sqrt{m},e}^d$ spanned by

$$\{e^{dx} f_1^{s_1/b_1} \cdots f_m^{s_m/b_m} x^q \partial \mid q \in \mathbb{Z}\}. \quad (1.6)$$

Note that $W_{(0,0,\dots,0)}$ is the Witt algebra $W(1)$ as defined in [3].

The two radical-homogeneous components $W_{(d,a_1,\dots,a_m)}$ and $W_{(d,r_1,\dots,r_m)}$ are equivalent if $a_1 - r_1, \dots, a_m - r_m \in \mathbb{Z}$. This defines an equivalence relation on $W_{\sqrt{m},e}^d$. Thus we note that the equivalent class of $W_{(d,a_1,\dots,a_m)}$ depends only on a_1, \dots, a_m . From now on $W_{(d,a_1,\dots,a_m)}$ will represent the radical homogeneous equivalent class of $W_{(d,a_1,\dots,a_m)}$ without ambiguity. It is possible to choose the minimal positive integers a_1, \dots, a_m for the radical homogeneous equivalent component $W_{(d,a_1,\dots,a_m)}$.

We give the lexicographic order on all the radical homogeneous equivalent components $W_{(d,a_1,\dots,a_m)}$ using $\mathbb{Z} \times \mathbb{Z}_+^m$.

The radical equivalent homogeneous component $W_{\sqrt{m},e}^d$ can be written as follows:

$$W_{\sqrt{m},e}^d = \sum_{(a_1, \dots, a_m) \in \mathbb{Z}_+^m} W_{(d,a_1,\dots,a_m)}. \quad (1.7)$$

Thus for any element $l \in W_{\sqrt{m},e}(\partial)$, l can be written uniquely as follows:

$$l = \sum_{(d,a_1,\dots,a_m) \in \mathbb{Z} \times \mathbb{Z}_+^m} l_{(d,a_1,\dots,a_m)}. \quad (1.8)$$

For any such element $l \in W_{\sqrt{m},e}(\partial)$, $H(l)$ is defined as the number of different homogeneous components of l as in (1.4), and $L_d(l)$ as the number of nonequivalent radical d -homogeneous components of l in (1.8). For each basis element $e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial$ of $W_{\sqrt{m},e}(\partial)$ (or $W_{\sqrt{m},e}^+(\partial)$), define $\deg_{\text{Lie}}(e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial) = t$. Since every element l of $W_{\sqrt{m},e}(\partial)$ is the sum of the standard basis element, we may define $\deg_{\text{Lie}}(l)$ as the highest power of each basis element of l . Note that the Lie algebra $W_{\sqrt{m},e}(\partial)$ is self-centralized, that is, the centralizer $C_l(W_{\sqrt{m},e}(\partial))$ of every element l in $W_{\sqrt{m},e}(\partial)$ is one dimensional [1]. We find the solution of

$$1^{1/3} = \gamma \quad (1.9)$$

in \mathbb{Z}_7 . Equation (1.9) implies that

$$1 \equiv \gamma^3 \pmod{7}. \quad (1.10)$$

The solutions of (1.10) are 1, 2, or 4. Thus $1^{1/3} = 1, 2, \text{ or } 4 \pmod{7}$. Thus the radical number in \mathbb{Z}_p is not uniquely determined generally. So we may not consider the Lie algebras in this paper over a field of characteristic p differently from the Lie algebras in [2, 3, 4]. It is easy to prove that the Lie algebra $W_{(0,\dots,0)}$ is simple [3].

2. Main results. We need several lemmas for [Theorem 2.5](#).

LEMMA 2.1. *For any element l in the (d, a_1, \dots, a_m) -radical-homogeneous component of $W_{\sqrt{m}}(\partial)$, and for any element $l_1 \in W_{(0,0,\dots,0)}$, $[l, l_1]$ is an element in the (d, a_1, \dots, a_m) -radical homogeneous equivalent component.*

The proof of [Lemma 2.1](#) is straightforward.

LEMMA 2.2. *A Lie ideal I of $W_{\sqrt{m},e}(\partial)$ which contains ∂ is $W_{\sqrt{m},e}(\partial)$.*

PROOF. Let I be the ideal in the lemma. The Lie subalgebra which has the standard basis $\{x^i\partial \mid i \in \mathbb{Z}_+\}$ is simple. Let I be any ideal of $W_{\sqrt{m},e}(\partial)$ which contains ∂ . Then for any $f\partial \in W_{\sqrt{m},e}(\partial)$,

$$[x\partial, f\partial] = x\partial(f)\partial - f\partial \in I. \quad (2.1)$$

On the other hand,

$$[\partial, xf\partial] = f\partial + x\partial(f)\partial \in I. \quad (2.2)$$

Thus by subtracting (2.2) from (2.1) we get $2f\partial \in I$. Therefore, we have proven the lemma, since $I \cap W_{(0,0,\dots,0)}$ contains nonzero elements and so $I \supset W_{(0,0,\dots,0)}$. \square

LEMMA 2.3. *A Lie ideal I of $W_{\sqrt{m},e}(\partial)$ which contains a nonzero element in $W_{(d,a_1,\dots,a_m)}$ is $W_{\sqrt{m},e}(\partial)$, for a fixed $(d, a_1, \dots, a_m) \in \mathbb{Z} \times \mathbb{Z}_+$.*

PROOF. Let I be a Lie ideal of $W_{\sqrt{m},e}(\partial)$ and l a nonzero element in the ideal I . Then we take an element $l_1 = e^{-dx} f_1^{-a_1/b_1} \cdots f_m^{-a_m/b_m} x^p \partial$ with p a sufficiently large positive integer such that $[l, l_1] \neq 0$. Then $[f\partial, [l, l_1]]$ is a nonzero element in $W_{(0,0,\dots,0)}$ by taking an element $f_1^{t_1} \cdots f_m^{t_m} \in \mathbb{F}[x]$, where t_1, \dots, t_m are sufficiently large integers. Thus $I \cap W_{(0,0,\dots,0)}$ contains nonzero elements, and hence, $\partial \in I \cap W_{(0,0,\dots,0)}$ by simplicity of $W_{(0,0,\dots,0)}$. Then the lemma follows from [Lemma 2.2](#). \square

Throughout this paper, $a \gg b$ means that a is a number sufficiently larger than b .

LEMMA 2.4. *Let I be any nonzero Lie ideal of $W_{\sqrt{m},e}(\partial)$. For any nonzero element $l \in I$, there is an element $x^s\partial$, $s \gg 0$, such that $[x^s\partial, l]$ is the sum of elements in $W_{\sqrt{m},e}(\partial)$ with $\deg_{\text{Lie}}([x^s\partial, l]) > 0$.*

PROOF. It is straightforward by choosing a sufficiently large positive integer s . \square

THEOREM 2.5. *The Lie algebra $W_{\sqrt{m},e}(\partial)$ is simple.*

PROOF. Let I be a nonzero Lie ideal of $W_{\sqrt{m},e}(\partial)$. Let l be a nonzero element of I . By [Lemma 2.4](#), we may assume that l has polynomial terms with positive powers for each basis element of l . We prove this theorem in several steps.

STEP 1. If l is in the 0-homogeneous component, then the theorem holds. We prove this step, by induction on the number $L_0(l)$ of nonequivalent radical-homogeneous components of the element l of I . If $L_0(l)$ is 1 and $l \in W_{(0,0,\dots,0)}$, then the theorem holds by [Lemmas 2.2, 2.3](#), and the fact that $W_{(0,0,\dots,0)}$ is simple.

Assume that $l \in W_{(0,0,\dots,0,a_r,\dots,a_m)}$ with $a_r \neq 0$. If we take an element $f_1^{h_r/k_r} \dots f_n^{h_m/k_m} x^{h_{m+1}} \partial$ such that $h_r \gg k_r, \dots, h_n \gg k_r$ and $(h_r + k_r)/k_r \in \mathbb{Z}_+, \dots, (h_m + k_m)/k_m \in \mathbb{Z}_+$, then we have $l_1 = [f_1^{h_r/k_r} \dots f_m^{h_m/k_m} x^{h_{m+1}} \partial, l] \neq 0$. This implies that l_1 is in $W_{(0,0,\dots,0)}$. Thus we have proven the theorem by [Lemma 2.2](#).

By induction, we may assume that the theorem holds for $l \in I$ such that $L_0(l) = k$, for some fixed nonnegative integer $k > 1$. Assume that $L_0(l) = k+1$. If l has a $W_{(0,0,\dots,0)}$ radical-homogeneous equivalent component, we take $l_2 \in W_{(0,0,\dots,0)}$ such that $[l, l_2]$ can be written as follows: $[l, l_2] = l_3 + l_4$ where l_3 is a sum of nonzero radical-homogeneous components, and $l_4 = f\partial$ with $f \in F[x]$. Thus we have the nonzero element

$$\partial, [\dots, [\partial, l] \dots] = l_2 \in I \quad (2.3)$$

which has no terms in the homogeneous equivalent component $W_{(0,0,\dots,0)}$, where we applied Lie brackets until l_2 has no terms in the radical homogeneous equivalent component $W_{(0,0,\dots,0)}$. Then $l_2 \in I$ such that $H(l_2) \leq k$. Therefore, we have proven the theorem by [Lemmas 2.2, 2.3](#), and induction. If l has no terms in the radical homogeneous equivalent component $(0,0,\dots,0)$, then l has a term in the radical homogeneous equivalent component $W_{(0,a_1,\dots,a_n)}$. Take an element $l_3 = f_1^{c_1/p_1} \dots f_m^{c_m/p_m} x^{c_{m+1}} \partial$ such that c_1, \dots, c_{m+1} are sufficiently large positive integers such that $c_1 + a_1 \in \mathbb{Z} \dots c_m + a_m \in \mathbb{Z}$, and which is in a radical homogeneous equivalent component $W_{(0,a_1,\dots,a_m)}$. Then $[l_3, l]$ is nonzero and which has a term in the radical homogeneous equivalent component $W_{(0,0,\dots,0)}$. So in this case we have proven the theorem by induction.

STEP 2. Assume that l is in the d -homogeneous component such that $0 \neq d$ and $L_0(l) = 1$, then the theorem holds. By taking $e^{-dx} x^t \partial$, we have $0 \neq [e^{-dx} x^t \partial, l] \in W_{(0,0,\dots,0)}$ by taking a sufficiently large positive integer t . Thus we have proven the theorem by [Step 1](#).

STEP 3. If l is the sum of $(k-1)$ nonzero homogeneous components and 0-homogeneous component, then the theorem holds. We prove the theorem by induction on the number of distinct homogeneous components by [Steps 1](#) and [2](#). Assume that we have proven the theorem when l has $(k-1)$ radical-homogeneous components. Assume that l has terms in $W_{(0,0,\dots,0)}$. By [Step 1](#), we have an element $l_1 \in I$, such that $l_1 = l_2 + f\partial$, where l_2 has $(k-1)$ homogeneous components and $f \in F[x]$. Then $0 \neq \partial, [\dots, [\partial, l_1] \dots] \in I$ has $(k-1)$ homogeneous components, where we applied the Lie bracket until it has no terms in $W_{(0,0,\dots,0)}$. Therefore, we have proven the theorem by induction.

Assume that l has a (k) homogeneous equivalent components. We may assume l has the terms which is in $0 \neq d$ -homogeneous component. By taking a sufficiently large positive integer r , we have $[e^{-dx} x^r \partial, l] \neq 0$ and it has (k) homogeneous components with a term in the radical-homogeneous component $W_{(0,0,\dots,0)}$. Therefore, we have proven the theorem by [Step 3](#). \square

COROLLARY 2.6. *The Lie algebra $W_{\sqrt{m},e}^+(\partial)$ is simple.*

PROOF. It is straightforward from [Theorem 2.5](#) without using [Lemma 2.4](#). \square

COROLLARY 2.7. *The Lie subalgebra $W_{\sqrt{m},e}^0$ of $W_{\sqrt{m},e}(\partial)$ is simple.*

PROOF. It is straightforward from Step 1 of Theorem 2.5. \square

PROPOSITION 2.8. *For any nonzero Lie automorphism θ of $W_{\sqrt{m},e}^+(\partial)$, $\theta(\partial) = \partial$ holds.*

PROOF. It is straightforward from the relation $\theta([\partial, x\partial]) = \theta(\partial)$ and the fact that $W_{\sqrt{m},e}^+(\partial)$ is self-centralized and \mathbb{Z} -graded. \square

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