

ON INCIDENCE ALGEBRAS AND DIRECTED GRAPHS

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The incidence algebra $I(X, \mathbb{R})$ of a locally finite poset (X, \leq) has been defined and studied by Spiegel and O'Donnell (1997). A poset (V, \leq) has a directed graph (G_v, \leq) representing it. Conversely, any directed graph G without any cycle, multiple edges, and loops is represented by a partially ordered set V_G . So in this paper, we define an incidence algebra $I(G, \mathbb{Z})$ for (G, \leq) over \mathbb{Z} , the ring of integers, by $I(G, \mathbb{Z}) = \{f_i, f_i^* : V \times V \rightarrow \mathbb{Z}\}$ where $f_i(u, v)$ denotes the number of directed paths of length i from u to v and $f_i^*(u, v) = -f_i(u, v)$. When G is finite of order n , $I(G, \mathbb{Z})$ is isomorphic to a subring of $M_n(\mathbb{Z})$. Principal ideals I_v of (V, \leq) induce the subdigraphs $\langle I_v \rangle$ which are the principal ideals \mathcal{P}_v of (G_v, \leq) . They generate the ideals $I(\mathcal{P}_v, \mathbb{Z})$ of $I(G, \mathbb{Z})$. These results are extended to the incidence algebra of the digraph representing a locally finite weak poset both bounded and unbounded.

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1. Introduction. The incidence algebra $I(X, \mathbb{R})$ of a locally finite poset X over a commutative ring \mathbb{R} with identity is discussed in much detail by Spiegel and O'Donnell [5]. Every poset (V, \leq) has a directed graph (G_v, \leq) , or (G, \leq) for short, representing it. The directed graph (G, \leq) is free of cycles and multiple arcs. It is natural for (G, \leq) to have an incidence algebra whose properties depend on those of the directed graph.

In this paper, we define and study an incidence algebra $I(G, \mathbb{Z})$ for (G, \leq) over the ring \mathbb{Z} of all integers. Section 2 contains the basic results and definitions used in this paper. Section 3 deals with $I(G, \mathbb{Z})$, the incidence algebra of (G, \leq) representing a finite poset (V, \leq) of cardinality n . Ideals of $I(G, \mathbb{Z})$ are given through the principal ideals of (G, \leq) which are the subdigraphs induced by the principal ideals of (V, \leq) . An extension of the results given in Section 3 to the digraph representing a locally finite weak poset (V, \leq) bounded or unbounded is the content of Section 4.

2. Definitions and basic results. We consider the directed graph (G_v, \leq) representing a locally finite partially ordered set (V, \leq) . The terminology introduced here is used throughout the paper.

DEFINITION 2.1. For any $u \leq v$ in (V, \leq) , $[u, v] = \{w : u \leq w \leq v\}$ is an interval of (V, \leq) . The length of $[u, v]$ denoted by $l[u, v]$ is the length of the longest chain in $[u, v]$. A poset (V, \leq) is locally finite if $l[u, v]$ is finite for each $[u, v]$ in (V, \leq) ; and (V, \leq) is bounded if there is a $k > 0$ such that $l[u, v] \leq k$ for all $[u, v]$ in (V, \leq) . Otherwise (V, \leq) is unbounded.

DEFINITION 2.2. The directed graph (G_v, \leq) associated with a locally finite poset (V, \leq) is defined as $(G_v, \leq) = (V, E)$ where $V = (V, \leq)$ and $E = \{\text{arcs } \vec{uv}; u < v\}$; (G_v, \leq) has no cycles nor multiple arcs.

LEMMA 2.3 (see [5]). *A finite poset (V, \leq) can be labelled $V = \{v_1, v_2, \dots, v_n\}$ so that $v_i \leq v_j$ implies that $i \leq j$.*

PROPOSITION 2.4 (see [4]). *The vertices of a finite directed graph $G = (V, E)$ can be labelled v_1, v_2, \dots, v_n such that $v_i v_j \in E$ implies that $i < j$, if and only if G has no cycles.*

NOTE 2.5. Lemma 2.3 and Proposition 2.4 have motivated the orientation of arcs in (G_v, \leq) .

DEFINITION 2.6 (see [5]). Let (X, \leq) be a locally finite poset and \mathbb{R} a commutative ring with identity. The incidence algebra $I(X, \mathbb{R})$ of X over \mathbb{R} is defined by

$$I(X, \mathbb{R}) = \{f : X \times X \rightarrow \mathbb{R} : f(x, y) = 0 \text{ if } x \not\leq y\}, \quad (2.1)$$

with operations defined by

- (i) $(f + g)(x, y) = f(x, y) + g(x, y)$;
- (ii) $(f \cdot g)(x, y) = \sum_{x \leq u \leq y} f(x, u)g(u, y)$;
- (iii) $(rf)(x, y) = rf(x, y)$, for all $r \in \mathbb{R}$; $f, g \in I(X, \mathbb{R})$.

DEFINITION 2.7. For any $v \in (V, \leq)$, let $I_v = \{u \in V : u \leq v\}$; I_v is called the principal ideal generated by v .

DEFINITION 2.8. An ideal \mathcal{I} of (G_v, \leq) is an induced subdigraph of G_v such that all directed paths with their terminal vertex in \mathcal{I} are in \mathcal{I} .

If I_v is a principal ideal of (V, \leq) , $\langle I_v \rangle$, the subdigraph induced by I_v is the principal ideal of G generated by v in (G_v, \leq) . Denote $\langle I_v \rangle$ by \mathcal{I}_v .

NOTATION 2.9. For the remaining part of the paper, the digraphs (G, \leq) and (G_∞, \leq) represent the finite poset (V, \leq) and the locally finite poset (V_∞, \leq) , respectively.

3. An incidence algebra for (G, \leq) . In this section, we define an incidence algebra $I(G, \mathbb{Z})$ for the digraph (G, \leq) representing the finite poset (V, \leq) . Subalgebras and ideals of $I(G, \mathbb{Z})$ are defined through principal ideals of (G, \leq) . Assume that $V = \{v_1, v_2, \dots, v_n\}$.

DEFINITION 3.1. For $u, v \in V$, let $p_k(u, v)$ denote the number of directed paths of length k from u to v and $p_k(v, u) = -p_k(u, v)$.

For $i = 0, 1, 2, \dots, n-1$, define $f_i, f_i^* : V \times V \rightarrow \mathbb{Z}$ by

$$f_i(u, v) = p_i(u, v), \quad f_i^*(u, v) = -p_i(u, v). \quad (3.1)$$

The incidence algebra $I(G, \mathbb{Z})$ of (G, \leq) over the commutative ring \mathbb{Z} with identity is defined by $I(G, \mathbb{Z}) = \{f_i, f_i^* : V \times V \rightarrow \mathbb{Z}, i = 0, 1, 2, \dots, n-1\}$ with operations defined as follows:

- (i) for $f \neq g$, $(f + g)(u, v) = f(u, v) + g(u, v)$;

- (ii) $(f \cdot g)(u, v) = \sum_w f(u, w)g(w, v)$;
- (iii) $(zf)(u, v) = zf(u, v)$, for all $z \in \mathbb{Z}$; $f, g \in I(G, \mathbb{Z})$.

REMARK 3.2. The function f_l is the digraph analogues of $\chi \in I(X, \mathbb{R})$ [5]. The matrix $[f_l, (v_i, v_j)]$ is the adjacency matrix of (G, \leq) and $f_l^k(v_i, v_j) = f_k(v_i, v_j)$. For any interval $[u, v]$ with $l[u, v] = k$, $f_l^k(u, v) = f_k(u, v) = 0$. For every $f \in I(G, \mathbb{Z})$, there is a constant $m \in \mathbb{Z}$ such that $f^m(u, v) = 0$, for all $(u, v) \in V \times V$.

DEFINITION 3.3. With each ideal $\mathcal{I}_v = \langle I_v \rangle$ of (G, \leq) we associate an incidence algebra,

$$\begin{aligned} I(\mathcal{I}_v, \mathbb{Z}) &= \{f \in I(G, \mathbb{Z}) : f : I_v \times I_v \rightarrow \mathbb{Z}\}, \\ \forall f \in I(\mathcal{I}_v, \mathbb{Z}), f(v_i, v_j) &= 0 \quad \forall (v_i, v_j) \notin I_v \times I_v. \end{aligned} \quad (3.2)$$

REMARK 3.4. If (H, \leq) is a subdigraph of (G, \leq) , then $I(H, \mathbb{Z})$ is a subalgebra of $I(G, \mathbb{Z})$ [5]. In particular, $I(\mathcal{I}_v, \mathbb{Z})$ is a subalgebra of $I(G, \mathbb{Z})$; $I(\mathcal{I}_v, \mathbb{Z})$ is called the subalgebra generated by the vertex v .

REMARK 3.5 (see [5]). If S is an ideal of \mathbb{Z} , $I(G, S) = \{f \in I(G, \mathbb{Z}) : f(u, v) \in S\}$ is a subalgebra of $I(G, \mathbb{Z})$.

PROPOSITION 3.6. For each principal ideal \mathcal{I}_v of (G, \leq) , $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of the ring $I(G, \mathbb{Z})$.

PROOF. Let \mathcal{I}_v be a proper principal ideal of (G, \leq) . Denote the elements of $I(G, \mathbb{Z})$ and $I(\mathcal{I}_v, \mathbb{Z})$ by f and f' , respectively. For all $f' \in I(\mathcal{I}_v, \mathbb{Z})$, there is a unique $f \in I(G, \mathbb{Z})$ such that $f'(u, w) = f(u, w)$ for all $(u, w) \in I_v \times I_v$; and $f'(u, w) = 0$ for all $(u, w) \notin I_v \times I_v$. \square

Hence for any $(u, w) \in V \times V$, $f, g \in I(G, \mathbb{Z})$, $g' \in I(\mathcal{I}_v, \mathbb{Z})$

$$(fg')(u, w) = \begin{cases} (f, g)(u, w), & \text{if } (u, w) \in I_v \times I_v, \\ 0, & \text{otherwise,} \end{cases} \quad (3.3)$$

with similar values for $(g'f)(u, w)$ also.

Consequently, fg' and $g'f \in I(\mathcal{I}_v, \mathbb{Z})$, and $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of $I(G, \mathbb{Z})$.

REMARK 3.7. The incidence algebra $I(G, \mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices over \mathbb{Z} [5]. In general, every ideal of $M_n(\mathbb{Z})$ has the form $M_n(S)$ for some ideal S of \mathbb{Z} [3].

PROPOSITION 3.8. Every ideal of $I(G, \mathbb{Z})$ has the form $I(\mathcal{I}_v, \mathbb{Z})$ for some principal ideal \mathcal{I}_v of (G, \leq) .

PROOF. Let S be a proper ideal of the ring $I(G, \mathbb{Z})$. Then $S = I(H, \mathbb{Z})$ for some subdigraph (H, \leq) of (G, \leq) . For all $f \in I(G, \mathbb{Z})$ and $g' \in S$, $fg' = g'f \in S$.

Consequently, there is an $h' \in S$ such that $fg' = h'$ satisfying $(fg')(u, v) = h'(u, v) = p_k(u, v)$ for some k and for all u, v in $V(H)$.

Then for all $(u, v) \in V(H) \times V(H)$, the number of paths of length k in H from u to v is the same as that in G .

Hence for any $v \in V(H)$, H contains all the directed paths terminating in v . Then $H = \mathcal{F}_v$ for some v . \square

4. The incidence algebra $I(G_\infty, \mathbb{Z})$. An extension of [Proposition 2.4](#) is obtained by the author for locally finite directed graphs [2]. This provides an isomorphism between $I(G_\infty, \mathbb{Z})$ and a subring of the ring of upper triangular matrices over \mathbb{Z} . Also we have extended [Propositions 3.6](#) and [3.8](#) to bounded as well as unbounded locally finite weak posets. It is assumed that the poset (V, \leq) is countable.

DEFINITION 4.1. A locally finite poset (V, \leq) is weak if only finitely many chains intersect at every element v in V .

DEFINITION 4.2. A directed graph $G = (V, E)$ is locally finite if every vertex is of a finite degree.

DEFINITION 4.3. A vertex v of the directed graph (G_∞, \leq) , such that $d(v) \neq 0$, is called a source or atom if there is no other vertex u such that $u \leq v$.

PROPOSITION 4.4 (see [2]). *Let $G = (V, E)$ be a locally finite, acyclic digraph where V is countable; V can be labelled $\{v_1, v_2, v_3, \dots\}$ such that $v_i \vec{v}_j \in E$ implies that $i < j$ if and only if*

- (i) *the set S of its sources is nonempty and finite;*
- (ii) *for each $v \in V$ and $s \in S$ such that $s \vec{v} \in E$, any maximal directed path terminating in v is of finite length.*

DEFINITION 4.5. Let (V, \leq) be a locally finite weak poset and (G_∞, \leq) the digraph representing (V, \leq) . The incidence algebra $I(G_\infty, \mathbb{Z})$ of (G_∞, \leq) over the ring \mathbb{Z} of integers is given by

$$I(G_\infty, \mathbb{Z}) = \{f_i, f_i^* : V \times V \rightarrow \mathbb{Z}\} \quad (4.1)$$

satisfying the operations given in [Definition 3.1](#).

DEFINITION 4.6. An infinite matrix $A = [a_{ij}]$ is row (column) finite if $a_{ij} \neq 0$ for finitely many $j(i)$.

NOTE 4.7. (i) Each $f \in I(G_\infty, \mathbb{Z})$ is represented by a matrix $[f]$ where $[f]_{i,j} = f(v_i, v_j)$ and $[f]$ is both row and column finite, $M_\infty(\mathbb{Z})$ denotes the ring of row and column finite matrices over \mathbb{Z} . The incidence algebra $I(G_\infty, \mathbb{Z})$ is isomorphic to a subring of $M_\infty(\mathbb{Z})$.

(ii) A characterization of infinite directed graphs for which $[f]$ is nilpotent is obtained by the author in [1].

PROPOSITION 4.8. *Let (V, \leq) be a weak, locally finite poset satisfying the following:*

- (i) *the set S of its atoms is nonempty and finite;*
- (ii) *for each $v \in V$, $s \in S$ such that $s \leq v$, any chain with v as upperbound is of finite length. Then $I(G_\infty, \mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices over \mathbb{Z} .*

PROOF. The digraph (G_∞, \leq) is locally finite. By [Proposition 4.4](#), $f(v_r, v_s) \geq 0$ for all $r \leq s$ and for all $f \in I(G_\infty, \mathbb{Z})$. Hence every f has a representation as an upper triangular matrix over \mathbb{Z} . \square

DEFINITION 4.9. In a bounded weak, locally finite poset (V, \leq) a principal ideal $I_v = \{u : u \leq v\}$. The corresponding principal ideal of (G_∞, \leq) is given by $\mathcal{I}_v = \langle I_v \rangle$.

PROPOSITION 4.10. Let (V, \leq) be any weak, locally finite bounded poset and \mathcal{I}_v a principal ideal of (G_∞, \leq) . Then

- (1) $I(\mathcal{I}_v, \mathbb{Z})$ is a subalgebra of $I(G_\infty, \mathbb{Z})$;
- (2) $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of $I(G_\infty, \mathbb{Z})$.

NOTE 4.11. This follows from [Propositions 3.6](#) and [4.4](#).

PROPOSITION 4.12 (see [\[5\]](#)). Let (V, \leq) be an unbounded weak poset. Then V contains a subpartially ordered set isomorphic to \mathbb{Z}^+ , \mathbb{Z}^- , or UCn where Cn denotes a chain of length n .

REMARK 4.13. When (V, \leq) is unbounded, the principal ideals of (V, \leq) and (G_∞, \leq) are not well defined. Hence [Proposition 4.4](#) is not true for unbounded posets, in general.

But there are unbounded, locally finite posets satisfying [Proposition 4.4](#). As an example we have $(V, \leq) = (\mathbb{Z}^+, \text{usual ordering})$.

Let (V, \leq) be an unbounded, locally finite, weak poset such that $I(G_\infty, \mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices. Then

- (i) principal ideals of (G_∞, \leq) are defined as $\mathcal{I}_v = \langle I_v \rangle$;
- (ii) for each \mathcal{I}_v , $I(\mathcal{I}_v, \mathbb{Z})$ is a subalgebra of $I(G_\infty, \mathbb{Z})$;
- (iii) for every \mathcal{I}_v , $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of $I(G_\infty, \mathbb{Z})$.

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