ON A CLASS OF CONTINUOUS FLOW

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ABSTRACT. This paper involves characterizations of a class of continuous flows. The flows considered are those in which the positive prolongation of each point coincides with the trajectory through the point. The characterizations are based on the theory of prolongation.

KEY WORDS AND PHRASES. Flow, prolongation, trajectory, periodic, critical.

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1.   INTRODUCTION.

The purpose of this paper is to study dynamical systems in which the positive prolongation of each point coincides with the trajectory through the point. Ahmad [1] studied similar flows--flows in which the positive prolongation of each point coincides with the closure of the positive semi-trajectory through the point. He referred to such flows as flows of characteristic 0^+. Other authors

have studied these flows or similar ones (see [1], [2], [3], [5], [6]). Knight
[6], for example, gave characterizations of flows of characteristic 0, i. e.,
flows in which the prolongation of each point coincides with the closure of the
trajectory through the point. The author in [7] characterized flows in which
the positive prolongation of each point coincides with the positive semi-trajec-
tory through the point.

In this paper we give several characterizations of flows where the posi-
tive prolongation of each point coincides with the trajectory through the point.
We show that several seemingly different flows are all equivalent, thus making
it unnecessary to study all such flows. We also give a simple characterization
of our flows for the case where the phase space is restricted to the two dimen-
sional space R^2.

2.   DEFINITIONS AND NOTATIONS.

We shall let R, R^+, R^- represent the real numbers, the non-negative   real
numbers, and the non-positive real numbers, respectively. By a dynamical system
or a continuous flow is meant a pair (X, π), where X is a topological space, re-
ferred to as the phase space, and $\pi: X \times R \to X$ is a mapping satisfying the follow-
ing three axioms:

(1) $\pi(x, 0) = x$,

(2) $\pi(\pi(x, t), s) = \pi(x, t+s)$,

(3) π is continuous.

For convenience, we shall let $\pi(x, t)$ be denoted by $x \cdot t$, or simply xt. For each
$x \in X$, $C(x) = x \cdot R$, $(C^+(x) = x \cdot R^+)$, $(C^-(x) = x \cdot R^-)$ represents the trajectory (pos-
itive semitrajectory) (negative semitrajectory) through x. A point x in X   is
called a critical point if $xt = x$ for all t in R. A non-critical point is called
a periodic point if there exists a number t, $t > 0$, in R satisfying $xt = x$.   A

subset M of X is said to be <u>positively invariant</u> (<u>negatively invariant</u>) (<u>in-</u><u>variant</u>) if $C^+(M) = M(C^-(M) = M)$ $(C(M) = M)$. For each x in X, we let $K(x) =$ $\overline{C(x)}$, $K^+(x) = \overline{C^+(x)}$, and $K^-(x) = \overline{C^-(x)}$. The <u>positive limit set</u> of a point x in X is denoted by $L^+(x)$, i. e., $L^+(x) = \{y \in X: xt_i \to y$ for some net (t_i) of R^+ with $t_i \to +\infty\}$. Similarly, the <u>negative limit set</u> and the <u>limit set</u> of x are denoted by $L^-(x)$ and $L(x)$, respectively. The <u>positive prolongation</u> of a point x of X is denoted by $D^+(x)$, i.e., $D^+(x) = \{y \in X:$ there exist nets (x_i) of X and (t_i) of R^+ such that $x_i \to x$, $x_i t_i \to y\}$. Similarly, $D^-(x)$ and $D(x)$ denote the <u>negative prolongation</u> and the <u>prolongation</u> of x, respectively. The <u>positive prolongational limit set</u> of a point x in X is denoted by $J^+(x)$, i.e., $J^+(x) = \{y \in X:$ there exist nets (x_i) of X and (t_i) of R^+ such that $x_i \to x$, $t_i \to +\infty$, $x_i t_i \to y\}$. Similarly, $J^-(x)$ and $J(x)$ denote the <u>negative prolonga-</u><u>tional limit set</u> and the <u>prolongational limit set</u>, respectively. A point x in X is said to be <u>positively Poisson stable</u> if $x \in L^+(x)$. The negative and bi-lateral versions are defined similarly. A point x in X is said to be <u>nonwan-</u><u>dering</u> if $x \in J^+(x)$. A point x in X is said to be <u>positively dispersive</u> if $J^+(x) = \emptyset$. The negative and bilateral versions are defined similarly.

For more information on the above concepts and related notions pertinent to this paper one is referred to the bibliography.

Throughout this paper, the phase space is always assumed to be Hausdorff.

3. <u>CHARACTERIZATIONS OF FLOWS SATISFYING $D^+(x) = C(x)$</u>.

THEOREM 3.1. Let (X, π) be any continuous flow. Then the following condi-tions are equivalent:

(1) $D^+(x) = C(x)$ for all x in X.

(2) $J^+(x) = C(x)$ for all x in X.

(3) $J^-(x) = C(x)$ for all x in X.

(4) $J(x) = C(x)$ for all x in X.

(5) $D(x) = C(x)$ for all x in X, and there are no positively dispersive

     points.

(6) $D^-(x) = C(x)$ for all x in X.

PROOF. Assume that (1) holds. Let $x \in X$. Then $J^+(x) \neq \emptyset$. For, $J^+(x) = \emptyset$

implies that $D^+(x) = C^+(x) \cup J^+(x) = C^+(x)$. It follows from (1) that $C^+(x) =$

$C(x)$. Therefore, $x \cdot (-1) = x \cdot t$ for some $t \in R^+$, and hence $x \cdot (t+1) = x$.   This

shows that x is either a critical or a periodic point; in either case $L^+(x) =$

$C^+(x) \neq \emptyset$ implies that $J^+(x) \neq \emptyset$, which contradicts the assumption that $J^+(x) =$

$\emptyset$. We note that $J^+(x) \subset D^+(x) = C(x)$. But, since $J^+(x)$ is an invariant set,

we must also have $C(x) \subset J^+(x)$. Hence, $J^+(x) = C(x)$, and (2) holds.

Assume that (2) holds. Let $x \in X$. Then $x \in J^+(x)$. Hence, $x \in J^-(x)$ since,

in general, $x \in J^+(y)$ implies $y \in J^-(x)$ for any two points x, y in X.   Since

$J^-(x)$ is invariant, we have $C(x) \subset J^-(x)$. Now, let $y \in J^-(x)$. Then $x \in J^+(y) =$

$C(y)$, and hence $y \in C(x)$. This shows that $J^-(x) \subset C(x)$. Therefore, $J^-(x) =$

$C(x)$, and (3) holds.

Assume that (3) holds. Let $x \in X$. Then $C(x) = J^-(x) \subset J(x)$. Now, let

$y \in J^+(x)$. Then, $x \in J^-(y) = C(y)$, and hence $y \in C(x)$. This shows that $J^+(x)$

$\subset C(x)$. Hence, we have $J(x) = J^+(x) \cup J^-(x) \subset C(x)$. Consequently, $J(x) =$

$C(x)$, and (4) holds.

Assume that (4) holds. Let $x \in X$. Obviously, $D(x) = C(x)$, since $D(x) =$

$C(x) \cup J(x)$. To see that x is not positively dispersive, i. e., $J^+(x) \neq \emptyset$,

we note that by (4), $J^+(x) = \emptyset$ implies that $x \in J^-(x)$. But $x \in J^-(x)$ implies

that $x \in J^+(x)$, contradicting $J^+(x) = \emptyset$. Thus, (5) holds.

Assume that (5) holds. Let $x \in X$. Then, $\emptyset \neq J^+(x) \subset D(x) = C(x)$. Since

$J^+(x)$ is a nonempty invariant subset of $C(x)$, we must have $C(x) \subset J^+(x)$. Hence,

$x \in J^+(x)$, and, consequently, $x \in J^-(x)$. This shows that $C(x) \subset J^-(x)$. There-

fore, $C(x) \subset D^-(x)$ since $J^-(x) \subset D^-(x)$. It is obvious from (5) that $D^-(x) \subset$

$C(x)$. Thus, we have $D^-(x) = C(x)$, and (6) holds.

Finally, assume that (6) holds. Let $x \in X$. We note that $J^-(x) = \emptyset$ would

imply that $C^-(x) = C(x)$ since $D^-(x) = C^-(x) \cup J^-(x)$. But, as was shown earlier,

$C^-(x) = C(x)$ would imply that x is a critical or a periodic point, thus contra-

dicting $J^-(x) = \emptyset$. Thus, we have $\emptyset \neq J^-(x) \subset D^-(x) = C(x)$. Again, since $J^-(x)$

is invariant, we must have $C(x) \subset J^-(x)$. This implies that $x \in J^-(x)$, and hence,

$x \in J^+(x)$. Therefore, $C(x) \subset J^+(x) \subset D^+(x)$. Now, let $y \in D^+(x)$. Then,

$x \in D^-(y) = C(y)$. But, $x \in C(y)$ implies that $y \in C(x)$. This shows that $D^+(x)$

$\subset C(x)$. Therefore, $D^+(x) = C(x)$, and hence (1) holds. This completes the proof

of the theorem.

We note that if $D(x) = C(x)$, then $D(x) = K(x)$ since $C(x) \subset K(x) \subset D(x)$.

Thus, the equivalence of statements (1) and (5) in the preceding theorem shows

that the class of flows that we are studying here form a subclass of flows of

characteristic 0 (recall [6] that a flow is said to have characteristic 0 if

$D(x) = K(x)$ for all x in the phase space). In addition, we showed in the pre-

ceding theorem that $D^+(x) = C(x)$ implies that $J^+(x) \neq \emptyset$. Since $J^+(x)$ is in-

variant, we must have $C(x) \subset J^+(x)$. Therefore, we have

COROLLARY 3.2. If $D^+(x) = C(x)$ for each x in X, then the flow is a non-

wandering flow of characteristic 0.

EXAMPLE 3.3. Consider the flow (R^2, π) defined by the system of differ-

ential equations

$$\dot{x} = 1$$

$$\dot{y} = 0$$

It is easy to verify that this flow satisfies the condition $D(x) = C(x)$ for

all x in R^2 and is of characteristic 0. But any point x has the property $D^+(x)$ ≠ C(x). This shows that our flows form a proper subclass of flows of characteristic 0.

REMARK 3.4. A trivial example of a flow of characteristic 0 that satisfies the condition $D^+(x)$ = C(x) is a global Poincaré center.

THEOREM 3.5. Let (X,π) be a dynamical system where the phase space X is either a locally compact metric space or a complete metric space. Further, assume that $D^+(x)$ = C(x) for each x in X. Then the set of periodic and critical points is dense in X.

PROOF. Obviously, if x is either periodic or critical, then x ε C(x) = $L^+(x)$ = $L^-(x)$. On the other hand, suppose that there is a point x, which is Poisson stable but neither periodic nor critical. If x ε $L^+(x)$, then since $L^+(x)$ is invariant, we have C(x) ⊂ $L^+(x)$ ⊂ $D^+(x)$ = C(x). Therefore, $L^+(x)$ = C(x). But this contradicts a known result (see, e. g., [4]) that if X is either locally compact and metric or complete metric, then every point that is neither periodic nor critical satisfies $\overline{L^+(x) - C(x)}$ = $L^+(x)$. A similar argument shows that if x ε $L^-(x)$, then x is either periodic or critical. Thus, we have shown that the set of points that are either periodic or critical coincides with the set of Poisson stable points. We further note that by Corollary 3.2, all points of X are non-wandering. The proof of our theorem now follows from a known result (see, e. g., [4]) that if the phase space X is either locally compact and metric or complete metric, and if all points are nonwandering, then the set of Poisson stable points is dense in X.

THEOREM 3.6. Let (X,π) be any dynamical system, where, $X \subset R^2$. Then $D^+(x) = C(x)$ for all x in X if and only if the flow has characteristic 0 and there are no positively dispersive points.

PROOF. The "only if" part follows from Corollary 3.2. Now, suppose that the flow has characteristic 0 and $J^+(x) \neq \emptyset$ for each x in X. We have $J^+(x) \subset D(x) = K(x)$. First let us assume that $L(x) = \emptyset$. Then, $K(x) = C(x)$, and hence $J^+(x) \subset C(x)$. But $\emptyset \neq J^+(x) \subset C(x)$ implies that $C(x) \subset J^+(x)$. Hence, $J^+(x) = C(x)$, which implies that $D^+(x) = C(x)$. Now, let us assume that $L(x) \neq \emptyset$. Let $y \in L(x)$. Then $y \in J(x)$ and, hence, $x \in J(y) \subset D(y) = K(y)$. But, since $L(x)$ is a closed invariant set, we must have $K(y) \subset L(x)$. Therefore, $x \in L(x)$. Hence x is either a critical or a periodic point. This implies that $D(x) = K(x) = C(x)$. Thus, $D^+(x) \subset C(x) = L^+(x) \subset D^+(x)$. This shows that $D^+(x) = C(x)$, and the proof is complete.

NOTATION. Let (R^2,π) be any continuous flow and S be the set of critical points. For any $s \in S$ we shall hence forth let

$$N_s = \{x \in R^2: x = s \text{ or } x \text{ is periodic and } S \cap \text{int } C(x) = \{s\}\}$$

REMARK 3.7. Knight showed in [5] that there are six basic types of planar flows having characteristic 0. These are

(1) parallelizable flows,

(2) flows having a global Poincaré center,

(3) flows where S consists of one local Poincaré center s, N_s is un-bounded and ∂N_s is a single trajectory. The restriction of the flow to $R^2 - N_s$ is parallelizable,

(4) flows similar to Example 3 of [5],

(5) flows similar to Example 3 of [5], except that $\partial N_{s_1} = \partial N_{s_2}$ where

$S = \{s_1, s_2\}$, and

(6) flows having only critical points.

From this and from theorem 3.6, it follows that there are three basic types of flows (R^2, π) satisfying $D^+(x) = C(x)$ for each $x \in R^2$. These are the flows in (2), (5) and (6).

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