ON THE HANKEL DETERMINANTS OF

CLOSE-TO-CONVEX UNIVALENT FUNCTIONS

K. INAYAT NOOR

Department of Mathematics
Kerman University
Kerman, Iran

ABSTRACT. The rate of growth of Hankel determinant for close-to-convex functions

is determined. The results in this paper are best possible.

KEY WORDS AND PHRASES. *Starlike and close-to-convex Functions, Hankel Determinant*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES.   30A32.

1.   INTRODUCTION.

Let K and S* be the classes of close-to-convex and starlike functions in

$\gamma = \{z: |z| < 1\}$. Let f be analytic in γ and $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$. The qth

Hankel determinant of f is defined for $q \geq 1$, $n \geq 1$ by

$$
H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots\cdots & a_{n+q-1} \\ a_{n+1} & \cdots\cdots\cdots\cdots & \\ \vdots & & \\ a_{n+q-1} & \cdots\cdots\cdots\cdots & a_{n+2q-2} \end{vmatrix}
$$

For $f \in S*$, Pommerenke [2] has solved the Hankel determinant problem completely. Following essentially the same method, we extend his results in this paper to the class K.

2. MAIN RESULTS.

THEOREM 1. Let $f \in K$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then, for $m = 0,1,...$, there are numbers γ_m and $c_{m\mu}$ ($\mu = 0, ..., m$) that satisfy $|c_{mo}| = |c_{mm}| = 1$ and

$$\sum_{k=0}^{\infty} \gamma_k \leq 3 \ , \ 0 \leq \gamma_m \leq \frac{2}{m+1} \qquad (2.1)$$

such that

$$\sum_{\mu=0}^{m} c_{m\mu} a_{n+\mu} = 0(1)n^{-1+\gamma_m} \qquad (n \to \infty).$$

The bounds (2.1) are best possible.

PROOF. Since $f \in K$, there exists $g \in S*$ such that, for $z \in \gamma$

$$zf'(z) = g(z)h(z), \ Reh(z) > 0 \qquad (2.2)$$

Now g can be represented as in [1], $g(z) = z \exp [\int_0^{2\pi} \log \frac{1}{1-ze^{-it}} d\mu(t)]$, where $\mu(t)$ is an increasing function and $\mu(2\pi) - \mu(0) = 2$. Let $\alpha_1 \geq \alpha_2 \geq ...$ be the jumps of $\mu(t)$, and $t=\theta_1, \theta_2,...$ be the values at which these jumps occur. We may assume that $\theta_1 = 0$. Then $\alpha_1 + \alpha_2 + ... \leq 2$ and $\alpha_1 + \alpha_2 + ... + \alpha_q = 2$ for some q if and only if g is of the form

$$g(z) = z \prod_{j=1}^{q} (1-e^{-i\theta_j} z)^{\frac{-2}{q}} \qquad (2.3)$$

We define ϕ_m by

$$\phi_m(z) = \prod_{\mu=1}^{m} (1-e^{i\theta_\mu} z),$$

and

$$\beta_m = \alpha_{m+1} \quad (m = 0,1,...)$$

We consider the three cases i.e.

(i) $0 \le \alpha_1 \le 1$, (ii) $1 < \alpha_1 < \frac{3}{2}$, (iii) $\frac{3}{2} \le \alpha_1 \le 2$

as in [2] and the first part, that is the bounds (2.1), follows similarly. For

the rest, we need the following which is well-known [2].

LEMMA. Let $\theta_1 < \theta_2 < \ldots < \theta_q < \theta_1 + 2\pi$, let $\lambda_1, \ldots, \lambda_q$ be real, and

$\lambda > 0$, $\lambda \ge \lambda_j$ (j=1, ..., q). If

$$\psi(z) = \prod_{j=1}^{q} (1-e^{-i\theta_j}z)^{-\lambda_j} = \sum_{n=1}^{\infty} b_n z^n \tag{2.4}$$

then $b_n = O(1) n^{\lambda-1}$ as $n \to \infty$.

We write

$$\phi_m(z) = \sum_{\mu=0}^{m} c_{m\mu} z^{m-\mu},$$

and

$$\phi_m(z)zf'(z) = \sum_{n=1}^{m} b_{mn} z^{n+m} + \sum_{n=1}^{\infty} (n+m)a_{mn} z^{n+m} \tag{2.5}$$

where

$$b_{mn} = \sum_{\nu=0}^{n} (n+\nu) c_{m-\nu} a_{n-\nu},$$

$$a_{mn} = \sum_{\mu=0} c_{m\mu} a_{n+\mu}, \quad |c_{mo}| = |c_{mm}| = 1.$$

There are two cases.

(a) Let g in (2.2) have the form (3); that is, $\alpha_1 + \alpha_2 + \ldots + \alpha_q = 2$.

With $\gamma_m = \beta_m$, it follows that $\gamma_m \le \frac{2}{m+1}$, $\gamma_0 + \gamma_1 + \ldots \le 3$ and $\lambda_m = \frac{2}{m+1}$ implies

$m = q-1$, $\alpha_1 = \ldots = \alpha_q = \frac{2}{q}$.

Now from (2.2), (2.5) and the Cauchy Integral formula, we have, with

$$B_m(r) = \frac{1}{r^{m+n}} \sum_{k=1}^{m} | b_{mk} | r^{k+m},$$

$$(n+m) |a_{mn}| \le \frac{1}{2\pi r^{n+m}} \int_0^{2\pi} |\phi_m(z)g(z)h(z)| d\theta + B_m(r). \tag{2.6}$$

Applying the Schwarz inequality, we have

$$(n+m)|a_{mn}| \leq \frac{1}{2\pi r^{n+m}} \left(\int_0^{2\pi} |\phi_m(z)g(z)|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} |h(z)|^2 d\theta \right)^{\frac{1}{2}} + B_m(r) .$$

When we write $[\phi_m(z)g(z)]^2$ in the form (2.4), the exponents $-\lambda_j$ satisfy $\lambda_j \leq 2\gamma_m$ (j=1, ... q: m > 0). Hence, using the Lemma, we have

$$\int_0^{2\pi} |\phi_m(z)g(z)|^2 d\theta \leq A \, n^{2\gamma_m - 1}, \quad (n \to \infty). \qquad (2.7)$$

Also

$$\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta = \sum_{n=0}^{\infty} |d_n|^2 r^{2n} \ (d_o=1), \ \mathrm{Re} h(z) > 0$$

But $|d_n| \leq 2$, $n \geq 1$, and so

$$\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \leq 1+4 \sum_{n=1}^{\infty} r^2 n = \frac{1+3r^2}{1-r^2} \leq An, \ n \geq 1 \qquad (2.8)$$

From (2.7) and (2.8), we have

$$(n+m)|a_{mn}| \leq An^{\gamma_m} \quad (n \to \infty)$$

i.e. $a_{mn} = O(1) \, n^{\gamma_m - 1} \qquad (n \to \infty).$

This proves the theorem in this case.

(b) Let g in (2.2) be not of the form (2.3). Then using arguments like those in [2], it follows that, for $z = re^{i\theta}$

$$\int_0^{2\pi} |\phi_m(z)g(z)h(z)| d\theta = O(1)(1-r)^{-\gamma_m} .$$

Hence from (2.6), we have

$$a_{mn} = O(1)n^{\gamma_m - 1} \quad (n \to \infty),$$

where a_{mn} is defined by (5).

The function f_o: $f_o(z) = z(1-z^q)^{-2/q} = \sum_{\nu=0}^{\infty} \binom{2/q+\nu-1}{\nu} z^{\nu q+1}$, shows that the

bounds (1) are best possible. We also note that except in the case where

$m=(q-1)$ and g in (2.2) is not of the form (2.3), one can choose $0 \le \gamma_m > \frac{2}{m+1}$

from theorem (1) and Pommerenke's method [2], we can now easily prove the

following

THEOREM 2. Let $f \in K$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Then for $q \ge 1$, $n \ge 1$,

$$H_q(n) = O(1)n^{2-q} \quad (n \to \infty)$$

This estimate is best possible. In particular, if g in (2.2) is not of the form

(2.3), there exists a $\delta = \delta(q,g) > 0$

such that $H_q(n) = O(1)n^{2-q-\delta} \quad (n \to \infty)$.

REFERENCES

[1] Pommerenke, Ch. On Starlike and Convex Functions, J. London Math. Soc. 37 (1962) 209-224.

[2] Pommerenke, Ch. On the Coefficients and Hankel Determinants of Univalent Functions, J. London Math. Soc. 41 (1966) 111-122.

Special Issue on
Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www .hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http:// mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São Josè dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk