

ON THE EQUATION $x^2 + 2^a \cdot 3^b = y^n$

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Received 1 December 1999 and in revised form 15 January 2000

We find all positive integer solutions (x, y, a, b, n) of $x^2 + 2^a \cdot 3^b = y^n$ with $n \geq 3$ and coprime x and y .

2000 Mathematics Subject Classification: 11D41, 11D61.

1. Introduction. The Diophantine equation $x^2 + C = y^n$, where x and y are positive integers, $n \geq 3$ and C is a given integer, has received considerable interest. The earliest reference seems to be an assertion by Fermat that he had shown that when $C = 2$, $n = 3$, the only solution is given by $x = 5$, $y = 3$; a proof was published by Euler in 1770. The first result for general n is due to Lebesgue [9] who proved that there are no solutions for $C = 1$. Ljunggren [10] solved this equation for $C = 2$, Nagell [13, 14] solved it for $C = 3, 4$, and 5 and Chao [5] proved that it has no solutions for $C = -1$. For an extensive list of references one should consult Cohn's beautiful paper [6] in which he develops a method by which he finds all solutions of the above equation for 77 of the values of $C \leq 100$. This equation was later solved for two additional values of $C \leq 100$ (namely, $C = 74$ and $C = 86$) by Mignotte and de Weger [12]. It is interesting to mention that the equation $x^2 + 7 = y^n$ is still unsolved.

In recent years, a different form of the above equation has been considered, namely, when C is no longer a fixed integer but a power of a fixed prime. Le [8] investigated the equation $x^2 + 2^m = y^n$. Arif and Muriefah solved the equation $x^2 + 3^m = y^n$ when m is odd (see [2]). They also gave partial results in the case when m is even (see [1]) but the general solution in the case m is even was found by Luca in [11].

For any nonzero integer k , let $P(k)$ be the largest prime dividing k . Let C_1 be any fixed positive constant. It follows, from the work of Bugeaud [4] and Turk [15], that if

$$x^2 + z = y^n \quad \text{with } (x, y) = 1, P(z) < C_1, \quad (1.1)$$

then $\max(|x|, |y|, n)$ is bounded by a constant computable in terms of C_1 alone.

In this paper, we find all solutions of (1.1) when $C_1 = 5$ and $z > 0$. More precisely, we find all solutions of the equation

$$x^2 + 2^a \cdot 3^b = y^n \quad \text{with } a, b \geq 0, n \geq 3, (x, y) = 1. \quad (1.2)$$

The proof uses the new result on the existence of primitive divisors of the Lucas numbers due to Bilu et al. [3] as well as a computational result of de Weger [7].

2. The result

THEOREM 2.1. *All positive solutions of the equation*

$$x^2 + 2^a \cdot 3^b = y^n \quad \text{with } a, b \geq 0, n \geq 3, (x, y) = 1 \quad (2.1)$$

have $n = 4$ or $n = 3$. For $n = 4$, the solutions are

$$(x, y) = (7, 3), (23, 5), (7, 5), (47, 7), (287, 17). \quad (2.2)$$

For $n = 3$, the solutions are

$$(x, y) = (5, 3), (11, 5), (10, 7), (17, 7), (46, 13), (35, 13), \\ (595, 73), (955, 97), (2681, 193), (39151, 1153). \quad (2.3)$$

In the statement of the theorem we have listed only the values of x , y , and n as the values of the parameters a and b that can be determined from the prime factor decomposition of $x^2 - y^n$ once x , y , and n are given.

From Lebesgue's result, we know that the equation $x^2 + 1 = y^n$ has no positive solutions for $n \geq 3$ and from the work of Arif, Muriefah, and Luca, we know that the only positive solutions of the equation $x^2 + 3^m = y^n$ with $(x, y) = 1$ are $(x, y, m, n) = (10, 7, 5, 3)$ and $(46, 13, 4, 3)$. From now on, we assume that $a > 0$. In particular, both x and y are odd.

3. The case $n \neq 3$ or 4. In this section, we show that it suffices to assume that $n \in \{3, 4\}$. Indeed, assume that $n \neq 4$. We may certainly assume that n is an odd prime. If $n \neq 3$, it follows that $n \geq 5$. Write $2^a \cdot 3^b = dz^2$ where $d \in \{1, 2, 3, 6\}$. Equation (2.1) can be written as

$$(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n. \quad (3.1)$$

Since x is odd and dz^2 is even, it follows that the two ideals $[(x + i\sqrt{d}z)]$ and $[(x - i\sqrt{d}z)]$ are coprime in the ring of integers of $\mathbf{Q}(i\sqrt{d})$. Since the class number of $\mathbf{Q}(i\sqrt{d})$ is 1 or 2 and $n \geq 5$ is prime, it follows that there exists an integer u and a root of unity ε in $\mathbf{Q}(i\sqrt{d})$ such that

$$x + i\sqrt{d}z = \varepsilon u^n, \quad x - i\sqrt{d}z = \overline{\varepsilon} \bar{u}^n. \quad (3.2)$$

Since ε is a root of unity belonging to a quadratic extension of \mathbf{Q} , it follows that $\varepsilon^k = 1$ for some $k \in \{1, 2, 3, 4, 6\}$. Since $n \geq 5$ is prime, it follows that up to a substitution one may assume that $\varepsilon = 1$ in system (3.2). From (3.2) with $\varepsilon = 1$, it follows that

$$2i\sqrt{d}z = u^n - \bar{u}^n. \quad (3.3)$$

Since certainly

$$\frac{u^n - \bar{u}^n}{u - \bar{u}} \in \mathbf{Z}, \quad (3.4)$$

we have that

$$P\left(\frac{u^n - \bar{u}^n}{u - \bar{u}}\right) < 5. \quad (3.5)$$

From (3.5), we find that the Lucas number given by formula (3.4) has no primitive divisor. From [3], it follows that there are at most 10 pairs (u, n) satisfying inequality (3.5) and all of them appear in [3, Table 1]. A quick investigation reveals that none of the u 's from [3, Table 1] belongs to $\mathbf{Q}(i\sqrt{d})$ for some $d \in \{1, 2, 3, 6\}$, which is the desired contradiction.

4. The case $n = 4$. Let $S = \{k \mid P(k) < 5\}$. Then, we have the following preliminary result.

LEMMA 4.1. *All solutions of the equation*

$$x^2 = k \pm l \quad \text{with } k, l > 0, k, l \in S, (k, l) = 1 \quad (4.1)$$

are

$$\begin{aligned} (x, k, l) = & (1, 2, 1), (2, 3, 1), (3, 8, 1), (5, 24, 1), (7, 48, 1), \\ & (17, 288, 1), (1, 4, 3), (1, 9, 8), (5, 16, 9), (5, 27, 2), (7, 81, 32). \end{aligned} \quad (4.2)$$

PROOF OF LEMMA 4.1. This lemma is a particular case of a result of de Weger [7, Chapter 7]. \square

THE PROOF OF THE THEOREM FOR $n = 4$. Rewrite (2.1) as

$$(y^2 - x)(y^2 + x) = 2^a \cdot 3^b. \quad (4.3)$$

Since $a > 0$ and $(x, y) = 1$, it follows that $(y^2 - x, y^2 + x) = 2$. Thus,

$$y^2 - x = k, \quad y^2 + x = l, \quad \text{with } k, l > 0, k, l \in S, (k, l) = 2. \quad (4.4)$$

Hence,

$$y^2 = \frac{k}{2} + \frac{l}{2}, \quad (4.5)$$

where $k/2, l/2 \in S$ are positive and coprime. By Lemma 4.1, we obtain that (4.5) has only 6 solutions. Five of them lead to solutions (2.2) of (2.1). One of the solutions of (4.5) leads to

$$2^2 + 2^2 \cdot 3 = 2^4, \quad (4.6)$$

which is not a convenient solution of (2.1) because $x = 2$ and $y = 2$ are not coprime.

The case $n = 4$ is therefore settled.

5. The case $n = 3$. We begin with another lemma.

LEMMA 5.1. *The only solutions of the equation*

$$3x^2 = k \pm l \quad \text{with } k, l > 0, k, l \in S, (k, l) \in \{1, 3\} \quad (5.1)$$

are

$$\begin{aligned} (x, k, l) = & (1, 2, 1), (1, 4, 1), (1, 6, 3), (2, 9, 3), \\ & (3, 24, 3), (5, 72, 3), (7, 144, 3), (17, 864, 3), \\ & (1, 12, 9), (1, 27, 24), (5, 48, 27), (5, 81, 6), (7, 243, 96). \end{aligned} \quad (5.2)$$

PROOF OF LEMMA 5.1. This lemma too is a particular instance of the more general computation of de Weger [7, Chapter 7]. \square

THE PROOF OF THE THEOREM FOR $n = 3$. Write again $2^a \cdot 3^b = dz^2$ where $d \in \{1, 2, 3, 6\}$. From arguments employed in Section 3, we know that there exist u and ε in $\mathbf{Q}(i\sqrt{d})$ such that $y = |u|^2$, ε is a root of unity and

$$x + i\sqrt{d}z = \varepsilon u^3, \quad x - i\sqrt{d}z = \overline{\varepsilon} \overline{u}^3. \quad (5.3)$$

Clearly,

$$2i\sqrt{d}z = \varepsilon u^3 - \overline{\varepsilon} \overline{u}^3. \quad (5.4)$$

We distinguish two cases.

CASE 1 ($\varepsilon = 1$). Equation (5.4) reads

$$2i\sqrt{d}z = u^3 - \overline{u}^3. \quad (5.5)$$

Assume first that $u = a + ib\sqrt{d}$ with a and b integers. Then, we get

$$2i\sqrt{d}z = (a + ib\sqrt{d})^3 - (a - ib\sqrt{d})^3 \quad (5.6)$$

or

$$2i\sqrt{d}z = 2i\sqrt{d}b(3a^2 - db^2). \quad (5.7)$$

Hence, $b \mid z$ and

$$3a^2 = db^2 \pm \frac{z}{b}. \quad (5.8)$$

Let $k = db^2$ and $l = z/b$. Notice that $k, l \in S$. Moreover, notice that $(k, l) \in \{1, 3\}$. Indeed, if $(k, l) \notin \{1, 3\}$, it follows that there exists a prime p such that $p \mid (k, l, a)$. In particular, $p \mid db^2$ and $p \mid a$, therefore $p \mid a^2 + db^2 = y$. Since $p \mid z$ and $2^a \cdot 3^b = dz^2$, we come to $p \mid 2^a \cdot 3^b$. It follows now that $p \mid (y^3 - 2^a \cdot 3^b) = x^2$ and therefore $p \mid x$. This contradicts the fact that x and y are coprime. Now all solutions of (5.8) are given by Lemma 5.1. For example, the solution

$$3 \cdot 1^2 = 2^1 + 1 \quad (5.9)$$

gives either $a = 1$, $d = 2$, $b = 1$, and $z = 1$ or $a = 1$, $d = 1$, $b = 1$, and $z = 2$. The first possibility yields $y = a^2 + db^2 = 1 + 2 = 3$ and $dz^2 = 2$, which leads to the solution $3^3 = 2 + 5^2$ of (2.1). The second possibility gives $y = a^2 + db^2 = 2$ and $dz^2 = 4$, which leads to the solution $2^3 = 2^2 + 2^2$ of (2.1). This is not an acceptable solution, since $x = 2$ and $y = 2$ are not coprime.

All the solutions of (2.1) for the case $n = 3$ except for $(x, y) = (10, 7)$ are obtained in this way by identifying a, b, d , and z from (5.8) via Lemma 5.1.

When $d = 3$, we also need to investigate the case in which

$$u = \frac{a + i\sqrt{3}b}{2} \quad (5.10)$$

for some odd integers a and b . From (5.5), we simply get that

$$16i\sqrt{3}z = (a + i\sqrt{3}b)^3 - (a - i\sqrt{3}b)^3 \quad (5.11)$$

or

$$16i\sqrt{3}z = 2i\sqrt{3}b(3a^2 - 3b^2). \quad (5.12)$$

It follows that b divides z and

$$3a^2 = 3b^2 \pm \frac{8z}{b}. \quad (5.13)$$

From Lemma 5.1, we derive that (5.13) has only two convenient solutions, namely, $3 \cdot 1^2 = 3 \cdot 3^2 - 8 \cdot 3$ and $3 \cdot 7^2 = 3 \cdot 3^4 - 8 \cdot 12$. These lead to the solutions $(x, y) = (10, 7)$ and $(595, 73)$ of (2.1).

CASE 2 ($\varepsilon \neq 1$). It is easy to see that the only case in which one may not be able to set $\varepsilon = 1$ in system (5.4) is when $d = 3$. In this case, one may assume that $\varepsilon = (1 + i\sqrt{3})/2$ and that $u = (a + i\sqrt{3}b)/2$ for some integers a and b such that $a \equiv b \pmod{2}$. Then (5.4) becomes

$$2i\sqrt{3}z = \left(\frac{1 + i\sqrt{3}}{2}\right) \cdot \left(\frac{a + i\sqrt{3}b}{2}\right)^3 - \left(\frac{1 - i\sqrt{3}}{2}\right) \cdot \left(\frac{a - i\sqrt{3}b}{2}\right)^3. \quad (5.14)$$

This equation is equivalent to

$$16z = a^3 + 3a^2b - 9ab^2 - 3b^3. \quad (5.15)$$

Assume first that both a and b are odd. Then, from (5.15), it follows that

$$16z = (a^3 - ab^2) + (3a^2b - 3b^3) - 8ab^2 = (a^2 - b^2)(a + 3b) - 8ab^2. \quad (5.16)$$

Since a and b are both odd, we obtain that $16 \mid (a^2 - b^2)(a + 3b)$. Equation (5.16) forces $16 \mid 8ab^2$, which is impossible.

Assume now that both a and b are even. Since $y = (a/2)^2 + 3(b/2)^2$ is odd, it follows that exactly one of the numbers $a/2$ and $b/2$ is even. Equation (5.15) now implies that

$$2z = \left(\frac{a}{2}\right)^3 + 3\left(\frac{a}{2}\right)^2\left(\frac{b}{2}\right) - 9\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)^2 - 3\left(\frac{b}{2}\right)^3. \quad (5.17)$$

However, (5.17) is now impossible, because precisely one of the numbers $a/2$ and $b/2$ is even and the other one is odd. Hence, this case can never occur.

The theorem is therefore completely proved. \square

ACKNOWLEDGEMENTS. This work was supported in part by the Alexander von Humboldt Foundation and in part by the programme KONTAKT ME 148 of the Czech Republic.

This paper was written when I visited the Mathematical Institute of the Czech Academy of Sciences in April, 1999. I would like to thank Michal Křížek both for the invitation and for helpful conversation on the topic as well as the institute for its hospitality. Michal also wrote a computer program which pointed out two solutions of (2.1) that I had originally missed. I thank him for that too.

Finally, I would also like to thank Yuri Bilu for sending me a copy of the preprint [3], an anonymous referee for making me aware of de Weger's work [7] and Benne de Weger for allowing me to consult his personal copy of [7].

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