

REPRESENTATION OF CERTAIN CLASSES OF DISTRIBUTIVE LATTICES BY SECTIONS OF SHEAVES

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ABSTRACT. Epstein and Horn ([6]) proved that a Post algebra is always a P-algebra and in a P-algebra, prime ideals lie in disjoint maximal chains. In this paper it is shown that a P-algebra L is a Post algebra of order $n \geq 2$, if the prime ideals of L lie in disjoint maximal chains each with $n-1$ elements. The main tool used in this paper is that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space. Also some properties of P-algebras are characterized in terms of the stalks.

KEY WORDS AND PHRASES. Post Algebra, P-algebra, B-algebra, Heyting Algebra, Stone Lattice, Boolean Space, Sheaf of Distributive Lattices Over a Boolean Space, Prime Ideals Lie in Disjoint Maximal Chains, Regular Chain Base.

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1. INTRODUCTION.

Epstein ([5]) proved that in a Post algebra of order $n \geq 2$ prime ideals lie in disjoint maximal chains each with $n - 1$ elements. He has also proved that if L is a finite distributive lattice and prime ideals of L lie in disjoint maximal chains each with $n-1$ elements, then L is a Post algebra of order n . Epstein and Horn ([6]) have shown that a Post algebra is always a P-algebra and in a P-algebra prime ideals lie in disjoint maximal chains. It is the main theme of this paper that a P-algebra L is a Post algebra of order $n \geq 2$, if the prime ideals of L lie in disjoint maximal chains each with $n-1$ elements.

The main tool used in this paper is the fact that every bounded distributive lattice is isomorphic with the lattice of all global sections of a sheaf of bounded distributive lattices over a Boolean space ([15] and [9]). It is well known that a P-algebra is always a (double) Heyting algebra, a (double) L-algebra, a pseudocomplemented lattice, a Stone lattice and a normal lattice. We characterize these properties of P-algebras in detail in terms of the stalks of the corresponding sheaf. We give another characterization of Post algebras by regular chain bases.

Throughout this paper, by L we mean a (nontrivial) bounded distributive lattice $(L, \vee, \wedge, 0, 1)$ and $B = B(L)$ the Boolean algebra of complemented elements of L . For any $a \in B$, we denote the complement of a by a' . For any $x, y \in L$, we denote the largest $z \in L$ such that $x \wedge z \leq y$ (if it exists) by $x \rightarrow y$ and the largest element $a \in B$ such that $x \wedge a \leq y$ (if it exists) by $x \Rightarrow y$. If, for every $x, y \in L$, $x \rightarrow y$ ($x \Rightarrow y$) exists, then we say that L is a Heyting algebra (respectively B algebra). Dually, we define $x \leftarrow y$ and $x \Leftarrow y$ respectively. If in a Heyting algebra (B-algebra), $(x \rightarrow y) \vee (y \rightarrow x) = 1$ ($(x \Rightarrow y) \vee (y \Rightarrow x) = 1$) for every $x, y \in L$, then we say that L is an L-algebra

(respectively BL-algebra). For any $x \in L$, if $x \rightarrow 0$ exists, then we say that x has the pseudocomplement and we usually write x^* for $x \rightarrow 0$. If x^* exists for each $x \in L$, then we say that L is pseudocomplemented. The dual of L is denoted by L^d . If both L and L^d are Heyting algebras (B-algebras, L-algebras, BL-algebras), then we say that L is a double Heyting algebra (respectively double B-algebra, double L-algebra, double BL-algebra). L is said to be a P-algebra if L is a BL-algebra. Epstein and Horn proved that L is a P-algebra if and only if L is a double L-algebra ([6], theorem 3.4). For the elementary properties of these types of lattices, we refer to ([2]) and ([6]).

By a sheaf of bounded distributive lattices we mean a triple (\mathcal{Y}, π, X) satisfying the following:

- i) \mathcal{Y} and X are topological spaces
- ii) $\pi : \mathcal{Y} \rightarrow X$ is a local homeomorphism
- iii) Each $\pi^{-1}(p)$, $p \in X$ is a bounded distributive lattice;
- iv) the maps $(x,y) \mapsto x \vee y$ and $(x,y) \mapsto x \wedge y$ from $\mathcal{Y} \times \mathcal{Y} = \{(x,y) \in \mathcal{Y} \times \mathcal{Y} / \pi(x) = \pi(y)\}$ into \mathcal{Y} are continuous and
- v) the maps $\hat{0} : p \mapsto 0(p)$ and $\hat{1} : p \mapsto 1(p)$ from $X \rightarrow Q$ are continuous, where $0(p)$ and $1(p)$ are the smallest and largest elements of $\pi^{-1}(p)$ respectively.

We call \mathcal{Y} the sheaf space X the base space and π the projection map. We write \mathcal{Y}_p for $\pi^{-1}(p)$, $p \in X$ and call \mathcal{Y}_p the stalk at p . By a (global) section of the sheaf (\mathcal{Y}, π, X) we mean a continuous map $\sigma : X \rightarrow \mathcal{Y}$ such that $\pi \circ \sigma = id_X$. For any sections σ and τ we write $|\langle \sigma, \tau \rangle|$ for the closed set $\{p \in X | \sigma(p) \neq \tau(p)\}$ and we call $|\langle \sigma, 0 \rangle|$ the support of σ and write $|\sigma|$ for $|\langle \sigma, 0 \rangle|$. For the preliminary results on sheaf theory, we refer to the pioneering work of Hofmann ([8]).

By $\text{Spec } L$, we mean the (Stone) space Y of all prime ideals of L with the hull-kernel topology; i.e., the topology for which $\{Y_x \mid x \in L\}$ is a base, where for any $x \in L$, $Y_x = \{P \in \text{Spec } L/x \notin P\}$. Throughout this paper X denotes $\text{Spec } B$ which is a Boolean space, i.e., a compact, Hausdorff and totally disconnected space. Since $a \mapsto X_a$ is a Boolean isomorphism of B onto the Boolean algebra of all clopen subsets of X , we identify a and X_a and write simply a for X_a . For any $p \in X$, \mathcal{J}_p be the quotient lattice L/θ_p where θ_p is the congruence on L given by

$$(x, y) \in \theta_p \iff x \wedge a = y \wedge a \text{ for some } a \in B-p,$$

and let \mathcal{Y} be the disjoint union of all \mathcal{J}_p , $p \in X$. For each $x \in L$, define $\hat{x} : X \rightarrow \mathcal{Y}$ by $\hat{x}(p) = \theta_p(x)$, the congruence class of θ_p containing x . Topologize \mathcal{Y} with the largest topology such that each \hat{x} , $x \in L$, is continuous. Define $\pi : \mathcal{Y} \rightarrow X$ by $\pi(s) = p$ if $s \in \mathcal{J}_p$. The following theorem is the main tool used in this paper and is due to Subrahmanyam ([15]) (see also [9]).

THEOREM 1.1 (\mathcal{Y}, π, X) described above is a sheaf of bounded distributive lattices in which each stalk has exactly two complemented elements, viz., $0(p)$ and $1(p)$.

1.2 For any $a \in B$, $p \in X$, $\hat{a}(p) = 1(p)$ if $p \in a$ and $\hat{a}(p) = 0(p)$ if $p \notin a$.

1.3 For any $x, y \in L$ and $a \in B$, $\hat{x}/a = \hat{y}/a$ if and only if $x \wedge a = y \wedge a$.

1.4 $x \mapsto \hat{x}$ is an isomorphism of L onto the lattice $\Gamma(x, \mathcal{Y})$ of all global sections of the sheaf (\mathcal{Y}, π, X) . We identify \hat{x} with x and write simply x for \hat{x} .

1.5 For any prime ideal P of L , there exists a unique $p \in X$ such that $\{x(p)/x \in P\}$ is a prime ideal of \mathcal{J}_p . On the other hand if Q is a prime ideal of \mathcal{J}_p , $p \in X$, then $\{x \in L/x(p) \in Q\}$ is a prime ideal of L . This

correspondence exhibits the set of all prime ideals of L as the disjoint union of the sets of prime ideals of the stalks. Moreover, if P and Q are prime ideals of distinct stalks \mathcal{F}_p and \mathcal{F}_q , then P and Q are incomparable, when they are regarded as prime ideals of L .

Throughout this paper, by a stalk \mathcal{F}_p , $p \in X$, we mean the stalks of the sheaf (\mathcal{F}, π, X) described above at p .

2. PSEUDOCOMPLEMENTED LATTICES.

It is well known that a bounded distributive lattice is a Heyting algebra if and only if it is relatively pseudocomplemented; i.e., each interval $[x, y]$, $x \leq y \in L$ is pseudocomplemented ([1]). Also the class of all distributive pseudocomplemented lattices and the class of all Heyting algebras are equationally definable (see [1] and [11]), when we regard the pseudocomplementation and $(x, y) \mapsto (x \rightarrow y)$ as unary and binary operations respectively in L . Thanks to the referee for suggesting a simpler proof of the following.

THEOREM 2.1. L is pseudocomplemented if and only if each stalk \mathcal{F}_p , $p \in X$ is pseudocomplemented and the pseudocomplementation $x \mapsto x^*$ is continuous and in this case, the pseudocomplement of $x(p)$ in \mathcal{F}_p is precisely $x^*(p)$ for all $x \in L$.

PROOF. Suppose L is pseudocomplemented. Then it is easily seen that for all x and p , $(x(p))_{\mathcal{F}_p}^*$ exists and is equal to $x^*(p)$. Then it is easy to show that the map $x \mapsto x^*$ is continuous. For the converse, if $x \in L$, the hypothesis implies that the map $f : x \rightarrow \mathcal{F}$ defined by $f(p) = (x(p))^*$ is a global section of \mathcal{F} . Therefore, $f = \hat{y}$ for some y and it is clear that $y = x^*$.

If L is a Heyting algebra, then each θ_a , $a \in B$, is compatible with the binary operation $(x, y) \mapsto (x \rightarrow y)$. For, if $a \in B$ and (x, y) and $(x_1, y_1) \in \theta$ then $(x \rightarrow x_1) \wedge y \wedge a = (x \rightarrow x_1) \wedge x \wedge a \leq x_1 \wedge a = y_1 \wedge a \leq y_1$, so that $(x \rightarrow x_1) \wedge a \leq (y \rightarrow y_1) \wedge a$. Similarly, we have $(y \rightarrow y_1) \wedge a \leq (x \rightarrow x_1) \wedge a$ and hence

$(x \rightarrow x_1, y \rightarrow y_1) \in \theta_a$. Hence the following theorem and its proof are analogous to the above.

THEOREM 2.2. L is a Heyting algebra if and only if each stalk $\mathcal{J}_p, p \in X$ is a Heyting algebra, and the operation $(s, t) \mapsto (s \rightarrow t)$ of $\mathcal{J}V\mathcal{J}$ into \mathcal{J} is continuous and in this case $x(p) \rightarrow y(p)$ in $\mathcal{J}_p, p \in X$, is equal to $(x \rightarrow y)(p)$ for all $x, y \in L$.

3. NORMAL LATTICES.

DEFINITION 3.1. (Cornish [4]). L is said to be normal if any two distinct minimal prime ideals of L are comaximal and L is said to be relatively normal if each interval $[x, y], x \leq y \in L$ is normal.

For any $x, y \in L$, let $(x, y)_L^*$ be the ideal $\{z \in L / x \wedge z \leq y\}$ of L . For any $x \in L$, we write $(x)_L^*$ for $(x, 0)_L^*$. Cornish ([4]) proved that L is normal if and only if $(x \wedge y)_L^* = (x)_L^* \vee (y)_L^*$ for all $x, y \in L$, and that L is relatively normal if and only if $(x \wedge y, z)_L^* = (x, z)_L^* \vee (y, z)_L^*$ for all $x, y, z \in L$ where \vee stands for the join operation in the lattice of all ideals of L .

THEOREM 3.2. (Speed [13]). A pseudocomplemented distributive lattice is normal if and only if it is a Stone lattice.

THEOREM 3.3. (Balbes and Horn [1]): A Heyting algebra is relatively normal if and only if it is an L -algebra.

THEOREM 3.4. L is normal if and only if each stalk $\mathcal{J}_p, p \in X$, is normal.

PROOF. Suppose L is normal and $p \in X$. Let $u, v \in \mathcal{J}_p$ so that $u = x(p)$ and $v = y(p)$ for some $x, y \in L$. Clearly $(u)_{\mathcal{J}_p}^* \vee (v)_{\mathcal{J}_p}^* \subseteq (u \wedge v)_{\mathcal{J}_p}^*$. Let $t(p) \in (u \wedge v)_{\mathcal{J}_p}^*, t \in L$. Since, $(x \wedge y \wedge t)(p) = x(p) \wedge y(p) \wedge t(p) = 0(p)$ there exists a $a \in B-p$ such that $x \wedge y \wedge t \wedge a = 0$, so that $t \in (x \wedge y \wedge a)_L^* = (x \wedge a)_L^* \vee (y \wedge a)_L^*$ and hence $t = t_1 \vee t_2$ for some $t_1 \in (x \wedge a)_L^*$ and $t_2 \in (y \wedge a)_L^*$. Now $t(p) = t_1(p) \vee t_2(p), t_1(p) \in (u)_{\mathcal{J}_p}^*$ and $t_2(p) \in (v)_{\mathcal{J}_p}^*$. Hence \mathcal{J}_p is normal. Conversely, suppose each stalk $\mathcal{J}_p, p \in X$ is normal. Let $x, y \in L$ and

$z \in (x \wedge y)_L^*$. For each $p \in X$, since $z(p) \in (x(p) \wedge y(p))_{\mathcal{F}_p}^* = (x(p))_{\mathcal{F}_p}^* \vee (y(p))_{\mathcal{F}_p}^*$, there exists $a \in B-p$, t and $s \in L$, such that $a \wedge z = a \wedge (t \vee s)$, $t \wedge x \wedge a = s \wedge y \wedge a = 0$. By the compactness of X , it follows that there exists $a_1, a_2, \dots, a_n \in B$ and $t_1, t_2, \dots, t_n, s_1, \dots, s_n \in L$ such that $\bigvee_{i=1}^n a_i = 1$, $a_i \wedge z = a_i \wedge (t_i \vee s_i)$, $t_i \wedge x \wedge a_i = 0 = s_i \wedge y \wedge a_i$. Now, Put $t = \bigvee_{i=1}^n (t_i \wedge a_i)$ and $s = \bigvee_{i=1}^n (s_i \wedge a_i)$ then, $z = \bigvee_{i=1}^n (z \wedge a_i) = \bigvee_{i=1}^n (a_i \wedge (t_i \vee s_i)) = t \vee s$ and $t \wedge x = \bigvee_{i=1}^n (t_i \wedge a_i \wedge x) = 0 = \bigvee_{i=1}^n (s_i \wedge a_i \wedge y) = s \wedge y$. Hence $(x \wedge y)_L^* \subseteq (x)_L^* \vee (y)_L^*$ and the otherside inclusion is obvious. Hence L is normal.

The proof of the following theorem is analogous to that of the above.

THEOREM 3.5. L is relatively normal if and only if each stalk $\mathcal{F}_p, p \in X$, is relatively normal.

DEFINITION 3.6. (Speed [12]). L is said to be a distributive $*$ lattice and denoted by $L \in \Delta^*$ if, for each $x \in L$, there exists $y \in L$ such that $(x)_L^{**} := \{u \in L / u \wedge v = 0 \text{ for every } v \in (x)_L^*\} = (y)_L^*$.

Speed ([12] proved that $L \in \Delta^*$ if and only if, for each $x \in L$, there exists $y \in L$, such that $x \wedge y = 0$ and $x \vee y$ is dense; i.e., $(x \wedge y)_L^* = \{0\}$.

THEOREM 3.7. $L \in \Delta^*$ if and only if (i) $\mathcal{F}_p \in \Delta^*$ for each $p \in X$ and (ii) $\{p \in X \mid x(p) \text{ is dense in } \mathcal{F}_p\}$ is open for each $x \in L$.

PROOF. Suppose $L \in \Delta^*$ and $x \in L$. There exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is dense in L . Let $p \in X$. Clearly, $x(p) \wedge y(p) = 0(p)$. Also, if $z \in L$, such that $((x(p) \vee y(p)) \wedge z(p) = 0(p)$, then $(x \vee y) \wedge z \wedge a = 0$ for some $a \in B-p$ and hence $z \wedge a = 0$, so that $z(p) = 0(p)$. Hence $x(p) \vee y(p)$ is dense in \mathcal{F}_p . Therefore $\mathcal{F}_p \in \Delta^*$. Now, suppose $x(p)$ is dense in \mathcal{F}_p . It follows that $y(p) = 0(p)$ and hence there exists $a \in B-p$ such that $y \wedge a = 0$.

We claim that $x(q)$ is dense in \mathcal{Y}_q for all $q \in a$. For, if $q \in a$ and $z(q) \in \mathcal{Y}_q$, $z \in L$, such that $x(q) \wedge z(q) = 0(q)$, then there exists $b \in B - q$ such that $x \wedge z \wedge b = 0$; so that $(x \vee y) \wedge z \wedge a \wedge b = 0$, and hence $z \wedge a \wedge b = 0$ and since $a \wedge b \in B - q$, $z(q) = 0(q)$. Conversely, suppose (i) and (ii) hold and $x \in L$. For each $p \in X$, by (i) and (ii), there exists $y \in L$ and $a \in B - p$ such that $x \wedge y \wedge a = 0$ and $(x \vee y)(q)$ is dense in \mathcal{Y}_q for all $q \in a$. By the usual compactness argument, there exists $y_1, y_2, \dots, y_n \in L$, $a_1, \dots, a_n \in B$ such that $\bigvee_{i=1}^n a_i = 1$, $a_i \wedge a_j = 0$ for $i \neq j$, $x \wedge y_i \wedge a_i = 0$ and $(x \vee y_i)(p)$ is dense in \mathcal{Y}_p for all $p \in a_i$. Now put $y = \bigvee_{i=1}^n (y_i \wedge a_i)$. Then $x \wedge y = 0$ and $x \vee y$ is dense in L . For, $(x \vee y) \wedge z = 0$ for some $z \in L$, then, for all $p \in a_i$, $0(p) = ((x \vee y) \wedge z)(p) = (x(p) \vee y(p)) \wedge z(p)$ and hence $z(p) = 0(p)$ for all $p \in a_i$ and therefore $z = 0$. Hence $L \in \Delta^*$.

4. STONE LATTICES.

For any $p \in X$, since the stalk \mathcal{Y}_p has exactly two complemented elements, \mathcal{Y}_p is a Stone lattice if and only if \mathcal{Y}_p is dense (i.e., if $x(p) \neq 0(p)$, then $(x(p))_{\mathcal{Y}_p}^* = \{0(p)\}$). Hence, by theorem 2.1, 3.2, and 3.4, we have the following.

THEOREM 4.1. Suppose L is pseudocomplemented. Then the following are equivalent.

- (i) L is a Stone lattice
- (ii) L is normal
- (iii) Each stalk \mathcal{Y}_p , $p \in X$, is a normal
- (iv) Each stalk \mathcal{Y}_p , $p \in X$, is a Stone lattice
- (v) Each stalk \mathcal{Y}_p , $p \in X$, is dense.

The following theorem is a consequence of theorem 2.2, 3.3 and 3.5.

THEOREM 4.2. Let L be a Heyting algebra. Then the following are equivalent.

- (i) L is an L -algebra
- (ii) L is relatively normal
- (iii) Each stalk $\mathcal{Y}_p, p \in X$, is relatively normal
- (iv) Each stalk $\mathcal{Y}_p, p \in X$, is an L -algebra.

Since L is an L -algebra if and only if it is relatively Stone lattice (Theorem 4.11 of [1]) (i.e., each interval is a Stone lattice) in view of theorem 4.1, one may suspect that if L is an L -algebra, then each stalk is relatively dense and hence a chain. This is not true (see 4.4 below), though the converse is proved in the following.

THEOREM 4.3. If L is a Heyting algebra and each stalk is a chain, then L is an L -algebra.

PROOF. If each stalk is a chain, then by theorem 1.5, the prime ideals of L lie in disjoint maximal chains and hence L is relatively normal lattice and hence the theorem follows from theorem 2.3.

EXAMPLE 4.4. Let B_4 be the 4-element Boolean algebra and A be the distributive lattice obtained by adjoining an external element to B_4 as the smallest element. Then A is an L -algebra which is not a chain (Thanks to the referee).

Epstein and Horn ([6]) proved that L is a Stone lattice if and only if L^d is pseudosupplemented and $0 \ll x \wedge y = (0 \ll x) \wedge (0 \ll y)$ for all $x, y \in L$. Now, these two necessary and sufficient conditions for L to be a Stone lattice can be viewed in terms of the stalks as follows.

THEOREM 4.5. L^d is pseudosupplemented if and only if $|x|$ is open for each $x \in L$ and in this case $|x| = 0 \ll x$ for all $x \in L$.

PROOF. Follows from Lemma 5.2.

For any $p \in X$, let (p) be the smallest ideal of L containing p . The proof of the following theorem is simple.

THEOREM 4.6. For any $p \in X$, the stalk \mathcal{J}_p is dense if and only if (p) is a prime ideal of L .

It can be easily seen that each stalk \mathcal{J}_p , $p \in X$, is dense if and only if $|x \wedge y| = |x| \cap |y|$ for all $x, y \in L$. Hence from theorem 4.5 and 4.6 and lemma 2.9 of ([7]), we have the following.

THEOREM 4.7. L is a Stone lattice if and only if $|x|$ is open for all $x \in L$ and each stalk \mathcal{J}_p , $p \in X$ is dense.

REMARK 4.8. Swamy and Rama Rao ([10]) proved that a lattice L is a Stone lattice if and only if L is isomorphic to the lattice of all global sections of a sheaf of dense bounded distributive lattices over a Boolean space in which each section has open support (see also [9]). It can be verified, that when L is a Stone lattice, then our sheaf (\mathcal{J}, π, X) coincides with the sheaf constructed in ([10]).

5. P-ALGEBRAS.

The following results interpret B-algebras in sheaf theoretic terminology.

LEMMA 5.1. Let $x, y \in L$. Then $x \Rightarrow y$ exists in B if and only if $\{p \in X / x(p) \leq y(p)\}$ is closed and in this case $x \Rightarrow y = \{p \in X / x(p) \leq y(p)\}$.

PROOF. For any $p \in X$, $x(p) \leq y(p)$ if and only if there exists $a \in B-p$ such that $x \wedge a \leq y$. If $x \Rightarrow y$ exists in B , then, for any $p \in X$, $x(p) \leq y(p)$ if and only if $p \in x \Rightarrow y$. Conversely, if $\{p \in X / x(p) \leq y(p)\}$ is closed, then there exists $a \in B$ such that $p \in a$ if and only if $x(p) \leq y(p)$ for all $p \in X$. Hence $a = x \Rightarrow y$.

The proof of the following is easy.

LEMMA 5.2. For any $x, y \in L$, $|(x, y)|$ is open if and only if there exists a largest element a of B such that $x \wedge a = y \wedge a$.

The following theorem is a consequence of the above lemmas.

THEOREM 5.3. The following are equivalent.

- 1) L is a dual B -algebra
- 2) For any $x, y \in L$, $\{a \in B / x \vee a = y \vee a\}$ is a principal filter of B .
- 3) For any $x, y \in L$, $\{a \in B / x \wedge a = y \wedge a\}$ is a principal ideal of B .
- 4) L is a B -algebra
- 5) $\{p \in X / x(p) \leq y(p)\}$ is closed for every $x, y \in L$.
- 6) $|(x, y)|$ is open for every $x, y \in L$.

THEOREM 5.4. Suppose L is a B -algebra. Then the following are equivalent.

- 1) L is a P -algebra; i.e. L is a BL -algebra
- 2) Each stalk is a chain
- 3) For every $x, y \in L$, there exists $a \in B$ such that $x \wedge a \leq y$ and $y \wedge a' \leq x$.
- 4) For every $x, y \in L$, there exists $a \in B$ such that $x \vee a \geq y$ and $y \vee a' \geq x$.

PROOF. $2 \Leftrightarrow 3$ is proved in ([15]) and $3 \Leftrightarrow 4$ is clear. $1 \Leftrightarrow 2$ follows from lemma 5.1.

6. POST ALGEBRAS.

The following definition is slightly different from that of Chang and Horn ([3]).

DEFINITION 6.1. By a generalized Post algebra, we mean the lattice $C(Z, C)$ of all continuous maps of a Boolean space Z into a discrete bounded chain C where, the operations are pointwise.

THEOREM 6.2. The following are equivalent

- 1) L is a generalized Post algebra.
- 2) There exists a chain C in L such that the natural map $c \mapsto c(p) : C \rightarrow \mathcal{J}_p$ is an isomorphism for all $p \in X$.
- 3) There exists a chain C and, for each $p \in X$, an order isomorphism $\alpha_p : C \rightarrow \mathcal{J}_p$ such that for any $c \in C$ and $x \in L$, $\{p \in X / \alpha_p(c) = x(p)\}$ is open in X .

PROOF. 1 \Rightarrow 2:

Let $L = C(Z, D)$ where Z is a Boolean space and D is a discrete bounded chain. It is well known that a $\triangleright \chi_a$ is a Boolean isomorphism of the algebra of all clopen subsets of Z onto B , the centre of L , where χ_a is the characteristic function on a . We identify χ_a with a . Also the Stone space X is homeomorphic with Z .

Let C be the set of all constant maps of Z into D . For any $d \in D$, let \bar{d} denote the constant map which maps every element of Z onto d . Then C is a chain in L . Let $p \in X$. Clearly, the natural map $\theta_p : C \rightarrow \mathcal{Y}_p = L/\theta_p$ is a homomorphism.

If $d_1, d_2 \in D$ such that $\bar{d}_1(p) = \bar{d}_2(p)$ then $\bar{d}_1 \wedge a = \bar{d}_2 \wedge a$ for some $a \in B-p$ and hence $d_1 = d_2$. Now, let $x \in L$. Then if $p \in x^{-1}(d)$ for some $d \in D$, since $x : Z \rightarrow D$ is continuous, $x^{-1}(d) \in B-p$ and since $x \wedge x^{-1}(d) = \bar{d} \wedge x^{-1}(d)$, it follows that $(x, \bar{d}) \in \theta_p$. Hence θ_p is an isomorphism.

2 \Rightarrow 3: If C is a chain in L and the natural map $\theta_p : C \rightarrow \mathcal{Y}_p$ is an isomorphism for every $p \in X$, then, for any $c \in C$ and $x \in L$. $\{p \in X / \alpha_p(c) = x(p)\} = \{p \in X / c(p) = x(p)\}$ which is open.

3 \Rightarrow 1: We first observe that since \mathcal{Y}_p is bounded and α_p is an isomorphism of C onto \mathcal{Y}_p , C is also bounded. Let $X = \text{Spec } B$. Define $\theta : L \rightarrow C(X, C)$ by $(\theta(x))(p) = \alpha_p^{-1}(x(p))$ for each $x \in L$ and $p \in X$. Let $c \in C$. Then

$$\begin{aligned} (\theta(x))^{-1}\{c\} &= \{p \in X / \alpha_p^{-1}(x(p)) = c\} \\ &= \{p \in X / \alpha_p(c) = x(p)\} \text{ is open by (3) and} \end{aligned}$$

hence $\theta(x)$ is continuous. Clearly θ is a homomorphism and one-one since α_p^{-1} is so. Now, we will show that θ is onto. Let $f \in C(X, C)$. Define $\sigma : X \rightarrow \mathcal{Y}$ by $\sigma(p) = \alpha_p(f(p))$ for every $p \in X$. We will show that σ is a section. Let $x \in L$ and $a \in B$, then

$$\begin{aligned} \sigma^{-1}(x(a)) &= \{p \in a \mid \alpha_p(f(p)) = x(p)\} \\ &= a \cap \bigcup_{c \in C} \{p \in X \mid f(p) = c\} \cap \{p \in X \mid \alpha_p(c) = x(p)\}. \end{aligned}$$

Since f is continuous, it follows that $\sigma^{-1}(x(a))$ is open. Since $\{x(a) \mid a \in B \text{ and } x \in L\}$ is a base for the topology on \mathcal{Y} , it follows that σ is continuous and clearly $\pi \circ \sigma = \text{id}_X$. Therefore, $\sigma = \hat{x}$ for some $x \in L$ and also $\theta(x) = f$. Hence θ is an isomorphism and therefore L is a generalized Post algebra.

THEOREM 6.3. Let $n \geq 2$ and L a P -algebra. Then the following are equivalent.

- 1) L is a Post algebra of order n .
- 2) $\text{Spec } L$ is the disjoint union of maximal chains each with $n-1$ elements.
- 3) Each stalk is a chain with n elements.

PROOF. $1 \Rightarrow 2$ is proved in ([5]).

Since L is a P -algebra, by theorem 4.4, each stalk $\mathcal{Y}_p, p \in X$, a chain. Also, by theorem 0.(5), $\text{Spec } L$ is the disjoint union of the chains $\text{Spec } \mathcal{Y}_p, p \in X$.

If $\text{Spec } L$ is the disjoint union of all maximal chains $C_\alpha, \alpha \in \Delta$ each with $n-1$ elements, then, for any $p \in X, \text{Spec } \mathcal{Y}_p = C_\alpha$ for some $\alpha \in \Delta$. Hence $\text{Spec } \mathcal{Y}_p$ has $n-1$ elements and therefore \mathcal{Y}_p has n elements and hence $(2) \Rightarrow (3)$.

Now, suppose each stalk is a chain with n elements and C_n is the n -element chain $1 < 2 < \dots < n$. For any $p \in X$, let $\mathcal{Y}_p = \{0(p) = x_{1p}(p) < x_{2p}(p) < \dots < x_{np}(p) = 1(p)\}$ where $x_{1p}, x_{2p}, \dots, x_{np} \in L$. Define for any $p \in X, \alpha_p: C_n \rightarrow \mathcal{Y}_p$ by $\alpha_p(i) = x_{ip}(p)$ for each $i \in C_n$. Clearly, α_p is an order isomorphism. Let $i \in C_n, x \in L$ and $p \in X$ such that $\alpha_p(i) = x(p)$. i.e., $x_{ip}(p) = x(p)$ so that there exists $a \in B-p$ such that $x_{ip}(q) = x(q)$ for all $q \in a$. Since L is a B -algebra and $x_{jp}(p) < x_{kp}(p)$ for all $j < k$, by theorem 5.3, there exists $b \in B-p$ such that $x_{jp}(q) < x_{kp}(q)$ for all $j < k$ and $q \in b$ and hence $x_{ip}(q) = x_{iq}(q)$ for all $i \in C_n$ and $q \in b$. Then $p \in a \wedge b \in B$ and

for any $q \in a \wedge b$, $\alpha_q(i) = x_{iq}(q) = x_{ip}(q) = x(q)$. Hence $\{p \in X / \alpha_p(i) = x(p)\}$ is open for each $i \in C_n$ and $x \in L$.

DEFINITION 6.4. By a chain base C for L we mean a chain C with 0 in L such that L is generated by the centre B and C ; i.e., every $x \in L$ can be

written in the form $\bigvee_{i=1}^n (a_i \wedge c_i)$ for some $a_i \in B$ and $c_i \in C$.

DEFINITION 6.5. A chain base C in L is said to be regular, if, for $c_1 \neq c_2 \in C$ and $a \in B$, $c_1 < c_2$ and $a \wedge c_2 \leq c_1$ imply $a = 0$.

It is proved in ([15]) that a bounded distributive lattice L is a generalized Post algebra if and only if there exists a regular chain base for L . Now, we characterize chain bases and regular chain bases in terms of the stalks. It is also proved in ([15]) that if C is a chain base for L , the natural map $\theta_p : C \rightarrow \mathcal{J}_p$, defined by $\theta_p(c) = c(p)$ is an epimorphism for all $p \in X$. We prove the converse in the following.

THEOREM 6.6. Let C be a chain in L and $0 \in C$. Then $\theta_p : C \rightarrow \mathcal{J}_p$ is an epimorphism for each $p \in X$, if and only if C is a chain base for L .

PROOF. Suppose θ_p is an epimorphism for each $p \in X$ and let $x \in L$. For each $p \in X$, there exists $c_p \in C$ such that $\theta_p(c_p) = x(p)$ i.e., $c_p(p) = x(p)$, so that there exists $a \in B-p$ such that $c_p \wedge a = x \wedge a$. Therefore, there exists a partition a_1, a_2, \dots, a_n of B and $c_1, c_2, \dots, c_n \in C$ such that $c_i \wedge a_i = x \wedge a_i$ so that $x = x \wedge 1 = x \wedge \bigvee_{i=1}^n a_i = \bigvee_{i=1}^n (x \wedge a_i) = \bigvee_{i=1}^n (c_i \wedge a_i)$.

Hence C is a chain base for L .

The following theorem is a straight forward verification.

THEOREM 6.7. Let C be a chain in L . Then the following are equivalent.

- 1) The natural map $\theta_p : C \rightarrow \mathcal{J}_p$ is one for all $p \in X$.
- 2) For any $c_1 \neq c_2 \in C$ and $a \in B$, $c_1 < c_2$ and $a \wedge c_2 \leq c_1$ imply $a = 0$.

- 3) For any $c_1 \neq c_2 \in C$ and $0 \neq a \in B$, $a \wedge c_1 \neq a \wedge c_2$.

By summarizing the above results, we have the following :

THEOREM 6.8. Suppose L is a bounded distributive lattice. Then the following are equivalent.

- 1) L is a generalized Post algebra
- 2) There exists a chain C in L such that the natural map $\theta_p: C \rightarrow \mathcal{Y}_p$ is an isomorphism for all $p \in X$.
- 3) There exists a chain C and for each $p \in X$, an order isomorphism $\alpha_p: C \rightarrow \mathcal{Y}_p$ such that for any $c \in C$ and $x \in L$, $\{p \in X / \alpha_p(c) = x(p)\}$ is open in X .
- 4) L has a regular chain base.

REMARK. The equivalence of (1) and (4) is established in ([15]) by using the Boolean extension techniques.

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