

## ON THE STRONGLY STARLIKENESS OF MULTIVALENTLY CONVEX FUNCTIONS OF ORDER $\alpha$

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ABSTRACT. The object of the present paper is to derive some sufficient conditions for strongly starlikeness of multivalently convex functions of order  $\alpha$  in the open unit disc.

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**1. Introduction.** Let  $\mathcal{A}(p)$  denote the class of the functions  $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$  which are analytic in the open unit disc  $\mathcal{E} = \{z : |z| < 1\}$ . A function  $f(z) \in \mathcal{A}(p)$  is called  $p$ -valently starlike if and only if the inequality

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > 0 \quad (1.1)$$

holds for  $z \in \mathcal{E}$ . A function  $f(z) \in \mathcal{A}(p)$  is called  $p$ -valently convex of order  $\alpha$  ( $0 < \alpha < p$ ) if and only if the inequality

$$1 + \operatorname{Re} \left\{ \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (1.2)$$

holds for  $z \in \mathcal{E}$ . We denote by  $\mathcal{C}(p, \alpha)$  the family of such functions. A function  $f(z) \in \mathcal{A}(p)$  is said to be strongly starlike of order  $\alpha$  ( $0 < \alpha \leq 1$ ) if and only if the inequality

$$\left| \arg \left\{ \frac{z f'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad (1.3)$$

holds for  $z \in \mathcal{E}$ . We also denote by  $\operatorname{STS}(p, \alpha)$  the family of functions which satisfy the above inequality for the argument. From the definition, it follows that if  $f(z) \in \operatorname{STS}(p, \alpha)$ , then we have

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > 0 \quad \text{in } \mathcal{E} \quad (1.4)$$

or  $f(z)$  is  $p$ -valently starlike in  $\mathcal{E}$  and therefore  $f(z)$  is  $p$ -valent in  $\mathcal{E}$  (see [1, Lemma 7]). Nunokawa [2, 3] proved the following theorems.

**THEOREM 1.1** (see [2]). *If  $f(z) \in \mathcal{A}(p)$  satisfies*

$$1 + \operatorname{Re} \left\{ \frac{z f''(z)}{f'(z)} \right\} < p + \frac{\alpha}{2}, \quad (1.5)$$

where  $0 < \alpha \leq 1$ , then  $f(z) \in \operatorname{STS}(p, \alpha)$ .

**THEOREM 1.2** (see [3]). *If  $f(z) \in \mathcal{A}(1)$  satisfies*

$$\left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \frac{\pi}{2} \alpha(\beta) \quad \text{in } \mathcal{E}, \quad (1.6)$$

then

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \beta \quad \text{in } \mathcal{E}, \quad (1.7)$$

where

$$\begin{aligned} \alpha(\beta) &= \beta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\beta q(\beta) \sin(\pi/2)(1-\beta)}{p(\beta) + \beta q(\beta) \cos(\pi/2)(1-\beta)} \right\}, \\ p(\beta) &= (1+\beta)^{(1+\beta)/2}, \quad q(\beta) = (1-\beta)^{(\beta-1)/2}. \end{aligned} \quad (1.8)$$

It is the purpose of the present paper to prove that if  $f(z) \in \mathcal{C}(1, 1 - (\alpha/2))$ , then  $f(z) \in \text{STS}(1, \alpha)$ .

In this paper, we need the following lemma.

**LEMMA 1.3.** *Let  $f(z) \in \mathcal{A}(1)$  be starlike with respect to the origin in  $\mathcal{E}$ . Let  $C(r, \theta) = \{f(te^{i\theta}) : 0 \leq t \leq r < 1\}$  and  $T(r, \theta)$  be the total variation of  $\arg f(te^{i\theta})$  on  $C(r, \theta)$ , so that*

$$T(r, \theta) = \int_0^r \left| \frac{\partial}{\partial t} \arg \{f(te^{i\theta})\} \right| dt. \quad (1.9)$$

Then

$$T(r, \theta) < \pi. \quad (1.10)$$

We owe this lemma to Sheil-Small [6, Theorem 1].

**2. Main theorem.** Our main theorem for the starlikeness of multivalently convex functions of order  $\alpha$  is the following.

**THEOREM 2.1.** *Let  $f(z) \in \mathcal{A}(1)$  and*

$$1 + \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 1 - \frac{\alpha}{2} \quad \text{in } \mathcal{E}, \quad (2.1)$$

where  $0 < \alpha \leq 1$ . Then

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in } \mathcal{E}, \quad (2.2)$$

or  $f(z)$  is strongly starlike of order  $\alpha$  in  $\mathcal{E}$ .

**PROOF.** We put

$$\frac{2}{\alpha} \left\{ 1 + \frac{zf''(z)}{f'(z)} - 1 + \frac{\alpha}{2} \right\} = \frac{zg'(z)}{g(z)}, \quad (2.3)$$

where  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ . From assumption (2.1), we have

$$\operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} > 0 \quad \text{in } \mathfrak{E}. \tag{2.4}$$

This shows that  $g(z)$  is starlike and univalent in  $\mathfrak{E}$ . With an easy calculation (cf. [4]), (2.3) gives us that

$$f'(z) = \left\{ \frac{g(z)}{z} \right\}^{\alpha/2}. \tag{2.5}$$

Since

$$f'(z) \neq 0, \quad 0 < |z| < 1, \tag{2.6}$$

we easily have

$$\frac{f(z)}{zf'(z)} = \int_0^1 \frac{f'(tz)}{f'(z)} dt = \int_0^1 t^{-\alpha/2} \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}^{\alpha/2} dt, \tag{2.7}$$

where  $z = re^{i\theta}$  and  $0 < r < 1$ . Since  $g(z)$  is starlike in  $\mathfrak{E}$ , from Lemma 1.3, we have

$$-\pi < \arg \left\{ g(tre^{i\theta}) \right\} - \arg \left\{ g(re^{i\theta}) \right\} < \pi \tag{2.8}$$

for  $0 < t \leq 1$ . Putting

$$\xi = \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}^{\alpha/2}, \tag{2.9}$$

we have

$$\arg s = \frac{\alpha}{2} \arg \left\{ \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right\}. \tag{2.10}$$

From (2.8) and (2.10),  $s$  lies in the convex sector

$$\left\{ s : |\arg s| \leq \frac{\pi}{2} \alpha \right\} \tag{2.11}$$

and the same is true of its integral mean of (2.7), (cf. [5, Lemma 1]). Therefore, we have

$$\left| \arg \left\{ \frac{f(z)}{zf'(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in } \mathfrak{E} \tag{2.12}$$

or

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \alpha \quad \text{in } \mathfrak{E}. \tag{2.13}$$

This shows that

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in } \mathfrak{E}, \tag{2.14}$$

which completes the proof of our main theorem. □

**REMARK 2.2.** This result is sharp for the case  $\alpha \rightarrow 0$  and  $\alpha = 1$ .

(a) For the case  $\alpha \rightarrow 0$ , put  $f(z) = z$ , then  $f(z)$  is a convex function of order  $1 - (\alpha/2) \rightarrow 1$  and  $f(z)$  then  $f(z)$  is a strongly starlike function of order  $\alpha \rightarrow 0$ .

(b) For the case  $\alpha = 1$ , put

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{1-z}. \quad (2.15)$$

Then we have

$$1 + \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2} \quad \text{in } \mathcal{E}, \quad (2.16)$$

and therefore  $f(z)$  is a convex function of order  $1/2$ . From (2.10), we easily have

$$f'(z) = \frac{1}{1-z}, \quad f(z) = \log \left\{ \frac{1}{1-z} \right\}. \quad (2.17)$$

Putting  $|z| = 1$ ,  $z = e^{i\theta}$ ,  $0 \leq \theta < 2\pi$ , then it follows that

$$\begin{aligned} \frac{z}{1-z} &= -\frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)}, \\ \log \left\{ \frac{1}{1-z} \right\} &= \log \left| \frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right| + i \arg \left\{ \frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right\}. \\ \lim_{\theta \rightarrow +0} \arg \left\{ \frac{zf'(z)}{f(z)} \right\} &= \lim_{\theta \rightarrow +0} \arg \left\{ \frac{z/(1-z)}{\log(1/(1-z))} \right\} \\ &= \lim_{\theta \rightarrow +0} \arg \left\{ -\frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right\} \\ &\quad - \lim_{\theta \rightarrow +0} \arg \left\{ \log \left| \frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right| + i \arg \left( \frac{1}{2} + i \frac{\cos(\theta/2)}{2\sin(\theta/2)} \right) \right\} \\ &= \frac{\pi}{2}. \end{aligned} \quad (2.18)$$

The above shows that the main theorem is sharp for the case  $\alpha \rightarrow 0$  and  $\alpha = 1$ .

Applying the same method as above and [2], we can obtain the following result.

**THEOREM 2.3.** If  $f(z) \in A(p)$  and satisfies

$$p - \frac{\alpha}{2} < 1 + \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} \quad \text{in } \mathcal{E}, \quad (2.19)$$

where  $0 < \alpha \leq 1$ , then  $f(z) \in \text{STS}(p, \alpha)$ .

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