IJMMS 27:7 (2001) 419–424
PII. S0161171201006081
http://ijmms.hindawi.com

ON n-FOLD FUZZY IMPLICATIVE/COMMUTATIVE IDEALS OF BCK-ALGEBRAS

YOUNG BAE JUN

(Received 3 November 2000)

Abstract. We consider the fuzzification of the notion of an n-fold implicative ideal, an n-fold (weak) commutative ideal. We give characterizations of an n-fold fuzzy implicative ideal. We establish an extension property for n-fold fuzzy commutative ideals.

2000 Mathematics Subject Classification. 06F35, 03G25, 03E72.

1. Introduction. Huang and Chen [1] introduced the notion of n-fold implicative ideals and n-fold (weak) commutative ideals. The aim of this paper is to discuss the fuzzification of n-fold implicative ideals, n-fold commutative ideals and n-fold weak commutative ideals. We show that every n-fold fuzzy implicative ideal is an n-fold fuzzy positive implicative ideal, and so a fuzzy ideal, and give a condition for a fuzzy ideal to be an n-fold fuzzy implicative ideal. Using the level set, we provide a characterization of an n-fold fuzzy implicative ideal. We also give a condition for a fuzzy ideal to be an n-fold fuzzy (weak) commutative ideal. We show that every n-fold fuzzy positive implicative ideal which is an n-fold fuzzy weak commutative ideal is an n-fold fuzzy implicative ideal. Finally, we establish an extension property for n-fold fuzzy commutative ideals.

2. Preliminaries. We include some elementary aspects of BCK-algebras that are necessary for this paper, and for more details we refer to [1, 2, 4, 5]. By a *BCK-algebra* we mean an algebra $(X;*,0)$ of type $(2,0)$ satisfying the axioms:

(I) $((x*y)*(x*z))*(z*y)=0$,

(II) $(x*(x*y))*y=0$,

(III) $x*x=0$,

(IV) $0*x=0$,

(V) $x*y=0$ and $y*x=0$ imply $x=y$, for all $x,y,z\in X$.

We can define a partial ordering $\leq$ on X by $x\leq y$ if and only if $x*y=0$. In any BCK-algebra X, the following hold:

(P1) $x*0=x$,

(P2) $x*y\leq x$,

(P3) $(x*y)*z=(x*z)*y$,

(P4) $(x*z)*(y*z)\leq x*y$,

(P5) $x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$.

Throughout, X will always mean a BCK-algebra unless otherwise specified. A nonempty subset I of X is called an *ideal* of X if it satisfies:

(I1) $0\in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

A nonempty subset I of X is said to be an *implicative ideal* of X if it satisfies:

(I1) $0 \in I$,

(I3) $(x * (y * x)) * z \in I$ and $z \in I$ imply $x \in I$.

A nonempty subset I of X is said to be a *commutative ideal* of X if it satisfies:

(I1) $0 \in I$,

(I4) $(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \to [0,1]$. For a fuzzy set μ in X and $t \in [0,1]$ define $U(\mu;t)$ to be the set $U(\mu;t) := \{x \in X \mid \mu(x) \geq t\}$.

A fuzzy set μ in X is said to be a *fuzzy ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Note that every fuzzy ideal μ of X is order reversing, that is, if $x \leq y$ then $\mu(x) \geq \mu(y)$.

A fuzzy set μ in X is called a *fuzzy implicative ideal* of X if it satisfies:

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F3) $\mu(x) \geq \min\{\mu((x * (y * x)) * z), \mu(z)\}$ for all $x, y, z \in X$.

A fuzzy set μ in X is called a *fuzzy commutative ideal* of X if it satisfies:

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F4) $\mu(x * (y * (y * x))) \geq \min\{\mu((x * y) * z), \mu(z)\}$ for all $x, y, z \in X$.

3. n-fold fuzzy implicative ideals.

For any elements x and y of a BCK-algebra X, $x * y^n$ denotes

$$(\cdots((x * y) * y) * \cdots) * y \tag{3.1}$$

in which y occurs n times. Huang and Chen [1] introduced the concept of n-fold implicative ideals as follows.

Definition 3.1 (see [1]). A subset A of X is called an *n-fold implicative ideal* of X if

(I1) $0 \in A$,

(I5) $(x * (y * x^n)) * z \in A$ and $z \in A$ imply $x \in A$ for every $x, y, z \in X$.

We consider the fuzzification of the concept of n-fold implicative ideal.

Definition 3.2. A fuzzy set μ in X is called an *n-fold fuzzy implicative ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F5) $\mu(x) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\}$ for every $x, y, z \in X$.

Notice that the 1-fold fuzzy implicative ideal is a fuzzy implicative ideal.

Theorem 3.3. *Every n-fold fuzzy implicative ideal is a fuzzy ideal.*

Proof. The condition (F2) follows from taking $y = 0$ in (F5). □

The following example shows that the converse of Theorem 3.3 may not be true.

Example 3.4. Let $X = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers, in which the operation $*$ is defined by $x * y = \max\{0, x - y\}$ for all $x, y \in X$. Then X is a BCK-algebra (see [1, Example 1.3]). Let μ be a fuzzy set in X given by $\mu(0) = t_0 > t_1 = \mu(x)$ for all $x (\neq 0) \in X$. Then μ is a fuzzy ideal of X. But μ is not a 2-fold fuzzy implicative ideal of X because

$$\mu(3) = t_1 < t_0 = \mu(0) = \min\{\mu((3 * (14 * 3^2)) * 0), \mu(0)\}. \tag{3.2}$$

We give a condition for a fuzzy ideal to be an n-fold fuzzy implicative ideal.

Theorem 3.5. *A fuzzy ideal μ of X is n-fold fuzzy implicative if and only if $\mu(x) \geq \mu(x * (y * x^n))$ for all $x, y \in X$.*

Proof. Necessity is by taking $z = 0$ in (F5). Suppose that a fuzzy ideal μ satisfies the inequality $\mu(x) \geq \mu(x * (y * x^n))$ for all $x, y \in X$. Then

$$\mu(x) \geq \mu(x * (y * x^n)) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\}. \tag{3.3}$$

Hence μ is an n-fold fuzzy implicative ideal of X. □

Theorem 3.6. *A fuzzy set μ in X is an n-fold fuzzy implicative ideal of X if and only if the nonempty level set $U(\mu; t)$ of μ is an n-fold implicative ideal of X for every $t \in [0, 1]$.*

Proof. Assume that μ is an n-fold fuzzy implicative ideal of X and $U(\mu; t) \neq \emptyset$ for every $t \in [0, 1]$. Then there exists $x \in U(\mu; t)$. It follows from (F1) that $\mu(0) \geq \mu(x) \geq t$ so that $0 \in U(\mu; t)$. Let $x, y, z \in X$ be such that $(x * (y * x^n)) * z \in U(\mu; t)$ and $z \in U(\mu; t)$. Then $\mu((x * (y * x^n)) * z) \geq t$ and $\mu(z) \geq t$, which imply from (F5) that

$$\mu(x) \geq \min\{\mu((x * (y * x^n)) * z), \mu(z)\} \geq t \tag{3.4}$$

so that $x \in U(\mu; t)$. Therefore $U(\mu; t)$ is an n-fold implicative ideal of X. Conversely, suppose that $U(\mu; t) (\neq \emptyset)$ is an n-fold implicative ideal of X for every $t \in [0, 1]$. For any $x \in X$, let $\mu(x) = t$. Then $x \in U(\mu; t)$. Since $0 \in U(\mu; t)$, we get $\mu(0) \geq t = \mu(x)$ and so $\mu(0) \geq \mu(x)$ for all $x \in X$. Now assume that there exist $a, b, c \in X$ such that

$$\mu(a) < \min\{\mu((a * (b * a^n)) * c), \mu(c)\}. \tag{3.5}$$

Selecting $s_0 = (1/2)(\mu(a) + \min\{\mu((a * (b * a^n)) * c), \mu(c)\})$, then

$$\mu(a) < s_0 < \min\{\mu((a * (b * a^n)) * c), \mu(c)\}. \tag{3.6}$$

It follows that $(a * (b * a^n)) * c \in U(\mu; s_0)$, $c \in U(\mu; s_0)$, and $a \notin U(\mu; s_0)$. This is a contradiction. Hence μ is an n-fold fuzzy implicative ideal of X. □

Definition 3.7 (see [3])**.** A fuzzy set μ in X is called an *n-fold fuzzy positive implicative ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F6) $\mu(x * y^n) \geq \min\{\mu((x * y^{n+1}) * z), \mu(z)\}$ for all $x, y, z \in X$.

Lemma 3.8 (see [3, Theorem 3.13]). *Let μ be a fuzzy set in X. Then μ is an n-fold fuzzy positive implicative ideal of X if and only if the nonempty level set $U(\mu;t)$ of μ is an n-fold positive implicative ideal of X for every $t \in [0,1]$.*

Lemma 3.9 (see [1, Theorem 2.5]). *Every n-fold implicative ideal is an n-fold positive implicative ideal.*

Using Theorem 3.6 and Lemmas 3.8 and 3.9, we have the following theorem.

Theorem 3.10. *Every n-fold fuzzy implicative ideal is an n-fold fuzzy positive implicative ideal.*

4. n-fold fuzzy commutative ideals

Definition 4.1 (see [1]). A subset A of X is called an *n-fold commutative ideal* of X if

(I1) $0 \in A$,

(I6) $(x*y)*z \in A$ and $z \in A$ imply $x*(y*(y*x^n)) \in A$ for all $x,y,z \in X$.

A subset A of X is called an *n-fold weak commutative ideal* of X if

(I1) $0 \in A$,

(I7) $(x*(x*y^n))*z \in A$ and $z \in A$ imply $y*(y*x) \in A$ for all $x,y,z \in X$.

We consider the fuzzification of n-fold (weak) commutative ideals as follows.

Definition 4.2. A fuzzy set μ in X is called an *n-fold fuzzy commutative ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F7) $\mu(x*(y*(y*x^n))) \geq \min\{\mu((x*y)*z), \mu(z)\}$ for all $x,y,z \in X$.

A fuzzy set μ in X is called an *n-fold fuzzy weak commutative ideal* of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F8) $\mu(y*(y*x)) \geq \min\{\mu((x*(x*y^n))*z), \mu(z)\}$ for all $x,y,z \in X$.

Note that the 1-fold fuzzy commutative ideal is a fuzzy commutative ideal. Putting $y = 0$ and $y = x$ in (F7) and (F8), respectively, we know that every n-fold fuzzy commutative (or fuzzy weak commutative) ideal is a fuzzy ideal.

Theorem 4.3. *Let μ be a fuzzy ideal of X. Then*

(i) *μ is an n-fold fuzzy commutative ideal of X if and only if*

$$\mu(x*(y*(y*x^n))) \geq \mu(x*y) \quad \forall x,y \in X. \tag{4.1}$$

(ii) *μ is an n-fold fuzzy weak commutative ideal of X if and only if*

$$\mu(y*(y*x)) \geq \mu(x*(x*y^n)) \quad \forall x,y \in X. \tag{4.2}$$

Proof. The proof is straightforward. □

Lemma 4.4 (see [3, Theorem 3.12]). *A fuzzy set μ in X is an n-fold fuzzy positive implicative ideal of X if and only if μ is a fuzzy ideal of X in which the following inequality holds:*

(F9) $\mu((x*z^n)*(y*z^n)) \geq \mu((x*y)*z^n)$ $\forall x,y,z \in X$.

Theorem 4.5. *If μ is both an n-fold fuzzy positive implicative ideal and an n-fold fuzzy weak commutative ideal of X, then it is an n-fold fuzzy implicative ideal of X.*

Proof. Let $x, y \in X$. Using Theorem 4.3(ii), Lemma 4.4, (P3), and (III), we have

$$\begin{aligned} \mu(x*(x*(y*x^n))) &\geq \mu((y*x^n)*((y*x^n)*x^n)) \\ &\geq \mu((y*(y*x^n))*x^n) \\ &= \mu((y*x^n)*(y*x^n)) \\ &= \mu(0). \end{aligned} \tag{4.3}$$

It follows from (F1) and (F2) that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x*(x*(y*x^n))), \mu(x*(y*x^n))\} \\ &\geq \min\{\mu(0), \mu(x*(y*x^n))\} \\ &= \mu(x*(y*x^n)) \end{aligned} \tag{4.4}$$

so from Theorem 3.5, μ is an n-fold fuzzy implicative ideal of X. □

Theorem 4.6 (extension property for n-fold fuzzy commutative ideals)**.** *Let μ and ν be fuzzy ideals of X such that $\mu(0) = \nu(0)$ and $\mu \subseteq \nu$, that is, $\mu(x) \leq \nu(x)$ for all $x \in X$. If μ is an n-fold fuzzy commutative ideal of X, then so is ν.*

Proof. Let $x, y \in X$. Taking $u = x*(x*y)$, we have

$$\begin{aligned} \nu(0) = \mu(0) &= \mu(u*y) \\ &\leq \mu(u*(y*(y*u^n))) \\ &\leq \nu(u*(y*(y*u^n))) \\ &= \nu((x*(x*y))*(y*(y*u^n))) \\ &= \nu((x*(y*(y*u^n)))*(x*y)). \end{aligned} \tag{4.5}$$

Since $x*(y*(y*x^n)) \leq x*(y*(y*u^n))$ and since ν is order reversing, it follows that

$$\begin{aligned} \nu(x*(y*(y*x^n))) &\geq \nu(x*(y*(y*u^n))) \\ &\geq \min\{\nu((x*(y*(y*u^n)))*(x*y)), \nu(x*y)\} \\ &\geq \min\{\nu(0), \nu(x*y)\} \\ &= \nu(x*y). \end{aligned} \tag{4.6}$$

Hence, by Theorem 4.3(i), ν is an n-fold fuzzy commutative ideal of X. □

Acknowledgement. This work was supported by Korea Research Foundation Grant (KRF-2000-005-D00003).

References

[1] Y. Huang and Z. Chen, *On ideals in BCK-algebras*, Math. Japon. **50** (1999), no. 2, 211–226. CMP 1 718 851. Zbl 938.06018.

[2] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1978), no. 1, 1–26. MR 80a:03081. Zbl 385.03051.

[3] Y. B. Jun and K. H. Kim, *On n-fold fuzzy positive implicative ideals of BCK-algebras*, Int. J. Math. Math. Sci. **26** (2001), no. 9, 525–537.

[4] Y. B. Jun and E. H. Roh, *Fuzzy commutative ideals of BCK-algebras*, Fuzzy Sets and Systems **64** (1994), no. 3, 401–405. MR 95e:06051. Zbl 846.06011.

[5] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa, Seoul, 1994. MR 95m:06041. Zbl 906.06015.

Young Bae Jun: Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea

E-mail address: ybjun@nongae.gsnu.ac.kr

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk