

A NOTE ON UNIFORMLY DOMINATED SETS OF SUMMING OPERATORS

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Received 26 May 2001

Let Y be a Banach space that has no finite cotype and p a real number satisfying $1 \leq p < \infty$. We prove that a set $\mathcal{M} \subset \Pi_p(X, Y)$ is uniformly dominated if and only if there exists a constant $C > 0$ such that, for every finite set $\{(x_i, T_i) : i = 1, \dots, n\} \subset X \times \mathcal{M}$, there is an operator $T \in \Pi_p(X, Y)$ satisfying $\pi_p(T) \leq C$ and $\|T_i x_i\| \leq \|T x_i\|$ for $i = 1, \dots, n$.

2000 Mathematics Subject Classification: 47B10.

1. Introduction. Let X and Y be Banach spaces and p a real number satisfying $1 \leq p < \infty$. A subset \mathcal{M} of $\Pi_p(X, Y)$ is called *uniformly dominated* if there exists a positive Radon measure μ defined on the compact space $(B_{X^*}, \sigma(X^*, X)|_{B_{X^*}})$ such that

$$\|Tx\|^p \leq \int_{B_{X^*}} |\langle x^*, x \rangle|^p d\mu(x^*) \quad (1.1)$$

for all $x \in X$ and all $T \in \mathcal{M}$. Since the appearance of Grothendieck-Pietsch's domination theorem for p -summing operators, there is a great interest in finding out the structure of uniformly dominated sets. We will denote by $\mathcal{D}_p(\mu)$ the set of all operators $T \in \Pi_p(X, Y)$ satisfying (1.1) for all $x \in X$. It is easy to prove that $\mathcal{D}_p(\mu)$ is absolutely convex, closed, and bounded (for the p -summing norm).

In [4], the authors consider the case $p = 1$ and prove that $\mathcal{M} \subset \Pi_p(X, Y)$ is uniformly dominated if and only if $\bigcup_{T \in \mathcal{M}} T^*(B_{Y^*})$ lies in the range of a vector measure of bounded variation and valued in X^* .

In [3], the following sufficient condition is proved: "let $\mathcal{M} \subset \Pi_p(X, Y)$ and $1 \leq p < \infty$. Suppose that there is a positive constant $C > 0$ such that, for every finite set $\{x_1, \dots, x_n\}$ of X , there exists $Q \in \mathcal{M}$ satisfying $\pi_p(Q) \leq C$ and

$$\sum_{i=1}^n \|Tx_i\|^p \leq \sum_{i=1}^n \|Qx_i\|^p \quad (1.2)$$

for all $T \in \mathcal{M}$. Then \mathcal{M} is uniformly dominated." They also prove that this condition is necessary in the rather particular case that $\mathcal{M} \subset \Pi_p(c_0, c_0)$ and $\mathcal{M} = \mathcal{D}_p(\mu)$ for some positive Radon measure μ on B_{ℓ_1} .

In this note, we obtain a necessary and sufficient condition for a set $\mathcal{M} \subset \Pi_p(X, Y)$ to be uniformly dominated, with the only restriction that Y is a Banach space without finite cotype. We refer to [1] for our operator terminology. If X is a Banach space, B_X will denote its closed unit ball; $\ell_a^p(X)$ ($\ell_w^p(X)$) will be the Banach space of the strongly (weakly) p -summable sequences.

2. Main result. We need the following characterization of uniformly dominated sets.

PROPOSITION 2.1. *Let $1 \leq p < \infty$ and $\mathcal{M} \subset \Pi_p(X, Y)$. The following statements are equivalent:*

- (a) \mathcal{M} is uniformly dominated.
- (b) For every $\varepsilon > 0$ and $(x_n) \in \ell_w^p(X)$, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n \geq n_0} \|T_n x_n\|^p < \varepsilon \quad (2.1)$$

for all sequences (T_n) in \mathcal{M} .

- (c) There exists a constant $C > 0$ such that

$$\sum_{i=1}^n \|T_i x_i\|^p \leq C^p \sup_{x^* \in B_{X^*}} \sum_{i=1}^n |\langle x^*, x_i \rangle|^p \quad (2.2)$$

for all $\{x_1, \dots, x_n\} \subset X$ and $\{T_1, \dots, T_n\} \subset \mathcal{M}$.

PROOF. (a) \Rightarrow (b). In a similar way as in the Pietsch factorization theorem [1], we can obtain, for all $T \in \mathcal{M}$, operators $U_T : L_p(\mu) \rightarrow \ell_\infty(B_{Y^*})$, $\|U_T\| \leq \mu(B_{X^*})^{1/p}$, and an operator $V : X \rightarrow L_\infty(\mu)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \searrow i_Y & \\ V \downarrow & & & & \ell_\infty(B_{Y^*}) \\ & & & & \\ L_\infty(\mu) & \xrightarrow{i_p} & L_p(\mu) & \xrightarrow{U_T} & \end{array} \quad (2.3)$$

Here i_p is the canonical injection from $L_\infty(\mu)$ into $L_p(\mu)$ and i_Y is the isometry from Y into $\ell_\infty(B_{Y^*})$ defined by $i_Y(y) = (\langle y^*, y \rangle)_{y^* \in B_{Y^*}}$. Given $\varepsilon > 0$ and $(x_n) \in \ell_w^p(X)$, we can choose $n_0 \in \mathbb{N}$ so that

$$\sum_{n \geq n_0} \|i_p \circ V(x_n)\|^p < \frac{\varepsilon}{\mu(B_{X^*})} \quad (2.4)$$

because $i_p \circ V$ is p -summing. Then, if (T_n) is a sequence in \mathcal{M} , we have

$$\begin{aligned} \sum_{n \geq n_0} \|T_n x_n\|^p &= \sum_{n \geq n_0} \|i_Y \circ T_n(x_n)\|^p \\ &= \sum_{n \geq n_0} \|U_{T_n} \circ i_p \circ V(x_n)\|^p \\ &\leq \mu(B_{X^*}) \sum_{n \geq n_0} \|i_p \circ V(x_n)\|^p \leq \varepsilon. \end{aligned} \quad (2.5)$$

(b) \Rightarrow (c). Using a standard argument, we can prove that \mathcal{M} is bounded for the operator norm. Hence, given $\hat{x} = (x_n) \in \ell_w^p(X)$, there exists $M_{\hat{x}} > 0$ such that

$$\sum_{n=1}^{\infty} \|T_n x_n\|^p \leq M_{\hat{x}} \quad (2.6)$$

for all (T_n) in \mathcal{M} . Then, we can consider the linear maps

$$\hat{T} : (x_n) \in \ell_w^p(X) \mapsto (T_n x_n) \in \ell_a^p(Y) \quad (2.7)$$

for each $\hat{T} = (T_n)$ in \mathcal{M} . They have closed graph; so, by the uniform boundedness principle, there exists $M > 0$ so that

$$\left(\sum_{n=1}^{\infty} \|T_n x_n\|^p \right)^{1/p} \leq M \epsilon_p(x_n) \quad (2.8)$$

for all $(x_n) \in \ell_w^p(X)$ and all (T_n) in \mathcal{M} (we wrote ϵ_p for the norm in $\ell_w^p(X)$).

(c) \Rightarrow (a). Given $A = \{T_1, \dots, T_n\} \subset \mathcal{M}$ and $B = \{x_1, \dots, x_n\} \subset X$, we define $f_{A,B} : B_{X^*} \rightarrow \mathbb{R}$ by

$$f_{A,B}(x^*) = C^p \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^p \right) - \sum_{i=1}^n \|T_i x_i\|^p \quad (2.9)$$

for all $x^* \in X^*$. We denote by \mathcal{P} the set of all functions $f_{A,B}$. It is clear that \mathcal{P} is convex and disjoint from the cone $\mathcal{N} = \{f \in \mathcal{C}(B_{X^*}) : f(x^*) < 0, \text{ for all } x^* \in B_{X^*}\}$. In a similar way as in the proof of Pietsch's domination theorem [1], we can show that there is a probability measure μ on B_{X^*} satisfying

$$\int_{B_{X^*}} (\|Tx\|^p - C^p |\langle x^*, x \rangle|^p) d\mu \leq 0 \quad (2.10)$$

for all $T \in \mathcal{M}$ and all $x \in X$. \square

As an application of this result, we can show a relatively compact set for the p -summing norm which is not uniformly dominated. Put $T_n = (1/n)e_n^* \otimes e_n$, $n \in \mathbb{N}$, where (e_n) and (e_n^*) are the unit basis of c_0 and ℓ_1 , respectively. As $\pi_1(T_n) = 1/n$, (T_n) is a null sequence in $\Pi_1(c_0, c_0)$, so (T_n) is relatively compact. To see that it is not uniformly dominated, we will use [Proposition 2.1](#): the sequence (e_n) is weakly summable but, for all $n \in \mathbb{N}$, we have

$$\sum_{k \geq n} \|T_k e_k\|_{\infty} = \sum_{k \geq n} \frac{1}{k}. \quad (2.11)$$

We are now ready to introduce our main result.

THEOREM 2.2. *Let Y be a Banach space that has no finite cotype, $\mathcal{M} \subset \Pi_p(X, Y)$, and $1 \leq p < \infty$. The following statements are equivalent:*

- (a) \mathcal{M} is uniformly dominated.
- (b) There is a constant $C > 0$ such that, for every $\{x_1, \dots, x_n\} \subset X$ and $\{T_1, \dots, T_n\} \subset \mathcal{M}$, there exists an operator $T \in \Pi_p(X, Y)$ satisfying $\pi_p(T) \leq C$ and

$$\|T_i x_i\| \leq \|Tx_i\|, \quad i = 1, \dots, n. \quad (2.12)$$

PROOF. (a) \Rightarrow (b). By hypothesis, there exists a positive Radon measure μ on B_{X^*} such that

$$\|Tx\| \leq \left(\int_{B_{X^*}} |\langle x^*, x \rangle|^p d\mu(x^*) \right)^{1/p} \quad (2.13)$$

for all $T \in \mathcal{M}$ and all $x \in X$. Since Y has no finite cotype, Y contains ℓ_∞^n 's uniformly. By [2], for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is an isomorphism J_n from ℓ_∞^n onto a subspace of Y satisfying $\|J_n^{-1}\| = 1$ and $\|J_n\| \leq 1 + \varepsilon$ for all $n \in \mathbb{N}$.

Given $\{x_1, \dots, x_n\} \subset X$ and $\{T_1, \dots, T_n\} \subset \mathcal{M}$, by (2.13) we have

$$\|T_i x_i\| \leq \left(\int_{B_{X^*}} |\langle x^*, x_i \rangle|^p d\mu(x^*) \right)^{1/p}, \quad i = 1, \dots, n. \quad (2.14)$$

For every $i = 1, \dots, n$, take $g_i \in L_q(\mu)$ such that $\|g_i\|_q = 1$ and

$$\left(\int_{B_{X^*}} |\langle x^*, x_i \rangle|^p d\mu(x^*) \right)^{1/p} = \int_{B_{X^*}} \langle x^*, x_i \rangle g_i(x^*) d\mu(x^*). \quad (2.15)$$

From (2.14) and (2.15), we obtain

$$\|T_i x_i\| \leq \int_{B_{X^*}} \langle x^*, x_i \rangle g_i(x^*) d\mu(x^*), \quad i = 1, \dots, n. \quad (2.16)$$

Put $y_i = J_n e_i$, being $(e_i)_{i=1}^n$ the unit basis of ℓ_∞^n . We define an operator $T : X \rightarrow Y$ by

$$Tx = \sum_{i=1}^n \left(\int_{B_{X^*}} \langle x^*, x \rangle g_i(x^*) d\mu(x^*) \right) y_i. \quad (2.17)$$

We first prove that $\|Tx\|^p \leq (\int_{B_{X^*}} |\langle x^*, x \rangle|^p d\mu(x^*)) (1 + \varepsilon)$ for all $x \in X$:

$$\begin{aligned} \|Tx\| &= \sup_{y^* \in B_{Y^*}} \left| \left\langle y^*, \sum_{i=1}^n \left(\int_{B_{X^*}} \langle x^*, x \rangle g_i(x^*) d\mu(x^*) \right) y_i \right\rangle \right| \\ &\leq \sup_{y^* \in B_{Y^*}} \sum_{i=1}^n \left(\int_{B_{X^*}} |\langle x^*, x \rangle| |g_i(x^*)| d\mu(x^*) \right) |\langle y^*, y_i \rangle| \\ &\leq \sup_{y^* \in B_{Y^*}} \sum_{i=1}^n \left(\int_{B_{X^*}} |\langle x^*, x \rangle|^p d\mu(x^*) \right)^{1/p} \left(\int_{B_{X^*}} |g_i(x^*)|^q d\mu(x^*) \right)^{1/q} |\langle y^*, y_i \rangle| \\ &\leq \left(\int_{B_{X^*}} |\langle x^*, x \rangle|^p d\mu(x^*) \right)^{1/p} \sup_{y^* \in B_{Y^*}} \sum_{i=1}^n |\langle y^*, y_i \rangle| \\ &\leq \left(\int_{B_{X^*}} |\langle x^*, x \rangle|^p d\mu(x^*) \right)^{1/p} \|J_n\| \\ &\leq \left(\int_{B_{X^*}} |\langle x^*, x \rangle|^p d\mu(x^*) \right)^{1/p} (1 + \varepsilon). \end{aligned} \quad (2.18)$$

Finally, we need to prove that $\|T_i x_i\| \leq \|Tx\|$ for $i = 1, \dots, n$. Put $y_i^* = e_i^* \circ J_n^{-1}$, $(e_i^*)_{i=1}^n$ being the unit basis of $(\ell_\infty^n)^* \simeq \ell_1^n$. Notice that $\|y_i^*\| \leq 1$ for $i = 1, \dots, n$. We

also denote by y_i^* a Hahn-Banach extension of $e_i^* \circ J_n^{-1}$ to Y . We have

$$\begin{aligned}
\|Tx_i\| &\geq |\langle y_i^*, Tx_i \rangle| \\
&= \left| \left\langle y_i^*, \sum_{j=1}^n \left(\int_{B_{X^*}} \langle x^*, x_i \rangle g_j(x^*) d\mu(x^*) \right) y_j \right\rangle \right| \\
&= \left| \sum_{j=1}^n \left(\int_{B_{X^*}} \langle x^*, x_i \rangle g_j(x^*) d\mu(x^*) \right) \langle y_i^*, y_j \rangle \right| \\
&= \left| \sum_{j=1}^n \left(\int_{B_{X^*}} \langle x^*, x_i \rangle g_j(x^*) d\mu(x^*) \right) \langle e_i^* \circ J_n^{-1}, J_n e_j \rangle \right| \\
&= \left| \sum_{j=1}^n \left(\int_{B_{X^*}} \langle x^*, x_i \rangle g_j(x^*) d\mu(x^*) \right) \langle e_i^*, e_j \rangle \right| \\
&= \int_{B_{X^*}} \langle x^*, x_i \rangle g_i(x^*) d\mu(x^*) \\
&\geq \|T_i x_i\|,
\end{aligned} \tag{2.19}$$

the last inequality is due to (2.16).

(b) \Rightarrow (a). It follows easily using Proposition 2.1(c). \square

REMARKS. (1) It is interesting to give an example of a uniformly dominated set \mathcal{M} for which there is no operator $T \in \mathcal{M}$ satisfying $\|T_i x_i\| \leq \|Tx_i\|$, $i = 1, \dots, n$, for some finite set $\{(x_i, T_i) : i = 1, \dots, n\} \subset X \times \mathcal{M}$. Let $X = \ell_1$ and $Y = \ell_\infty$ and consider the set $\mathcal{M} = \{T_\beta : \beta \in B_{\ell_2}\}$, $T_\beta : \ell_1 \rightarrow \ell_\infty$ being defined by $T_\beta(\alpha) = (\alpha_n \beta_n)$ for all $\alpha = (\alpha_n) \in \ell_1$. Obviously, \mathcal{M} is a uniformly dominated subset of $\Pi_1(\ell_1, \ell_\infty)$.

By contradiction, suppose the following condition holds: "there is a constant $C > 0$ such that, for every finite set $\{(x_i, T_i) : i = 1, \dots, n\} \subset X \times \mathcal{M}$, there exists $T \in \mathcal{M}$ satisfying $\|T_i x_i\| \leq C \|Tx_i\|$, $i = 1, \dots, n$." Put $x_i = e_i$ and $T_i = T_{\beta_i}$ for $i = 1, \dots, n$, where $(e_i)_{i=1}^\infty$ is the unit basis of ℓ_1 and $\beta_i = (1/\sqrt{i}, 0, \dots)$. Take $T_Y \in \mathcal{M}$ such that

$$\|T_i x_i\| \leq C \|T_Y x_i\|, \quad i = 1, \dots, n; \tag{2.20}$$

this yields

$$\frac{1}{\sqrt{i}} \leq C |y_i|, \quad i = 1, \dots, n. \tag{2.21}$$

Then we have

$$1 \geq \sum_{i=1}^{\infty} |y_i|^2 \geq \sum_{i=1}^n |y_i|^2 \geq \frac{1}{C^2} \sum_{i=1}^n \frac{1}{i}. \tag{2.22}$$

So, we have obtained the inequality $\sum_{i=1}^n 1/i \leq C^2$ for all $n \in \mathbb{N}$ which allows us to state that such an operator T cannot exist.

(2) Notice that, in the above example, \mathcal{M} is absolutely convex and weakly compact in $\Pi_1(\ell_1, \ell_\infty)$. Then, \mathcal{M} is absolutely convex, closed, and uniformly dominated but $\mathcal{M} \neq \mathcal{D}_1(\mu)$ for every admissible positive Radon measure μ .

(3) Finally, we give an example of a bounded set \mathcal{M} of 2-summing operators that does not have property (b) in [Theorem 2.2](#). Consider the set \mathcal{M} of all 2-summing operators $T_\beta : c_0 \rightarrow \ell_\infty$ defined by $T_\beta(\alpha) = (\alpha_n \beta_n)$ for all $\alpha = (\alpha_n) \in c_0$, where $\beta = (\beta_n)$ runs over the unit ball of ℓ_2 . We have $T_\beta = i \circ S_\beta$, i being the identity map from ℓ_2 into ℓ_∞ and $S_\beta : c_0 \rightarrow \ell_2$ defined by $S_\beta(\alpha) = (\alpha_n \beta_n)$. Since ℓ_2 has cotype 2, it follows that S_β is 2-summing [\[1\]](#). Nevertheless, \mathcal{M} does not satisfy property (b) in the above theorem. By contradiction, suppose that there is a constant $C > 0$ such that (b) holds. Again, we take $\tilde{\beta}_i = (1/\sqrt{i}, \dots, 1/\sqrt{i}, 0, \dots)$ for all $i \in \mathbb{N}$. By hypothesis, there exists $T \in \Pi_2(c_0, \ell_\infty)$ such that $\pi_2(T) \leq C$ and $\|T_{\tilde{\beta}_i} e_i\| \leq \|Te_i\|$ for $i = 1, \dots, n$. Then we have

$$\sum_{i=1}^n \frac{1}{i} = \sum_{i=1}^n \|T_{\tilde{\beta}_i} e_i\|^2 \leq \sum_{i=1}^n \|Te_i\|^2 \leq C^2 \quad (2.23)$$

for all $n \in \mathbb{N}$. Hence, \mathcal{M} does not have property (b) in [Theorem 2.2](#).

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