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APPROXIMATION ON THE SEMI-INFINITE INTERVAL

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ABSTRACT. The approximation of a function $f \in C[a,b]$ by Bernstein polynomials is well-known. It is based on the binomial distribution. O. Szasz has shown that there are analogous approximations on the interval $[0,\infty)$ based on the Poisson distribution. Recently R. Mohapatra has generalized Szasz' result to the case in which the approximating function is

$$\alpha e^{-ux} \sum_{k=N}^{\infty} \frac{(ux)^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} f\left(\frac{k\alpha}{u}\right)$$

The present note shows that these results are special cases of a Tauberian theorem for certain infinite series having positive coefficients.

KEYWORDS AND PHRASES. *Szasz operators, Borel summability, Tauberian theorems.*

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1. INTRODUCTION.

Let us denote the class of functions f such that $f \in C[0,\infty)$ and for which

$\lim_{t\to\infty} f(t)$ exists by $C_{L,\infty}$. The subclass for which $\lim_{t\to\infty} f(t) = 0$ we shall denote by C_∞.

It is known that if $f \in C_{L,\infty}$ then

$$\lim_{u\to\infty} \alpha e^{-xu} \sum_{k=N}^{\infty} \frac{(xu)^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} f(\frac{k\alpha}{u}) = f(x) \tag{1}$$

for each $x \in (0,\infty)$. Here $\alpha > 0$, β is a real number and N is a positive integer exceeding $-\beta/\alpha$. This result was proved in [1] and is a generalization of a result due to O. Szász [2] which was the special case $\alpha = \beta = 1$, $N = 0$.

The proof of (1) depends heavily on a result due to D. Borwein [3], namely that

$$\lim_{u\to\infty} \alpha e^{-u} \sum_{k=N}^{\infty} \frac{u^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} = 1 \tag{2}$$

and it is the purpose of the present note to show that the deduction of (1) from (2) is a special case of a general theorem about infinite series. This theorem is of the Tauberian type and the method of proof which we give is of rather wide applicability. Our result is

THEOREM. Suppose that $f \in C_{L,\infty}$. Let $a_k \geq 0$, let K be a constant and let $\{v_k\}$ be a strictly increasing sequence of positive numbers. Then

$$\lim_{u\to\infty} e^{-u} \sum_{k=0}^{\infty} a_k u^{v_k} = 1 \tag{3}$$

$$\text{implies} \quad \lim_{u\to\infty} e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k} f(\frac{v_k+K}{u}) = f(x)$$

for each $x \in (0,\infty)$.

2. PROOF OF THE THEOREM

Since the result is trivially true if f is a constant function there is no loss of generality in supposing $f \in C_\infty$

instead of $f \in C_{L,\infty}$. As usual we will denote by $||f||$ the norm of f in the space C_∞, namely $||f|| = \sup_{[0,\infty)} |f(x)|$. Now for each $x \in (0,\infty)$

$$\overline{\lim_{u\to\infty}}\ e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k} f\left(\frac{v_k+K}{u}\right)$$

defines a linear functional on C_∞ which we will denote by $\overline{\ell}_x$. And if $\overline{\lim}$ is replaced by $\underline{\lim}$ the corresponding linear functional will be denoted by $\underline{\ell}_x$.

First we consider $\overline{\ell}_x$. Since

$$\left|e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k} f\left(\frac{v_k+K}{u}\right)\right| \le ||f||\ e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k}$$

we see, on letting $u\to\infty$, that $|\overline{\ell}_x(f)| \le ||f||$. Hence $\overline{\ell}_x$ is a bounded linear functional on C_∞ and so we will have

$$\overline{\ell}_x(f) = \int_0^\infty f(t)\ d\alpha_x(t)$$

for some function $\alpha_x \in BV[0,\infty)$, and we shall take α_x as having been normalized in the usual way. Now if we take $f(t) = e^{-\lambda t}$ $(\lambda > 0)$ it is a simple matter to see that $\overline{\ell}_x(e^{-\lambda t}) = e^{-\lambda x}$. In this calculation the hypothesis (3) is used in the form

$$\lim_{u\to\infty} e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k} = 1 \qquad (x > 0).$$

Hence

$$\overline{\ell}_x(e^{-\lambda t}) \equiv \int_0^\infty e^{-\lambda t}\ d\alpha_x(t) = e^{-\lambda x} \qquad (\lambda > 0).$$

By a well known theorem [4] this determines the normalized function α_x uniquely and by inspection it is seen to be

$$\alpha_x(t) = \begin{cases} 0 & (0 \le t < x) \\ \frac{1}{2} & (t = x) \\ 1 & (x < t) \end{cases}$$

Hence for $f \in C_\infty$ we have

$$\overline{\ell}_x(f) = \int_0^\infty f(t)\, d\alpha_x(t) = f(x) \ .$$

Now all of the above analysis involving $\overline{\ell}_x$ could be repeated with $\underline{\ell}_x$ instead. The same function α_x would be obtained and so we have

$$\underline{\ell}_x(f) = \overline{\ell}_x(f) = f(x)$$

That is to say, if $x > 0$

then $\lim_{u\to\infty} e^{-xu} \sum_{k=0}^{\infty} a_k(xu)^{v_k} f(\frac{v_k+K}{u})$ exists

and equals $f(x)$. This concludes the proof of the theorem.

We conclude with two remarks. The above theorem is about point-wise convergence whereas in [1] and [2] the uniform convergence of a set of functions $P_u(x)$ to $f(x)$ at each point $x_o \in [0,\infty)$ was considered. For the definition of this type of convergence we refer the reader to either of these sources but, when $f \in C_{L,\infty}$, to go from pointwise convergence to this other type of convergence is, any way, a simple matter. Secondly, we mention that in [1] the result (1) was stated for $x \in [0,\infty)$ but except in the case $N\alpha+\beta = 1$ the point $x = 0$ should be omitted.

REFERENCES

1. Mohapatra, R.N., A Note on Approximation of Continuous Functions by Generalized Szász Operators, Nanta Mathematica, 10 (1977), 181-184.

2. Szász, O., Generalization of S. Bernstein's Polynomial to the Infinite Interval, J. Nat. Bur. Standards, 45 (1950), 239-245.

3. Borwein, D., Relations Between Borel-type Methods of Summability, J. London Math. Soc., 35 (1960), 65-70.

4. Widder, D.V., "The Laplace Transform," Princeton (1941).

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