IJMMS 27:8 (2001) 471–476
PII. S0161171201006482
http://ijmms.hindawi.com
© Hindawi Publishing Corp.

SOME APPLICATIONS OF MINIMAL OPEN SETS

FUMIE NAKAOKA and NOBUYUKI ODA

(Received 16 January 2001)

ABSTRACT. We characterize minimal open sets in topological spaces. We show that any nonempty subset of a minimal open set is pre-open. As an application of a theory of minimal open sets, we obtain a sufficient condition for a locally finite space to be a pre-Hausdorff space.

2000 Mathematics Subject Classification. 54A05, 54D99.

1. Introduction. Let X be a topological space. We call a nonempty open set U of X a minimal open set when the only open subsets of U are U and $\emptyset$.

In this paper, we study fundamental properties of minimal open sets and apply them to obtain some results on pre-open sets (cf. [2]) and pre-Hausdorff spaces.

In Section 2, we characterize minimal open sets, that is, we show that a nonempty open set U is a minimal open set if and only if $\mathrm{Cl}(U) = \mathrm{Cl}(S)$ for any nonempty subset S of U. This result implies that any nonempty subset S of a minimal open set U is a pre-open set.

In Section 3, we study minimal open sets in locally finite spaces. The results of this section are closely related to the work of James [1], and these results will be used in the next scetion.

In Section 4, we apply the theory of minimal open sets to study pre-open sets. Our first main result of this section is a property of the set of all minimal open sets in any nonempty finite open set which is not a minimal open set. This result enables us to prove a generalization of Theorem 2.5, when U is a nonempty finite open set, in Theorem 4.4. Theorem 4.5 shows that our theory of minimal open set is useful to study pre-open sets.

Finally, we show that some conditions on minimal open sets implies pre-Hausdorffness of a space, that is, if any minimal open set of a locally finite space X has two elements at least, then X is a pre-Hausdorff space.

2. Minimal open sets. Let (X,τ) be a topological space.

DEFINITION 2.1. A nonempty open set U of X is said to be a minimal open set if and only if any open set which is contained in U is $\emptyset$ or U.

LEMMA 2.2. (1) *Let U be a minimal open set and W an open set. Then $U \cap W = \emptyset$ or $U \subset W$.*

(2) *Let U and V be minimal open sets. Then $U \cap V = \emptyset$ or $U = V$.*

Proof. (1) Let W be an open set such that $U \cap W \neq \emptyset$. Since U is a minimal open set and $U \cap W \subset U$, we have $U \cap W = U$. Therefore $U \subset W$.

(2) If $U \cap V \neq \emptyset$, then we see that $U \subset V$ and $V \subset U$ by (1). Therefore $U = V$. □

Proposition 2.3. *Let U be a minimal open set. If x is an element of U, then $U \subset W$ for any open neighborhood W of x.*

Proof. Let W be an open neighborhood of x such that $U \not\subset W$. Then $U \cap W$ is an open set such that $U \cap W \subsetneq U$ and $U \cap W \neq \emptyset$. This contradicts our assumption that U is a minimal open set. □

Proposition 2.4. *Let U be a minimal open set. Then*

$$U = \cap\{W \mid W \text{ is an open neighborhood of } x\} \tag{2.1}$$

for any element x of U.

Proof. By Proposition 2.3 and the fact that U is an open neighborhood of x, we have $U \subset \cap\{W \mid W \text{ is an open neighborhood of } x\} \subset U$. Therefore we have the result. □

Theorem 2.5. *Let U be a nonempty open set. Then the following three conditions are equivalent:*

(1) *U is a minimal open set.*

(2) *$U \subset \mathrm{Cl}(S)$ for any nonempty subset S of U.*

(3) *$\mathrm{Cl}(U) = \mathrm{Cl}(S)$ for any nonempty subset S of U.*

Proof. (1)⇒(2). Let S be any nonempty subset of U. By Proposition 2.3, for any element x of U and any open neighborhood W of x, we have

$$S = U \cap S \subset W \cap S. \tag{2.2}$$

Then, we have $W \cap S \neq \emptyset$ and hence x is an element of $\mathrm{Cl}(S)$. It follows that $U \subset \mathrm{Cl}(S)$.

(2)⇒(3). For any nonempty subset S of U, we have $\mathrm{Cl}(S) \subset \mathrm{Cl}(U)$. On the other hand, by (2), we see $\mathrm{Cl}(U) \subset \mathrm{Cl}(\mathrm{Cl}(S)) = \mathrm{Cl}(S)$. Therefore we have $\mathrm{Cl}(U) = \mathrm{Cl}(S)$ for any nonempty subset S of U.

(3)⇒(1). Suppose that U is not a minimal open set. Then there exists a nonempty open set V such that $V \subsetneq U$ and hence there exists an element $a \in U$ such that $a \notin V$. Then we have $\mathrm{Cl}(\{a\}) \subset V^c$, the complement of V. It follows that $\mathrm{Cl}(\{a\}) \neq \mathrm{Cl}(U)$. □

A subset M of a space (X,τ) is called a *pre-open* set if $M \subset \mathrm{Int}\,\mathrm{Cl}(M)$. The family of all pre-open sets in (X,τ) will be denoted by $\mathrm{PO}(X,\tau)$, (cf. [2]).

A space (X,τ) is called *pre-Hausdorff* if for each $x, y \in X$, $x \neq y$ there exist subsets $U, V \in \mathrm{PO}(X,\tau)$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Theorem 2.6. *Let U be a minimal open set. Then any nonempty subset S of U is a pre-open set.*

Proof. By Theorem 2.5(2), we have $\mathrm{Int}\,U \subset \mathrm{Int}\,\mathrm{Cl}(S)$. Since U is an open set, we have $S \subset U = \mathrm{Int}(U) \subset \mathrm{Int}\,\mathrm{Cl}(S)$. □

Theorem 2.7. *Let U be a minimal open set and M a nonempty subset of X. If there exists an open neighborhood W of M such that $W \subset \mathrm{Cl}(M \cup U)$, then $M \cup S$ is a pre-open set for any nonempty subset S of U.*

Proof. By Theorem 2.5(3), we have $\mathrm{Cl}(M \cup S) = \mathrm{Cl}(M) \cup \mathrm{Cl}(S) = \mathrm{Cl}(M) \cup \mathrm{Cl}(U) = \mathrm{Cl}(M \cup U)$. Since $W \subset \mathrm{Cl}(M \cup U) = \mathrm{Cl}(M \cup S)$ by assumption, we have $\mathrm{Int}(W) \subset \mathrm{Int}\,\mathrm{Cl}(M \cup S)$. Since W is an open neighborhood of M, namely W is an open set such that $M \subset W$, we have $M \subset W = \mathrm{Int}(W) \subset \mathrm{Int}\,\mathrm{Cl}(M \cup S)$. Moreover we have $\mathrm{Int}(U) \subset \mathrm{Int}\,\mathrm{Cl}(M \cup U)$, for $\mathrm{Int}(U) = U \subset \mathrm{Cl}(U) \subset \mathrm{Cl}(M) \cup \mathrm{Cl}(U) = \mathrm{Cl}(M \cup U)$. Since U is an open set, we have $S \subset U = \mathrm{Int}\,U \subset \mathrm{Int}\,\mathrm{Cl}(M \cup U) = \mathrm{Int}\,\mathrm{Cl}(M \cup S)$. Therefore $M \cup S \subset \mathrm{Int}\,\mathrm{Cl}(M \cup S)$. □

Corollary 2.8. *Let U be a minimal open set and M a nonempty subset of X. If there exists an open neighborhood W of M such that $W \subset \mathrm{Cl}(U)$, then $M \cup S$ is a pre-open set for any nonempty subset S of U.*

Proof. By assumption, we have $W \subset \mathrm{Cl}(M) \cup \mathrm{Cl}(U) = \mathrm{Cl}(M \cup U)$. So by Theorem 2.7, we see that $M \cup S$ is a pre-open set. □

The condition of Theorem 2.7, namely $W \subset \mathrm{Cl}(M \cup S)$, does not necessarily imply the condition of Corollary 2.8, namely $W \subset \mathrm{Cl}(S)$. We have the following example.

Example 2.9. Let $X = \{a,b,c,d\}$ with topology $\theta = \{\varnothing, \{d\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$, $U = \{a,b\}$ and $M = W = \{d\}$. Then $W = \{d\} \subset \mathrm{Cl}(\{a,b\} \cup \{d\}) = \mathrm{Cl}(M \cup U)$ and $W = \{d\} \not\subset \mathrm{Cl}(\{a,b\}) = \mathrm{Cl}(U)$.

Theorem 2.10. *Let U be a minimal open set and x an element of $X - U$. Then $W \cap U = \varnothing$ or $U \subset W$ for any open neighborhood W of x.*

Proof. Since W is an open set, we have the result by Lemma 2.2. □

Corollary 2.11. *Let U be a minimal open set and x an element of $X - U$. Define $U_x \equiv \cap\{W \mid W$ is an open neighborhood of $x\}$. Then $U_x \cap U = \varnothing$ or $U \subset U_x$.*

Proof. If $U \subset W$ for any open neighborhood W of x, then $U \subset \cap\{W \mid W$ is an open neighborhood of $x\}$. Therefore $U \subset U_x$. Otherwise there exists an open neighborhood W of x such that $W \cap U = \varnothing$. Then we have $U \cap U_x = \varnothing$. □

3. Finite open sets. In this section, we study some properties of minimal open sets in finite open sets and locally finite spaces.

Theorem 3.1. *Let V be a nonempty finite open set. Then there exists at least one (finite) minimal open set U such that $U \subset V$.*

Proof. If V is a minimal open set, we may set $U = V$. If V is not a minimal open set, then there exists an (finite) open set V_1 such that $\varnothing \neq V_1 \subsetneq V$. If V_1 is a minimal open set, we may set $U = V_1$. If V_1 is not a minimal open set, then there exists an (finite) open set V_2 such that $\varnothing \neq V_2 \subsetneq V_1 \subsetneq V$. Continuing this process, we have a sequence of open sets

$$V \supsetneq V_1 \supsetneq V_2 \cdots \supsetneq V_k \supsetneq \cdots . \qquad (3.1)$$

Since V is a finite set, this process repeats only finitely. Then, finally we get a minimal open set $U = V_n$ for some positive integer n. □

A topological space is said to be a *locally finite space* if each of its elements is contained in a finite open set.

Corollary 3.2. *Let X be a locally finite space and V a nonempty open set. Then there exists at least one (finite) minimal open set U such that $U \subset V$.*

Proof. Since V is a nonempty set, there exists an element x of V. Since X is a locally finite space, we have a finite open set V_x such that $x \in V_x$. Since $V \cap V_x$ is a finite open set, we get a minimal open set U such that $U \subset V \cap V_x \subset V$ by Theorem 3.1. □

Theorem 3.3. *Let V_λ be an open set for any $\lambda \in \Lambda$ and W a nonempty finite open set. Then $W \cap (\cap_{\lambda\in\Lambda} V_\lambda)$ is a finite open set.*

Proof. We see that there exists an integer n such that $W \cap (\cap_{\lambda\in\Lambda} V_\lambda) = W \cap (\cap_{i=1}^{n} V_{\lambda_i})$ and hence we have the result. □

Theorem 3.4. *Let V_λ be an open set for any $\lambda \in \Lambda$ and W_μ a nonempty finite open set for any $\mu \in \mathcal{M}$. Let $S = \cup_{\mu\in\mathcal{M}} W_\mu$. Then $S \cap (\cap_{\lambda\in\Lambda} V_\lambda)$ is an open set.*

Proof. Since W_μ is a finite open set, by Theorem 3.3, we have $W_\mu \cap (\cap_{\lambda\in\Lambda} V_\lambda)$ is a finite open set for any $\mu \in \mathcal{M}$. Since

$$S \cap (\cap_{\lambda\in\Lambda} V_\lambda) = (\cup_{\mu\in\mathcal{M}} W_\mu) \cap (\cap_{\lambda\in\Lambda} V_\lambda) = \cup_{\mu\in\mathcal{M}} (W_\mu \cap (\cap_{\lambda\in\Lambda} V_\lambda)), \tag{3.2}$$

we have the result. □

Corollary 3.5 (see [1]). *Any locally finite space is an Alexandroff space.*

4. Applications. Let U be a nonempty finite open set. We see, by Lemma 2.2 and Corollary 3.2, that there exists a positive integer k such that $\{U_1, U_2, \dots, U_k\}$ is the set of all minimal open sets in U. Then it satisfies the following two conditions:

(a) $U_i \cap U_j = \varnothing$ for any i, j with $1 \le i, j \le k$, and $i \ne j$.

(b) If U' is a minimal open set in U, then there exists i with $1 \le i \le k$ such that $U' = U_i$.

Theorem 4.1. *Let U be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal open sets in U and x an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Define $U_x \equiv \cap\{W \mid W$ is an open neighborhood of $x\}$. Then there exists a positive integer i of $\{1, \dots, n\}$ such that $U_i \subset U_x$.*

Proof. Assume that $U_i \not\subset U_x$ for any positive integer i of $\{1, \dots, n\}$. Then we have $U_i \cap U_x = \varnothing$ for any minimal open set U_i in U by Corollary 2.11. Since U_x is a nonempty finite open set by Theorem 3.3, there exists a minimal open set U' such that $U' \subset U_x$ by Theorem 3.1. Since $U' \subset U_x \subset U$, we have U' is a minimal open set in U. By assumption, we have $U_i \cap U' \subset U_i \cap U_x = \varnothing$ for any minimal open set U_i. Therefore $U' \ne U_i$ for any positive integer i of $\{1, 2, \dots, n\}$. This contradicts our assumption. □

Proposition 4.2. *Let U be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal open sets in U and x an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Then there exists a positive integer i of $\{1, \dots, n\}$ such that $U_i \subset W_x$ for any open neighborhood W_x of x.*

Proof. Since $W_x \supset \cap\{W \mid W$ is an open neighborhood of $x\}$, we have the result by Theorem 4.1. □

Theorem 4.3. *Let U be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal open sets in U and x an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Then there exists a positive integer i of $\{1, \dots, n\}$ such that x is an element of $\mathrm{Cl}(U_i)$.*

Proof. By Proposition 4.2, there exists a positive integer i of $\{1, \dots, n\}$ such that $U_i \subset W$ for any open neighborhood W of x. Therefore $U_i \cap W \supset U_i \cap U_i \neq \emptyset$ for any open neighborhood W of x. Therefore we have the result. □

The following result is a generalization of Theorem 2.5, when U is a nonempty finite open set.

Theorem 4.4. *Let U be a nonempty finite open set and U_i a minimal open set in U for each $i \in \{1, 2, \dots, n\}$. Then the following three conditions are equivalent:*
(1) *$\{U_1, U_2, \dots, U_n\}$ is the set of all minimal open sets in U.*
(2) *$U \subset \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subsets S_i of U_i for $i \in \{1, 2, \dots, n\}$.*
(3) *$\mathrm{Cl}(U) = \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subsets S_i of U_i for $i \in \{1, 2, \dots, n\}$.*

Proof. (1)⇒(2). If U is a minimal open set, then this is the result of Theorem 2.5(2). Otherwise U is not a minimal open set. If x is any element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$, we have $x \in \mathrm{Cl}(U_1) \cup \mathrm{Cl}(U_2) \cup \cdots \cup \mathrm{Cl}(U_n)$ by Theorem 4.3. Therefore

$$\begin{aligned} U \subset \mathrm{Cl}(U_1) \cup \mathrm{Cl}(U_2) \cup \cdots \cup \mathrm{Cl}(U_n) &= \mathrm{Cl}(S_1) \cup \mathrm{Cl}(S_2) \cup \cdots \cup \mathrm{Cl}(S_n) \\ &= \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n) \end{aligned} \tag{4.1}$$

by Theorem 2.5(3).

(2)⇒(3). For any nonempty subset S_i of U_i with $i \in \{1, 2, \dots, n\}$, we have $\mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n) \subset \mathrm{Cl}(U)$. On the other hand, by (2), we see

$$\mathrm{Cl}(U) \subset \mathrm{Cl}(\mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)) = \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n). \tag{4.2}$$

Therefore we have $\mathrm{Cl}(U) = \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subset S_i of U_i with $i \in \{1, 2, \dots, n\}$.

(3)⇒(1). Suppose that V is a minimal open set in U and $V \neq U_i$ for $i \in \{1, 2, \dots, n\}$. Then we have $V \cap \mathrm{Cl}(U_i) = \emptyset$ for each $i \in \{1, 2, \dots, n\}$. It follows that any element of V is not contained in $\mathrm{Cl}(U_1 \cup U_2 \cup \cdots \cup U_n)$. This contradicts the condition (3) because $V \subset U \subset \mathrm{Cl}(U) = \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$. □

Let U be a nonempty finite open set, $\{U_1, U_2, \dots, U_n\}$ the set of all minimal open sets in U and x_i an element of U_i for each $i \in \{1, 2, \dots, n\}$. Then we see that the set $\{x_1, x_2, \dots, x_n\}$ is a pre-open set by Theorem 4.4. Moreover, we have the following result.

Theorem 4.5. *Let U be a nonempty finite open set and $\{U_1,U_2,\dots,U_n\}$ the set of all minimal open sets in U. Let S be any subset of $U-(U_1\cup U_2\cup\cdots\cup U_n)$ and S_i be any nonempty subset of U_i for each $i\in\{1,2,\dots,n\}$. Then $S\cup S_1\cup S_2\cdots\cup S_n$ is a pre-open set.*

Proof. By Theorem 4.4(2), we have

$$U\subset \mathrm{Cl}\,(S_1\cup S_2\cdots\cup S_n)\subset \mathrm{Cl}\,(S\cup S_1\cup S_2\cdots\cup S_n). \tag{4.3}$$

Since U is an open set, then we have

$$S\cup S_1\cup S_2\cdots\cup S_n\subset U=\mathrm{Int}(U)\subset \mathrm{Int}\,\mathrm{Cl}\,(S\cup S_1\cup S_2\cdots\cup S_n). \tag{4.4}$$

Then we have the result. □

Theorem 4.6. *Let X be a locally finite space. If any minimal open set of X has two elements at least, then X is a pre-Hausdorff space.*

Proof. Let x, y be elements of X such that $x\neq y$. Since X is a locally finite space, there exists finite open sets U and V such that $x\in U$ and $y\in V$. By Theorem 3.1, there exists the set $\{U_1,U_2,\dots,U_n\}$ of all minimal open sets in U and the set $\{V_1,V_2,\dots,V_m\}$ of all minimal open sets in V.

Case 1. If there exists i of $\{1,2,\dots,n\}$ and j of $\{1,2,\dots,m\}$ such that $x\in U_i$ and $y\in V_j$, then, by Theorem 2.6, $\{x\}$ and $\{y\}$ are disjoint pre-open sets which contains x and y, respectively.

Case 2. If there exists i of $\{1,2,\dots,n\}$ such that $x\in U_i$ and $y\notin V_j$ for any j of $\{1,2,\dots,m\}$, then we find an element y_j of V_j for each j such that $\{x\}$ and $\{y,y_1,y_2,\dots,y_m\}$ are pre-open sets and $\{x\}\cap\{y,y_1,y_2,\dots,y_n\}=\varnothing$ by Theorems 2.6, 4.5 and the assumption.

Case 3. If $x\notin U_i$ for any i of $\{1,2,\dots,n\}$ and $y\notin V_j$ for any j of $\{1,2,\dots,m\}$, then we find elements x_i of U_i and y_j of V_j for each i, j such that $\{x,x_1,x_2,\dots,x_n\}$ and $\{y,y_1,y_2,\dots,y_m\}$ are pre-open sets and $\{x,x_1,x_2,\dots,x_n\}\cap\{y,y_1,y_2,\dots,y_m\}=\varnothing$ by Theorem 4.5 and the assumption. We remark that we use the assumption that any minimal open set of X has at least two elements for the case $U_i=V_j$ for some i and j in the argument of cases (2) and (3).

Therefore X is a pre-Hausdorff space. □

References

[1] I. M. James, *Alexandroff spaces*, Rend. Circ. Mat. Palermo (2) Suppl. (1992), no. 29, 475–481, International Meeting on Topology in Italy (Italian) (Lecce, 1990/Otranto, 1990). MR 94g:54020. Zbl 793.54006.

[2] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt (1982), no. 53, 47–53. MR 87c:54002. Zbl 571.54011.

Fumie Nakaoka: Department of Applied Mathematics, Fukuoka University, Nanakuma Jonan-ku, Fukuoka 814-0180, Japan

E-mail address: fumie@fukuoka-u.ac.jp

Nobuyuki Oda: Department of Applied Mathematics, Fukuoka University, Nanakuma Jonan-ku, Fukuoka 814-0180, Japan

E-mail address: odanobu@cis.fukuoka-u.ac.jp

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk