

## PEANO COMPACTIFICATIONS AND PROPERTY S METRIC SPACES

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**ABSTRACT.** Let  $(X, d)$  denote a locally connected, connected separable metric space. We say the  $X$  is S-metrizable provided there is a topologically equivalent metric  $\rho$  on  $X$  such that  $(X, \rho)$  has Property S, i.e. for any  $\epsilon > 0$ ,  $X$  is the union of finitely many connected sets of  $\rho$ -diameter less than  $\epsilon$ . It is well-known that S-metrizable spaces are locally connected and that if  $\rho$  is a Property S metric for  $X$ , then the usual metric completion  $(\tilde{X}, \tilde{\rho})$  of  $(X, \rho)$  is a compact, locally connected, connected metric space, i.e.  $(\tilde{X}, \tilde{\rho})$  is a Peano compactification of  $(X, \rho)$ . There are easily constructed examples of locally connected connected metric spaces which fail to be S-metrizable, however the author does not know of a non-S-metrizable space  $(X, d)$  which has a Peano compactification. In this paper we conjecture that: If  $(P, \rho)$  a Peano compactification of  $(X, \rho|_X)$ ,  $X$  must be S-metrizable. Several (new) necessary and sufficient for a space to be S-metrizable are given, together with an example of non-S-metrizable space which fails to have a Peano compactification.

KEY WORDS AND PHRASES. Property S metrics, Peano spaces, compactifications.

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## 1. INTRODUCTION.

Throughout this note let  $(X, d)$  denote a locally connected, connected separable metric space. We say that  $X$  is S-metrizable provided there is a topologically equivalent metric  $\rho$  on  $X$  such that  $(X, \rho)$  has Property S, i.e. for any  $\epsilon > 0$ ,  $X$  is the union of finitely many connected sets of  $\rho$ -diameter less than  $\epsilon$ . It is well-known that S-metrizable spaces are locally connected and that if  $\rho$  is a Property S metric for  $X$ , then the usual metric completion  $(\tilde{X}, \tilde{\rho})$  of  $(X, \rho)$  is a compact, locally connected, connected metric space, i.e.  $(\tilde{X}, \tilde{\rho})$  is a Peano compactification of  $(X, \rho)$  [8, p.154].

Property S metric spaces  $(X, \rho)$  have been studied extensively in [1, 2, 3, 4, 8]. There are easily constructed examples of locally connected, connected metric spaces which fail to be S-metrizable, however the author does not know of a non-S-metrizable space  $(X, d)$  which has a Peano compactification. We therefore ask:

QUESTION 1. If  $(P, \rho)$  is a Peano compactification of  $(X, \rho|_X)$ , must  $X$  be S-metrizable?

2. DEFINITIONS AND BASIC RESULTS A space  $Z$  is an extension of a space  $Y$  if  $Y$  is a dense subspace of  $Z$ . If  $Z$  is an extension of  $Y$ , we say that  $Y$  is locally connected in  $Z$  if  $Z$  has a basis consisting of regions (that is, open connected sets) whose intersections with  $Y$  are regions in  $Y$ .  $Z$  is a perfect extension of  $Y$  if  $Z$  is an extension of  $Y$  and whenever a closed subset  $H$  of  $Y$  separates two sets  $A, B \subset Y$  in  $Y$ , the set  $\text{cl}_Z H$  (the closure of  $H$  in  $Z$ ) separates  $A, B$  in  $Z$ . [6]

For completeness we include the following:

THEOREM 2.1 [6]. Let  $Z$  be an extension of  $X$ . Then  $X$  is locally connected in  $Z$  if and only if  $Z$  is a perfect locally connected extension of  $X$ .

THEOREM 2.2 [6]. Let  $(X, d)$  be a metric space. Then  $X$  is S-metrizable if

and only if  $X$  has a metrizable compactification  $Z$  in which it is locally connected.

**THEOREM 2.3 [6].** A topological space is  $S$ -metrizable if and only if it has a perfect locally connected metrizable compactification.

**THEOREM 2.4 [6].** Let  $X$  be a space having a perfect  $S$ -metrizable extension. Then  $X$  is  $S$ -metrizable.

**THEOREM 2.5 [5].** Let  $X$  be a separable, locally connected, connected rim compact metric space. Then  $X$  is  $S$ -metrizable.

**THEOREM 2.6 [6].** Every countable product of  $S$ -metrizable connected spaces  $X_1, X_2, \dots$ , is  $S$ -metrizable.

### 3. RELATED RESULTS AND QUESTIONS.

**THEOREM 3.1.** Let  $(P, d)$  be a Peano space and let  $X$  be a dense, locally connected, connected subset of  $P$ . Then there exists a  $G_\delta$ -subset  $Y$  of  $P$  containing  $X$  such that  $X$  is locally connected in  $Y$  (as an extension of  $X$ ).

**PROOF.** Let  $n$  be a positive integer and define  $Z_n = \{y \in P: \text{ if } U \text{ is an open connected subset of } P \text{ containing } y \text{ and } \delta(U) < 2^{-n}, \text{ then } U \cap X \text{ is not connected}\}$ . (Here  $\delta(U)$  denotes the  $d$ -diameter of  $U$ ). We first assert that  $Z_n$  is closed. For suppose  $y_1, y_2, \dots$ , is a sequence in  $Z_n$  which converges to  $y \in (P \setminus Z_n)$ . Since  $y \notin Z_n$ , there exists an open connected subset  $U$  of  $P$  containing  $y$  and  $\delta(U) < 2^{-n}$  and  $U \cap Z_n \neq \emptyset$  and this is a contradiction. Hence  $Z_n$  is closed.

We next assert  $Z_n \cap X = \emptyset$ . For let  $x \in X$  and let  $V$  be an open connected subset of  $X$  such that  $\delta(\text{cl} V) < 2^{-n}$ . Then  $U = \text{int cl} V$  is open in  $P$  and contains  $x$  and  $\delta(U) < 2^{-n}$ . Furthermore,  $U \cap X$  is connected since  $V \subseteq U \cap X \subseteq \text{cl} V$  and  $V$  is connected. Thus  $x \notin Z_n$  and  $Z_n \cap X = \emptyset$ .

Clearly  $Z_1 \subset Z_2 \subset Z_3 \dots$  is a monotonically increasing sequence and if for each  $i \geq 1$ ,  $Y_i = P \setminus Z_i$ ,  $Y = \bigcap_{i=1}^{\infty} Y_i$  is a connected  $G_\delta$ -subset of  $P$  which contains  $X$ .

We now assert that  $X$  is locally connected in  $Y$ , as an extension of  $X$ . For let  $\epsilon > 0$  and let  $y \in Y$ . Then there exists a positive integer  $n$  so that  $\epsilon > 2^{-n}$ ,

and since  $y \notin Z_n$ , there exists an open connected subset  $U$  of  $P$  with  $\delta(U) < 2^{-n}$  and such that  $U \cap X$  is connected. This implies that  $W = \text{int}_Y \text{cl}_Y U$  is an open connected subset of  $Y$ . Thus  $Y$  has a basis consisting of regions whose intersection with  $X$  is connected. This completes the proof.

**COROLLARY 3.1.1.** Every dense, locally connected, connected  $G_\delta$ -subset of a Peano continuum is  $S$ -metrizable if and only if dense, locally connected, connected subset of a Peano continuum is  $S$ -metrizable.

**PROOF.** This follows from (2.1), (2.4) and (3.1).

Since every nested intersection of countably many sets can be represented as an inverse limit space and since every  $Y_i$  above is  $S$ -metrizable, by (2.5), we ask:

**QUESTION 2.** If  $\{Y_i, f_{i,j}, \mathbb{N}\}$  is an inverse limit sequence of  $S$ -metrizable spaces and continuous maps (bicontinuous injections), must  $Y_\infty = \text{inv lim } \{Y_i, f_{i,j}, \mathbb{N}\}$  be  $S$ -metrizable?

Of course an affirmative answer to Question 2 would yield an affirmative answer to Question 1.

**THEOREM 3.2.** Let  $(X, d)$  be a locally connected, connected separable metric space, let  $\beta X$  denote the Stone-Ćech compactification of  $X$ . Then  $X$  is  $S$ -metrizable if and only if there exists a Peano compactification  $P$  of  $X$  such that  $\beta f$ , the continuous extension of the identity injection  $f: X \rightarrow P$  to  $\beta X$ , is monotone.

**PROOF.** Recall that a map between compact Hausdorff spaces is monotone if every point inverse is connected. Suppose that  $(X, d)$  is  $S$ -metrizable, say  $\rho$  is an  $S$ -metric for  $X$ . By (2.3), there exists a Peano compactification  $P$  of  $X$  and  $X$  is locally connected in  $P$ . Let  $\beta f: \beta X \rightarrow P$  be the continuous extension of the identity map  $f: X \rightarrow P$  to  $\beta X$ . We need to show that for  $y \in P$ ,  $\beta f^{-1}(y)$  is connected. But since  $P$  is a metric space and  $X$  is locally connected in  $P$ , there exists a neighborhood basis for  $y$  in  $P$ ,  $\{U_i\}_{i=1}^\infty$  such that for  $i \in \mathbb{N}$ ,  $\text{cl } U_{i+1} \subseteq U_i$  and

$U_i \cap X$  is connected. Then, if  $\beta f^{-1}(U_i) = W_i$ ,  $\beta f^{-1}(U_i \cap X) = f^{-1}(U_i \cap X)$  is connected and  $W_i \cap X = \beta f^{-1}(U_i \cap X)$ . Thus by (1.4) of [7],  $W_i$  is connected. It then follows that  $\beta f^{-1}(y) = \bigcap_{i=1}^{\infty} \text{cl } W_i$  is connected and that completes the proof of the necessity.

Now suppose  $(P, \rho)$  is a Peano compactification of  $X$  and  $\beta f: \beta X \rightarrow P$  is a monotone map. Let  $y \in P$  and let  $V$  be an open connected subset of  $P$  containing  $y$ . Since  $\beta f$  is monotone,  $\beta f^{-1}(V) = W$  is a connected open subset of  $\beta X$ . Again, by (1.4) of [7],  $W \cap X$  is connected. This implies that  $\beta f(W \cap X) = f(W \cap X) = V \cap X$  is connected and so  $X$  is locally connected in  $P$ . By (2.3),  $S$  is  $S$ -metrizable.

4. AN EXAMPLE. This is an example which fails to be  $S$ -metrizable, however it also fails to have a Peano compactification.

Let  $L_i$  be the line in  $\mathbb{R}^2$  defined by  $L_i = \{(x, y) : y = x/i, 0 \leq x \leq 1\}$  and let  $X = \bigcup_{i=1}^{\infty} L_i$  with the relative topology inherited from  $\mathbb{R}^2$ . We first assert that  $X$  is not  $S$ -metrizable. For in any (Hausdorff) compactification  $Z$  of  $X$ ,  $U_i = L_i \setminus \{(0, 0)\}$  is an open subset of  $Z$  and since  $A = \{(0, 0)\}$  is compact,  $A$  and  $B = \bigcup_{i=1}^{\infty} \{(1, i^{-1})\}$  are subsets of  $X$  whose closures are disjoint in  $Z$ . Thus if  $Z$  is a metric space with metric  $r$  and the distance from  $A$  to  $\text{cl}_Z B$  is  $\epsilon$ , then  $\epsilon > 0$ . It then follows that no finite collection of connected sets with  $r$ -diameter less than  $\epsilon/2$  fails to cover  $Z$ . Thus  $r$  is not a Property  $S$  metric for  $Z$  and  $X$  is not  $S$ -metrizable.

We will now show that  $X$  fails to have a locally connected metric compactification. Suppose  $(Z, r)$  is a locally connected metric compactification of  $X$ . Let  $U$  and  $V$  be open subsets of  $Z$  containing  $(0, 0)$  such that  $\text{cl } U \subseteq V \subseteq (Z \setminus \text{cl } B)$  ( $B$  is defined above). Then each  $L_i$  intersects  $\text{bd } U$  and  $\text{bd } V$  and contains a subarc  $S_i$  such that  $S_i \subseteq (\text{cl } V \setminus U)$  and  $S_i$  meets each of  $\text{bd } V$  and  $\text{bd } U$  in a single point, say  $S_i \cap \text{bd } V = \{a_i\}$  and  $S_i \cap \text{bd } U = \{b_i\}$ . Without loss of generality we may suppose that  $\{a_i\}_{i=1}^{\infty}$  converges to a point  $a \in \text{bd } V$  and  $\{b_i\}_{i=1}^{\infty}$  converges to a point  $b \in \text{bd } U$ . Then  $L = \limsup \{S_i : i \in \mathbb{N}\}$  is a connected set subset of  $\text{cl } V \setminus U$  meeting  $\text{bd } U$  and  $\text{bd } V$  [8, p. 14]. Then since every point of  $L \setminus (\text{bd } U \cup \text{bd } V)$  is a limit

point of  $\bigcup_{i=1}^{\infty} S_i$  and each  $S_i$  is a component of  $\text{cl } W \setminus U$ ,  $Z$  fails to be locally connected at any point of  $L \setminus (\text{bd } U \cup \text{bd } V)$ . Thus  $X$  fails to have a Peano compactification.

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