Internat. J. Math. & Math. Sci.
Vol. 3 No. 3 (1980) 491-504

ON CERTAIN GROUPS OF FUNCTIONS

J.S. YANG
Department of Mathematics,
Computer Science, and Statistics
University of South Carolina
Columbia, South Carolina 29208 U.S.A.

(Received October 18, 1979)

ABSTRACT. Let C(X,G) denote the group of continuous functions from a topological space X into a topological group G with the pointwise multiplication and the compact-open topology. We show that there is a natural topology on the collection of normal subgroups Δ(X) of C(X,G) of the $M_p = \{f \in C(X,G): f(p) = e\}$ which is analogous to the hull-kernel topology on the commutative Banach algegra C(X) of all continuous real or complex-valued functions on X. We also investigate homomorphisms between groups C(X,G) and C(Y,G).

KEY WORDS AND PHRASES. *Continuous functions, topological group, compact-open topology, hull-kernel topology, normal subgroups, S-pair, S-topology, Banach algebra, structure space.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES: *Primary 54C35, 54H10, Secondary 22A05, 20F30.*

1. INTRODUCTION AND NOTATION.

Suppose X is a compact topological space and suppose C(X) is the algebra of all continuous real or complex-valued functions on X with the usual pointwise operations and the supremum norm. Then C(X) is a regular commutative Banach algebra with identity and X is homeomorphic to the maximal ideal space $\Delta(C(X))$ of the algebra C(X), where $\Delta(C(X))$ is endowed with the Gel'fand topology which coincides with the hull-kernel topology since C(X) is regular, [3]. If X is a topological space and G is a topological group, let C(X,G) be the topological group of all continuous functions from X into G under pointwise multiplication and the compact-open topology. In Section 2 of this paper, we study spaces of normal subgroups of C(X,G). There is a natural topology, analogous to the hull-kernel topology in Banach algebra, for the collection of normal subgroups of the form $M_p = M_p(X,G) = \{f \in C(X,G): f(p) = e\}$, where e is the identity element of G; the resulting topological space will be denoted by $\Delta(X)$. We show that, with some mild restriction on X and G, X is homeomorphic to $\Delta(X)$, and that $\Delta(X^*)$ is the one-point compactification of $\Delta(X)$, where X* is the one-point compactification of the locally compact space X. Some theorems on homomorphisms and extension of homomorphisms in C(X,G) are considered in Section 3. We also prove a correct version of a theorem originally stated in [7, theorem 8].

All spaces considered in this paper are assumed to be Hausdorff unless specified. For topological spaces X and Y, the function space $F \subset C(X,Y)$ is understood to be endowed with the compact-open topology whenever it is referred to topologically. $I_0(X,G)$, or simply I_0 if no confusion should occur, will denote the identity element of the group C(X,G).

2. THE STRUCTURE SPACES.

For a topological space X and a topological group G, let $\Gamma = C(X,G)$. If X is compact and G is a Lie group, then Γ and M_p, $p \in X$, are in general ℓ_2 -manifolds (c.f. [1]). It is easy to see that M_a is locally contractible at I_0, where $a \in X$, if X is a locally compact group locally contractible at a, and that every M_p, $p \in X$, is n-simple for every positive integer n if X is a locally compact contractible space. It is also easy to see that the topological group Γ is a group with equal left and right uniformities if so is the group G, and that, if G is the projective limit of the inverse system of topological groups $\{(G_\alpha, f_{\beta\alpha}): \alpha, \beta \in A\}$, then M_p is the projective limit of the inverse system $\{(M_p(X,G_\alpha), f^p_{\beta\alpha}): \alpha, \beta \in A\}$, where $f^p_{\beta\alpha}(f) = f_{\beta\alpha} \circ f$ for every $f \in M_p(X,G)$.

Throughout this paper the spaces X and G will be subject to the following condition.

DEFINITION 1 [6]. A pair (X,G) of a topological space X and a topological group G is called an S-pair if for each closed subset A of X and $x \notin A$, there exists $f \in \Gamma$ such that $f(x) \neq e$ and $Z(f) = \{x: f(x) = e\} \supset A$.

It is clear that (X,G) is an S-pair if X is completely regular and G is path connected or if X is zero-dimensional. It is also clear that X is completely regular if (X,G) is an S-pair, and that $(\pi_{\alpha\in A} X_a, \pi_{\alpha\in A} G_a)$ is also an S-pair whenever (X_α, G_α) is an S-pair for each $\alpha \in A$. Magill called a space X a V-space, [4], if for points p, q, x, and y of X, where $p \neq q$, there exists a continuous function f of X into itself such that $f(p) = x$ and $f(q) = y$, and has shown that every completely regular path connected space and every zero-dimensional space is a V-space. It is easy to see that $(\pi_{\alpha\in A} X_a, G)$ is an S-pair if each (X_a,G), $a \in A$, is an S-pair and if G is a V-space. If G is a topological group such that (G,G) is an S-pair, G may not be a V-space. For example, let G_1 be the additive group of real numbers with the usual topology and let G_2 be any non-trivial finite group with the discrete topology, then $(G_1 \times G_2, G_1 \times G_2)$ is an S-pair since (G_1,G_1) and (G_2,G_2) are S-pairs. Since the topological group $G_1 \times G_2$ is not connected with the identity component isomorphic to G_1, $G_1 \times G_2$ is not a V-space as it follows from [4, Theorem 3.5]. It is pointed out in [7] that X is hemicompact and G is metrizable if (X,G) is an S-pair, G is a V-space, and Γ is first countable.

It is well-known (c.f. [2]) that, for every topological space X, there exists a completely regular space Y such that C(Y) is (algebraically) isomorphic to C(X), where C(Z) is the ring of continuous real-valued function on the space Z. Using the similar argument mutatis mutandis as used in the construction of the space Y, it is a straightforward to see that, for every topological space X and a topological group G, there is a completely regular space Y_G such that $C(Y_G,G)$ is continuously isomorphic to C(X,G), and that, in the case G is path connected, (Y_G,G) is an S-pair and the associated space Y_G is independent of the group G within the category of path connected topological groups. The latter means that $Y_{G_1} = Y_{G_2}$ whenever G_1 and G_2 are path connected groups. It follows from the construction of the space Y_G that $X = Y_G$ if (X,G) is an S-pair.

Because of the remarks just made above, we shall now assume that (X,G) is an S-pair.

For a collection $\sum$ of normal subgroups of $\Gamma = C(X,G)$, we define "*" as follows: If $U \subset \sum$ and $U \neq \phi$, let $U^* = \{M \in \sum : M \supset \cap U\}$, let $\phi^* = \phi$.

THEOREM 1. "*" is a closure operator on $\sum$ if and only if whenever $M \in \sum$ and $M \supset M_1 \cap M_2$, where M_1 and M_2 are intersections of some subsets of $\sum$, then either $M \supset M_1$ of $M \supset M_2$.

PROOF: It is clear that $U^* \supset U$, $(U^*)^* = U^*$, $\phi^* = \phi$, and that $U^* \cup V^* \subset (U \cup V)^*$ for subsets U and V of $\sum$. Hence "*" is a closure operator if and only if $U^* \cup V^* \supset (U \cup V)^*$ for subsets U and V of $\sum$. Now if $M_1 = \cap U$, and $M_2 = \cap V$, then $(U \cap V)^* = \{M \in \sum : M \supset M_1 \cap M_2\}$. Hence we have the theorem.

DEFINITION 2. If "*" is a closure operator on $\sum$, we shall refer the resulting topology, not necessarily Hausdorff, on $\sum$ as the S-topology, and the resulting space will be referred to as a G-structure space, or simply structure space, of the space X.

COROLLARY. If $\sum$ admits the S-topology, so is every subset of $\sum$.

REMARK 2. If G is path connected, we may speak of structure spaces for the space X without referring to the group since C(X,G) and C(X,R) are isomorphic in this case.

LEMMA 3. If a collection of normal subgroup $\sum$ of Γ admits the S-topology, then a subset A of $\sum$ is closed if and only if there exists a normal subgroup M_0 of Γ which is the intersection of some subset of $\sum$ such that $A = \{M \in \sum: M \supset M_0\}$. In fact, $M_0 = \cap A$.

PROOF: Suppose $A \subset \sum$ is closed, then $A = \bar{A} = \{M \in \sum: M \supset \cap A = M_0\}$.

Conversely, suppose that there exists a normal subgroup M_0 of Γ, where $M_0 = \cap U$ for some $U \subset \sum$, such that $A = \{M \in \sum: M \supset M_0\}$. Then $\bar{A} = \{M \in \sum: M \supset \cap A\} = A$. Hence A is closed.

THEOREM 4. If a collection of normal subgroups $\sum$ of Γ admits the S-topology, then $\sum$ is Hausdorff if and only if for M_1, $M_2 \in \sum$, $M_1 \neq M_2$, there are I_1 and I_2, where $I_1 = \cap U_1$, $I_2 = \cap U_2$ and U_1, $U_2 \subset \sum$, such that $M_1 \supset I_1$, $M_2 \supset I_2$, $M_1 \not\supset I_2$, $M_2 \not\supset I_1$, and $I_1 \cap I_2 = \cap\sum$.

PROOF: Suppose that $\sum$ is Hausdorff, and let M_1, $M_2 \in \sum$, $M_1 \neq M_2$. Thus there are disjoint open sets U_1 and U_2 in $\sum$ such that $M_1 \subset U_1$, and $M_2 \subset U_2$. If $A_1 = \sum - U_2$, $A_2 = \sum - U_1$, then A_1 and A_2 are closed and $M_1 \in A_1$, $M_2 \in A_2$. Using Lemma 3, we have $A_i = \{M \in \sum: M \supset \cap A_i\}$, $i = 1,2$. If we let $I_i = \cap A_i$, $i = 1,2$, then $M_1 \supset I_1$, $M_2 \supset I_2$, $M_1 \not\supset I_2$, $M_2 \not\supset I_1$ and $I_1 \cap I_2 = \cap\sum$.

Conversely, assume that the stated property holds, and let M_1, $M_2 \in \sum$ such that $M_1 \neq M_2$. Then there are subsets U_1 and U_2 of $\sum$ such that if $I_i = \cap U_i$, $i = 1,2$, $M_1 \supset I_1$, $M_2 \supset I_2$, $M_1 \not\supset I_2$, $M_2 \not\supset I_1$, and $I_1 \cap I_2 = \cap\sum$. Let $B_i = \{M \in \sum: M \supset I_i\}$, $i = 1,2$. Then B_i are closed by Lemma 3, $M_1 \in B_1$, $M_2 \in B_2$, $M_1 \notin B_2$ and $M_2 \notin B_1$. If we let $V_2 = \sum - B_1$, $V_1 = \sum - B_2$, then $M_1 \in V_1$, $M_2 \in V_2$, and $V_1 \cap V_2 = \phi$. To see that $V_1 \cap V_2 = \phi$, it suffices to show that if $M \in \sum$, then either $M \in B_1$ or $M \in B_2$. Now $M \in \sum$ implies $M \supset \cap \sum = I_1 \cap I_2$. This means that either $M \supset I_1$ or $M \supset I_2$ since $\sum$ admits the S-topology. Hence $M \in B_1$ or $M \in B_2$. This completes the proof.

If we denote by $\Delta(X)$ the collection of all normal subgroups of Γ of the form $M_p = \{f \in C(X,G)\colon\ f(p) = e\}$, $p \in X$, then the following theorem states that $\Delta(X)$ admits the S-topology and that the S-topology is Hausdorff if (X,G) is an S-pair.

THEOREM 5. $\Delta(X)$ admits the Hausdorff S-topology.

PROOF: Let U and V be subsets of $\Delta(X)$, and let $O_1 = \{P \in X\colon\ M_p \in U\}$ and $O_2 = \{q \in X\colon\ M_q \in V\}$. It is, by Theorem 1, sufficient to show that, if $M_q \supset (\bigcap_{p\in O_1} M_p) \cap (\bigcap_{k\in O_2} M_k)$, then either $M_q \supset \bigcap_{p\in O_1} M_p$ and $M_q \supset \bigcap_{k\in O_2} M_k$. Suppose otherwise, then there exist $f \in \bigcap_{p\in O_1} M_p - M_q$ and $g \in \bigcap_{k\in O_2} M_k - M_q$. This implies that $q \notin \bar{O}_1$ and $q \notin \bar{O}_2$. For if $q \in \bar{O}_1$, then there is a net $\{q_\alpha\}$ in O_1 such that $q_\alpha \to q$. Then $f(q\ ) \to f(q)$, and hence $f(q) = e$ since $f(q\ ) = e$ for each α. Similarly, $q \notin \bar{O}_2$. Hence $q \notin \overline{O_1 \cup O_2}$. But (X,G) is an S=pair, let $h \in \Gamma$ such that $\overline{O_1 \cup O_2} \subset Z(h)$ but $h(q) \neq e$. This would show that $h \in (\bigcap_{p\in O_1} M_p) \cap (\bigcap_{k\in O_2} M_k)$ but $h \notin M_q$, a contradiction. Hence either $M_q \supset (\bigcap_{p\in O_1} M_p)$ or $M_q \supset (\bigcap_{k\in O_2} M_k)$, and $\Delta(X)$ admits the S-topology.

Next to show that the S-topology is Hausdorff. Let $M_p, M_q \in \Delta(X)$, where $p \neq q$. Since X is T_2, let O_1 and O_2 be open sets in X such that $P \in O_1$, $q \in O_2$ and $O_1 \cap O_2 = \phi$. If $C_2 = X - O_1$ and $C_1 = X - O_2$, then $p \in C_1$ and $q \in C_2$. If $I_1 = \bigcap_{k\in C_1} M_k$ and $I_2 = \bigcap_{k\in O_2} M_k$, then $I_1 \cap I_2 = \cap\Delta(X)$ since $C_1 \cup C_2 = X$, $M_p \supset I_1$, and $M_q \supset I_2$. To see that $M_p \not\supset I_2$ note that $p \notin C_2$, hence there exists $f \in \Gamma$ such that $F(C_2) = e$ but $f(p) \neq e$. Thus $f \in \bigcap_{k\in C_2} M_k$ but $f \notin M_p$. This shows that $M_p \not\supset I_2$. Similarly, we have $M_q \not\supset I_1$. This completes the proof that $\Delta(X)$ is T_2, by Theorem 4.

Note that the S-topology defined above for $\Delta(X)$ is analogous to the hull-kernel topology, which coincides with the Gel'fand topology, on the maximal ideal space of the commutative Banach algebra $C(X)$.

For each $\alpha \in I$, let A_α be a closed set of a structure space $\sum$. Then, by Lemma 3, there exists a normal subgroup M_α of Γ which is the intersection of some subset of $\sum$ such that $A_\alpha = \{M \in \sum: M \supset M_\alpha\}$. If we denoted by $[\bigcup_{\alpha\in I} M_\alpha]$ the normal subgroup of Γ generated by $\bigcup_{\alpha\in I} M_\alpha$, then we have the following lemma whose proof is straightforward and hence omitted.

LEMMA 6. $\bigcap_{\alpha\in I} A_\alpha = \{M \in \sum: M \supset [\bigcup_{\alpha\in I} M]\}$

THEOREM 7. A structure space $\sum$ of X is compact if and only if every collection of normal subgroups $\{N_\alpha\}_{\alpha\in I}$ of Γ, each of which is the intersection of some subset of $\sum$, such that $[\bigcup_{\alpha\in I} N_\alpha] \not\subset M$ for each $M \in \sum$ has a finite subcollection. $\{N_{\alpha_1}, N_{\alpha_2}, \ldots, N_{\alpha_n}\}$ such that $[\bigcup_{i=1}^{n} N_{\alpha_i}] \not\subset M$ for each $M \in \sum$.

PROOF: Suppose $\sum$ is compact, and let $\{N_\alpha\}_{\alpha\in I}$ be a collection of normal subgroups of Γ, each of which is the intersection of some subset of $\sum$, such that $[\bigcup_{\alpha\in I} N_\alpha] \not\subset M$ for each $M \in \sum$. If, for each $\alpha \in I$, let $A_\alpha = \{M \in \sum: M \supset N_\alpha\}$, then A_α is closed in $\sum$, Lemma 3, and $\bigcap_{\alpha\in I} A_\alpha = \{M \in \sum: M \supset [\bigcup_{\alpha\in I} N_\alpha]\} = \phi$. Hence, by the compactness of $\sum$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\bigcap_{i=1}^{n} A_{\alpha_i} = \phi$; i.e., there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\{M \in \sum: M \supset [\bigcup_{i=1}^{n} N_{\alpha_i}]\} = \phi$. Hence $[\bigcup_{i=1}^{n} N_{\alpha_i}] \not\subset M$ for each $M \in \sum$.

Conversely, suppose that $\sum$ has the stated property, and let $\{A_\alpha\}_{\alpha\in I}$ be a collection of closed sets with the finite intersection property, where $A_\alpha = \{M \in \sum: M \supset N_\alpha\}$ and N is the intersection of some subset of $\sum$. Suppose that $\bigcap_{\alpha\in I} A_\alpha = \phi$. Then $\{M \in \sum: M \supset [\bigcup_{\alpha\in I} N_\alpha] = \phi$, hence $[\bigcup_{\alpha\in I} N_\alpha] \not\subset M$ for each $M \in \sum$. Thus, by the hypothesis, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $[\bigcup_{i=1}^{n} N_{\alpha_i}] \not\subset M$ for each $M \in \sum$. This would imply that $\bigcap_{i=1}^{n} A_{\alpha_i} = \phi$, a contradiction. Hence $\sum$ is compact.

COROLLARY. A structure space $\sum$ of X is compact if every normal subgroup N of Γ not contained in any element of $\sum$ contains a finitely generated normal subgroup of N not contained in any element of $\sum$.

PROOF: Assume that the stated property holds in $\sum$, and let $\{N_\alpha\}_{\alpha\in I}$ be a collection of normal subgroups of Γ, each of which is the intersection of some subset of $\sum$, such that $[\bigcup_{\alpha\in I} N_\alpha] \not\subset M$ for every M in $\sum$. Let $N = [\bigcup_{\alpha\in I} N_\alpha]$. Then $N \not\subset M$ for every M in $\sum$, thus N contains a finitely generated normal subgroup B such that $B \not\subset M$ for each $M \in \sum$. Let $B = [a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_n}]$, where $a_{\alpha_i} \in N_{\alpha_i}$, $i = 1, 2, \ldots, n$. Then $[\bigcup_{i=1}^{n} N_{\alpha_i}] \not\subset M$ for every M in $\sum$. Hence $\sum$ is compact by Theorem 7.

We shall call a normal subgroup N of Γ free if there is no $p \in X$ such that $f(p) = e$ for each $f \in N$.

COROLLARY. $\Delta(X)$ is compact if every free normal subgroup N of Γ contains a finitely generated free normal subgroup.

THEOREM 8. The mapping ψ: $X \to \Delta(X)$ defined by $\psi(x) = M_x$, $x \in X$, is a homeomorphism.

PROOF: Clearly, ψ is one-to-one and onto.

For the continuity of ψ, let $A \subset \Delta(X)$ be closed. Then there exists a normal subgroup M_0 of Γ such that $A = \{M_x \in \Delta(X): M_x \supset M_0\}$. We shall see that $\psi^{-1}(A)$ is closed. For this purpose, let $\{x_\alpha\}$ be a net in $\psi^{-1}(A)$ converging to $x \in X$. Then $M_{x_\alpha} \supset M_0$ for each α. If $M_x \not\supset M_0$, there exists $f \in M_0 - M_x$ which would imply that $f(x) \neq e$, a contradiction since $f(x_\alpha) \to f(x)$ and $f(x) = e$ for each α.

Next to show that ψ is a closed map. Let C be closed in X, and let $M_0 = \bigcap_{x\in C} M_x$. We claim that $\psi(C) = \{M_x \in \Delta(X): M_x \supset M_0\}$, which would imply that $\psi(C)$ is closed. It is clear that $\psi(C) \subset \{M_x \in \Delta(X): M_x \supset M_0\}$. Now let $M_x \in \{M_x \in \Delta(X): M_x \supset M_0\}$. Then $M_x \supset M_0$. Suppose $x \notin C$, then there exists $f \in C(X,G)$ such that $f(C) = e$ but $f(x) \neq e$. Hence $f \in M_0$ but $f \notin M_x$, a contradiction. Thus $x \in C$, and we have $M_x \in \psi(C)$.

If A is a commutative Banach algebra without identity, and if A(e) is the algebra obtained by adjoining an identity to A, then the maximal ideal space

$\Delta(\dot{A}(e))$, with the Gel'fand topology, is the one-point compactification of $\Delta(A)$. Using the previous results, we can also state the following theorem whose proof is now trivial.

THEOREM 9. If X is a locally compact space and X* its one-point compactification, then $\Delta(X^*)$ is the one-point compactification of $\Delta(X)$, and $\Delta(X^*) = \Delta(X) \cup \{M_\infty\}$, where $M_\infty = \{f \in C(X^*,G): f(\infty) = e\}$.

3. <u>HOMOMORPHISMS OF C(X,G)</u>.

In this section, we shall study homomorphisms of the group C(X,G) into the group C(Y,G) which leads us to have another version of a theorem originally announced in [7]. We shall also, at the end of the section, consider extensions of homomorphisms of the group C(X,G). All pairs (Z,G) are again assumed to be S-pairs.

DEFINITION 3. (1) A homomorphism ϕ of the group C(Y,G) into the group C(X,G) is said to be a constant-preserving if ϕ maps every constant function on Y into the corresponding constant function on X.

(2) A homomorphism ϕ of the group C(Y,G) into the group C(X,G) which has the property that $\phi^{-1}(\Delta(X)) \subset \Delta(Y)$ is called an F-homomorphism.

It is easy to construct an example of a homomorphism ϕ: $C(Y,G) \to C(X,G)$ which is an F-homomorphism but is not constant-preserving. The following example [5], shows that the converse does not hold either.

EXAMPLE. Let Y = [0,1] be the closed unit interval, and let X = $([-1,1] \times \{0\}) \cup (\{0\} \times (0,1])$ considered as a subspace of R^2. For each $f \in C(Y,R)$, define $\phi(f) \in C(X,R)$ by

$$\phi(f)(t,0) = f(\tfrac{1}{4}(t+1)), \quad t \in [-1,1]$$

$$\phi(f)(0,s) = f(\tfrac{1}{2}(s+1)) + f(\tfrac{1}{2}), \quad s \in [0,1]$$

Then ϕ is a constant-preserving isomorphism of C(Y,R) onto C(X,R). If $g \in C(X,R)$, then

$$\phi^{-1}(g)(y) = \begin{cases} g(4y-1,0) & y \in [0,\frac{1}{2}] \\ g(0,2y-1) + g(1,0) - g(0,0), & y \in [\frac{1}{2},1] \end{cases}$$

Now choose $g \in C(X,R)$ such that $Z(g) = \{(0,\frac{1}{2})\}$ and that $g(1,0) - g(0,0) > 0$, then $g \in M_{(0,\frac{1}{2})}$, but $Z(\phi^{-1}(g)) = \phi$. Hence ϕ is not an F-homomorphism.

THEOREM 10. Suppose that ϕ: $C(Y,G) \to C(X,G)$ is a continuous constant-preserving F-homomorphism of $C(Y,G)$ into $C(X,G)$. Then

(1) ϕ induces a one-to-one continuous map of $\Delta(X)$ into $\Delta(Y)$, and

(2) ϕ induces a continuous map j of X into Y such that $j(x) = y$ if and only if $\phi(g)(x) = g(y)$ for each $g \in C(Y,G)$.

PROOF. (1) for each $x \in X$, let hx: $C(X,G) \to G$ be the evaluation map defined by $hx(f) = f(x)$, $f \in C(X,G)$, and let $M_x = \ker hx$. Define $h(x)$: $C(Y,G) \to G$ by $h(x) = hx \circ \phi$, $x \in X$. Then $\ker h(x) = \phi^{-1}(M_x)$, hence $\ker h(x) = M_y$ for some $y \in Y$. Such an y is unique and we have $h(x) = hy$. Now we define a mapping Φ: $(X) \to \Delta(Y)$ by $\Phi(M_x) = M_y$.

Clearly Φ is one-to-one. For the continuity of Φ, let $A = \{M_y \in \Delta(Y)\colon M_y \supset M_1\}$, where M_1 is the intersection of some subset U of $\Delta(Y)$, be ä closed set in $\Delta(Y)$. We claim that $\Phi^{-1}(A) = \{M_x \in \Delta(X)\colon M_x \supset \phi(M_1)\}$ and that $\phi(M_1)$ is the intersection of the subset $\phi(U)$ of $\Delta(X)$. In fact, let $M_x \supset \phi(M_1)$. Then $\phi^{-1}(M_x) \supset M_1$. If $\Phi(M_x) = M_y$, $M_y = \ker hy = \ker (hx \circ \phi) = \phi^{-1}(M_x \supset M_1$, hence $M_y \in A$, thus $M_x = \Phi^{-1}(M_y) \in \Phi^{-1}(A)$. Conversely, let $M_z \in \Phi^{-1}(A)$. Then $\Phi(M_z) \in A$. If $\Phi(M_z) = M_y \in A$ for some $y \in Y$, then $Hy = hz \circ \phi$, hence $\phi(M_1) \subset M_z$. It is easy to see that $\phi(M_1)$ is the intersection of the subset $\phi(U)$ of $\Delta(X)$. Therefore $\Phi^{-1}(A)$ is closed in $\Delta(X)$, and Φ is continuous.

(2) Let the mapping j: $X \to Y$ be defined by $j = \psi_y^{-1} \circ \Phi \circ \psi x$, where

ψ_z: $Z \to \Delta(Z)$ is the mapping of Theorem 8. Then clearly j is continuous and $j(x) = y$ if and only if $\Phi(M_x) = M_y$. To see that $j(x) = y$ if and only if $\phi(g)(x) = g(y)$ for every $g \in C(Y,G)$, let $j(x) = y$. Then $\Phi(M_x) = M_y$. Thus $\ker(hx \circ \phi) = M_y$. Let $g \in C(Y,G)$. If $g \in M_y$, $hx \circ \phi(g) = e$, and we have $\phi(g)(x) = g(y)$. If $g \notin M_y$, there exists $c \in G$ such that $g \in \underline{c}M_y$, where $\underline{c}$ is the constant mapping of X into c, hence $g = \underline{c}k$ for some $k \in My$. Now $hx \circ \phi(g) = hx \circ \phi(\underline{c}k) = hx(\underline{c}\ \phi(k)) = c\phi(k)(x) = c$, while $g(y) = ck(y) = c$. Hence $\phi(g)(x) = g(y)$ for each $g \in C(Y,G)$. Conversely, if $\phi(g)(x) = g(y)$ for each $g \in C(Y,G)$, then, for $g \in C(Y,G)$, $hx \circ \phi(g) = \phi(g)(x) = g(y) = hy(g)$. Thus $\Phi(Mx) = My$, and we have that $j(x) = y$.

REMARK: It is easy to see that, if the mapping ϕ in Theorem 10 is an onto map, then Φ is an embedding.

THEOREM 11. A continuous homomorphism ϕ of $C(Y,G)$ into $C(X,G)$ is a constant-preserving F-homomorphism if and only if there exists $f \in C(X,Y)$ such that $\phi(k) = k \circ f$ for every $k \in C(Y,G)$.

PROOF: It is clear that a homomorphism ϕ of the form $\phi(k) = k \circ f$ for every $k \in C(Y,G)$ is a constant-preserving F-homomorphism. Conversely, if ϕ is a constant-preserving F-homomorphism, and if j is the continuous map of X into Y as defined in Theorem 10, then, for each $k \in C(Y,G)$, $\phi(k)(x) = k(y) = k \circ j(x)$, where $j(x) = y$, Hence $\phi(k) = k \circ j$ for each $k \in C(Y,G)$.

COROLLARY. A homomorphism ϕ of $C(Y,G)$ into $X(X,G)$ is a constant-preserving F-homomorphism if and only if there exists $f \in C(X,Y)$ such that $\phi(k) = k \circ f$ for every $k \in C(Y,G)$.

PROOF: Note that the group topologies for $C(Y,G)$ and $C(X,G)$ are not relevant in the proof of Theorem 10. Hence take discrete topologies for the groups $C(Y,G)$ and $C(X,G)$, then apply the proof of Theorem 11.

As a consequence of the discussions made above, we can now state a correct version of the theorem originally stated in [7, Theorem 8] in the following.

THEOREM 12. If there exists an isomorphism ϕ between groups $C(Y,G)$ and $C(X,G)$ which is constant-preserving such that both ϕ and ϕ^{-1} are F-homomorphisms, then X and Y are homeomorphic.

PROOF: It is clear that ϕ^{-1} is also constant-preserving if ϕ is. Applying the above corollary to ϕ and ϕ^{-1}, there exist functions $j \in C(X,Y)$ and $\ell \in C(Y,X)$ such that $\phi(k) = k \circ j$ for each $k \in C(Y,G)$ and $\phi^{-1}(k) = k \circ \ell$ for each $k \in C(X,G)$. Consequently, we have that $\ell \circ j(x) = x$ and $j \circ \ell(y) = y$ for $x \in X$ and $y \in Y$. To see this suppose that there exists $x \in X$ such that $\ell \circ j(x) \neq x$, then we have $f \in C(X,G)$ such that $f(\ell \circ j(x)) \neq f(x)$ or $f \circ \ell \circ j(x) \neq f(x)$. Hence $(\phi^{-1}(f) \circ j)(x) \neq f(x)$, and thus $\phi(\phi^{-1}(f))(x) \neq f(x)$ which leads to $f(x) \neq f(x)$. Similarly, $j \circ \ell(y) = y$. Hence j is a homeomorphism of X onto Y.

For topological spaces X and Y, it is clear that the space $C(X,Y)$ may be embedded into the space $C(X \times Z,Y)$ as a retract for any space Z, and that every homomorphism of the topological group $C(X,G)$ into a topological group L may be extended to a homomorphism of the topological group $C(X \times Y,G)$ into L for any topological group L. We shall conclude this paper with the following result concerning an extension of F-homomorphisms.

THEOREM 13. Suppose A is a closed subset of X. Then every constant-preserving F-homomorphism h of the topological group $C(G,G)$ into the topological group $C(A,G)$ may be extended to a homomorphism H of the same kind from the topological group $C(G,G)$ into the topological group $C(X,G)$ such that $I \circ H = h$ if every continuous function $f\colon A \to G$ may be continuously extended to all of X, where $I\colon C(X,G) \to C(A,G)$ be the map defined by $I(f) = f \circ i$ for $f \in C(X,G)$, i being the inclusion map of A into X.

PROOF: For necessity, let $f\colon$ A G be any continuous function, and let $f^*\colon$ $C(G,G)$ $C(A,G)$ be the natural homomorphism induced by f, namely $f^*(k) =$

$k \circ f$ for each $k \in C(G,G)$. Then f^* is a constant-preserving F-homomorphism, by Theorem 11. Hence there exists a constant-preserving F-homomorphism H of the topological group $C(G,G)$ into $C(X,G)$ such that $I \circ H = f^*$. Let $\theta \in C(X,G)$ such that $H(k) = k \circ \theta$ for every $k \in C(G,G)$. If i_d denotes the identity map of G into itself, then, for $a \in A$, $\theta(a) = (i_d \circ \theta)(a) = H(i_d)(a) = H(i_d)(i(a)) = H(i_d) \circ i(a) - I(H(i_d))(a) = (I \circ H)(i_d)(a) = f^*(i_d)(a)\ (i_d \circ f)(a) = f(a)$. Hence θ is an extension of f to all of X.

For sufficiency, assume that every continuous function $f\colon A \to G$ may be extended continuously to all of X, and let $h\colon C(G,G) \to C(A,G)$ be a constant-preserving homomorphism of the topological group $C(G,G)$ into the topological group $C(A,G)$. Then there exists $f \in C(A,G)$ such that $h(k) = k \circ f$ for every $k \in C(G,G)$. If we denote by f the extension of $\hat{f}$ to all of X, define a function H: $C(G,G) \to C(X,G)$ by $H(k) = k \circ \hat{f}$ for each $k \in C(G,G)$. Then H is a constant-preserving F-homomorphism and $I \circ H = h$. This completes the proof.

In particular, if X is a normal space, and A a closed subset of X, then every constant-preserving F-homomorphism h of the topological group $C(R,R)$ into the topological group $C(Z,R)$ may be extended to a homomorphism H of the same kind from the topological group $C(R,R)$ into the topological group $C(X,G)$ such that $I \circ H = h$.

REFERENCES

1. Geoghegan, R., On spaces of homeomorphisms, embeddings, and functions I, Topology, Vol. 11 (1972), 159-177.

2. Gillman, L. and Jerrison, M., Rings of continuous functions, Van Nostrand, N. Y. (1960).

3. Larsen, R., Banach algebra, Dekker, N. Y. (1973).

4. Magill, K. D. Jr., Some Homomorphism theorems for a class of semigroups, Proc. London Math. Soc., Vol. 15 (1965), 517-526.

5. Nickolas, P., Free topological groups and function spaces, Preprint.

6. Yang, J. S., Transformation groups of automorphisms of C(X,G), Proc. Amer. Math. Soc. Vol. 39 (1973), 619-624; erratum, ibid Vol. 48 (1975), 518.

7. Yang, J. S., On isomorphic groups and homeomorphic spaces, Proc. Amer. Math. Soc. Vol. 43 (1974), 431-438; erratum, ibid, Vol. 48 (1975), 518.

Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São Josè dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk