

ON IMAGINABLE T -FUZZY SUBALGEBRAS AND IMAGINABLE T -FUZZY CLOSED IDEALS IN BCH-ALGEBRAS

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ABSTRACT. We inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a t -norm T , we introduce the notion of (imaginable) T -fuzzy subalgebras and (imaginable) T -fuzzy closed ideals, and obtain some related results. We give relations between an imaginable T -fuzzy subalgebra and an imaginable T -fuzzy closed ideal. We discuss the direct product and T -product of T -fuzzy subalgebras. We show that the family of T -fuzzy closed ideals is a completely distributive lattice.

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1. Introduction. In 1983, Hu et al. introduced the notion of a BCH-algebra which is a generalization of a BCK/BCI-algebra (see [6, 7]). In [4], Chaudhry et al. stated ideals and filters in BCH-algebras, and studied their properties. For further properties on BCH-algebras, we refer to [2, 3, 5]. In [8], the first author considered the fuzzification of ideals and filters in BCH-algebras, and then described the relation among fuzzy subalgebras, fuzzy closed ideals and fuzzy filters in BCH-algebras. In this paper, we inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a t -norm T , we introduce the notion of (imaginable) T -fuzzy subalgebras and (imaginable) T -fuzzy closed ideals, and obtain some related results. We give relations between an imaginable T -fuzzy subalgebra and an imaginable T -fuzzy closed ideal. We discuss the direct product and T -product of T -fuzzy subalgebras. We show that the family of T -fuzzy closed ideals is a completely distributive lattice.

2. Preliminaries. By a *BCH-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

- (H1) $x * x = 0$,
- (H2) $x * y = 0$ and $y * x = 0$ imply $x = y$,
- (H3) $(x * y) * z = (x * z) * y$,

for all $x, y, z \in X$.

In a BCH-algebra X , the following statements hold:

- (P1) $x * 0 = x$.

(P2) $x * 0 = 0$ implies $x = 0$.

(P3) $0 * (x * y) = (0 * x) * (0 * y)$.

A nonempty subset A of a BCH-algebra X is called a *subalgebra* of X if $x * y \in A$ whenever $x, y \in A$. A nonempty subset A of a BCH-algebra X is called a *closed ideal* of X if

(i) $0 * x \in A$ for all $x \in A$,

(ii) $x * y \in A$ and $y \in A$ imply that $x \in A$.

In what follows, let X denote a BCH-algebra unless otherwise specified. A *fuzzy set* in X is a function $\mu : X \rightarrow [0, 1]$. Let μ be a fuzzy set in X . For $\alpha \in [0, 1]$, the set $U(\mu; \alpha) = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a *level set* of μ .

A fuzzy set μ in X is called a *fuzzy subalgebra* of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}, \quad \forall x, y \in X. \quad (2.1)$$

DEFINITION 2.1 (see [1]). By a *t-norm* T on $[0, 1]$, we mean a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

(T1) $T(x, 1) = x$,

(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$,

(T3) $T(x, y) = T(y, x)$,

(T4) $T(x, T(y, z)) = T(T(x, y), z)$, for all $x, y, z \in [0, 1]$.

In what follows, let T denote a *t-norm* on $[0, 1]$ unless otherwise specified. Denote by Δ_T the set of elements $\alpha \in [0, 1]$ such that $T(\alpha, \alpha) = \alpha$, that is,

$$\Delta_T := \{\alpha \in [0, 1] \mid T(\alpha, \alpha) = \alpha\}. \quad (2.2)$$

Note that every *t-norm* T has a useful property:

(P4) $T(\alpha, \beta) \leq \min(\alpha, \beta)$ for all $\alpha, \beta \in [0, 1]$.

3. Fuzzy closed ideals

DEFINITION 3.1 (see [8]). A fuzzy set μ in X is called a *fuzzy closed ideal* of X if

(F1) $\mu(0 * x) \geq \mu(x)$ for all $x \in X$,

(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

THEOREM 3.2. Let D be a subset of X and let μ_D be a fuzzy set in X defined by

$$\mu_D(x) = \begin{cases} \alpha_1 & \text{if } x \in D, \\ \alpha_2 & \text{if } x \notin D, \end{cases} \quad (3.1)$$

for all $x \in X$ and $\alpha_1 > \alpha_2$. Then μ_D is a fuzzy closed ideal of X if and only if D is a closed ideal of X .

PROOF. Assume that μ_D is a fuzzy closed ideal of X . Let $x \in D$. Then, by (F1), we have $\mu(0 * x) \geq \mu(x) = \alpha_1$ and so $\mu(0 * x) = \alpha_1$. It follows that $0 * x \in D$. Let $x, y \in X$ be such that $x * y \in D$ and $y \in D$. Then $\mu_D(x * y) = \alpha_1 = \mu_D(y)$, and hence

$$\mu_D(x) \geq \min\{\mu_D(x * y), \mu_D(y)\} = \alpha_1. \quad (3.2)$$

Thus $\mu_D(x) = \alpha_1$, that is, $x \in D$. Therefore D is a closed ideal of X .

Conversely, suppose that D is a closed ideal of X . Let $x \in X$. If $x \in D$, then $0 * x \in D$ and thus $\mu_D(0 * x) = \alpha_1 = \mu_D(x)$. If $x \notin D$, then $\mu_D(x) = \alpha_2 \leq \mu_D(0 * x)$. Let $x, y \in X$. If $x * y \in D$ and $y \in D$, then $x \in D$. Hence

$$\mu_D(x) = \alpha_1 = \min\{\mu_D(x * y), \mu_D(y)\}. \quad (3.3)$$

If $x * y \notin D$ and $y \notin D$, then clearly $\mu_D(x) \geq \min\{\mu_D(x * y), \mu_D(y)\}$. If exactly one of $x * y$ and y belong to D , then exactly one of $\mu_D(x * y)$ and $\mu_D(y)$ is equal to α_2 . Therefore, $\mu_D(x) \geq \alpha_2 = \min\{\mu_D(x * y), \mu_D(y)\}$. Consequently, μ_D is a fuzzy closed ideal of X . $\square$

Using the notion of level sets, we give a characterization of a fuzzy closed ideal.

THEOREM 3.3. *A fuzzy set μ in X is a fuzzy closed ideal of X if and only if the nonempty level set $U(\mu; \alpha)$ of μ is a closed ideal of X for all $\alpha \in [0, 1]$.*

We then call $U(\mu; \alpha)$ a *level closed ideal* of μ .

PROOF. Assume that μ is a fuzzy closed ideal of X and $U(\mu; \alpha) \neq \emptyset$ for all $\alpha \in [0, 1]$. Let $x \in U(\mu; \alpha)$. Then $\mu(0 * x) \geq \mu(x) \geq \alpha$, and so $0 * x \in U(\mu; \alpha)$. Let $x, y \in X$ be such that $x * y \in U(\mu; \alpha)$ and $y \in U(\mu; \alpha)$. Then

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{\alpha, \alpha\} = \alpha, \quad (3.4)$$

and thus $x \in U(\mu; \alpha)$. Therefore $U(\mu; \alpha)$ is a closed ideal of X . Conversely, suppose that $U(\mu; \alpha) \neq \emptyset$ is a closed ideal of X . If $\mu(0 * a) < \mu(a)$ for some $a \in X$, then $\mu(0 * a) < \alpha_0 < \mu(a)$ by taking $\alpha_0 := 1/2(\mu(0 * a) + \mu(a))$. It follows that $a \in U(\mu; \alpha_0)$ and $0 * a \notin U(\mu; \alpha_0)$, which is a contradiction. Hence $\mu(0 * x) \geq \mu(x)$ for all $x \in X$. Assume that there exist $x_0, y_0 \in X$ such that

$$\mu(x_0) < \min\{\mu(x_0 * y_0), \mu(y_0)\}. \quad (3.5)$$

Taking $\beta_0 := 1/2(\mu(x_0) + \min\{\mu(x_0 * y_0), \mu(y_0)\})$, we get $\mu(x_0) < \beta_0 < \mu(x_0 * y_0)$ and $\mu(x_0) < \beta_0 < \mu(y_0)$. Thus $x_0 * y_0 \in U(\mu; \beta_0)$ and $y_0 \in U(\mu; \beta_0)$, but $x_0 \notin U(\mu; \beta_0)$. This is impossible. Hence μ is a fuzzy closed ideal of X . $\square$

THEOREM 3.4. *Let μ be a fuzzy set in X and $\text{Im}(\mu) = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$, where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{D_k \mid k = 0, 1, 2, \dots, n\}$ be a family of closed ideals of X such that*

(i) $D_0 \subseteq D_1 \subseteq \dots \subseteq D_n = X$,

(ii) $\mu(D_k^*) = \alpha_k$, where $D_k^* = D_k \setminus D_{k-1}$ and $D_{-1} = \emptyset$ for $k = 0, 1, \dots, n$.

Then μ is a fuzzy closed ideal of X .

PROOF. For any $x \in X$ there exists $k \in \{0, 1, \dots, n\}$ such that $x \in D_k^*$. Since D_k is a closed ideal of X , it follows that $0 * x \in D_k$. Thus $\mu(0 * x) \geq \alpha_k = \mu(x)$. To prove that μ satisfies condition (F2), we discuss the following cases: if $x * y \in D_k^*$ and $y \in D_k^*$, then $x \in D_k$ because D_k is a closed ideal of X . Hence

$$\mu(x) \geq \alpha_k = \min\{\mu(x * y), \mu(y)\}. \quad (3.6)$$

If $x * y \notin D_k^*$ and $y \notin D_k^*$, then the following four cases arise:

- (i) $x * y \in X \setminus D_k$ and $y \in X \setminus D_k$,
- (ii) $x * y \in D_{k-1}$ and $y \in D_{k-1}$,
- (iii) $x * y \in X \setminus D_k$ and $y \in D_{k-1}$,
- (iv) $x * y \in D_{k-1}$ and $y \in X \setminus D_k$.

But, in either case, we know that $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. If $x * y \in D_k^*$ and $y \notin D_k^*$, then either $y \in D_{k-1}$ or $y \in X \setminus D_k$. It follows that either $x \in D_k$ or $x \in X \setminus D_k$. Thus $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Similarly for the case $x * y \notin D_k^*$ and $y \in D_k^*$, we have the same result. This completes the proof. $\square$

THEOREM 3.5. *Let Λ be a subset of $[0, 1]$ and let $\{D_\lambda \mid \lambda \in \Lambda\}$ be a collection of closed ideals of X such that*

- (i) $X = \cup_{\lambda \in \Lambda} D_\lambda$,
- (ii) $\alpha > \beta$ if and only if $D_\alpha \subsetneq D_\beta$ for all $\alpha, \beta \in \Lambda$.

Define a fuzzy set μ in X by $\mu(x) = \sup\{\lambda \in \Lambda \mid x \in D_\lambda\}$ for all $x \in X$. Then μ is a fuzzy closed ideal of X .

PROOF. Let $x \in X$. Then there exists $\alpha_i \in \Lambda$ such that $x \in D_{\alpha_i}$. It follows that $0 * x \in D_{\alpha_j}$ for some $\alpha_j \geq \alpha_i$. Hence

$$\mu(x) = \sup\{\alpha_k \in \Lambda \mid \alpha_k \leq \alpha_i\} \leq \sup\{\alpha_k \in \Lambda \mid \alpha_k \leq \alpha_j\} = \mu(0 * x). \quad (3.7)$$

Let $x, y \in X$ be such that $\mu(x * y) = m$ and $\mu(y) = n$, where $m, n \in [0, 1]$. Without loss of generality we may assume that $m \leq n$. To prove μ satisfies condition (F2), we consider the following three cases:

$$(1^\circ) \lambda \leq m, \quad (2^\circ) m < \lambda \leq n, \quad (3^\circ) \lambda > n. \quad (3.8)$$

Case (1°) implies that $x * y \in D_\lambda$ and $y \in D_\lambda$. It follows that $x \in D_\lambda$ so that

$$\mu(x) = \sup\{\lambda \in \Lambda \mid x \in D_\lambda\} \geq m = \min\{\mu(x * y), \mu(y)\}. \quad (3.9)$$

For the case (2°) , we have $x * y \notin D_\lambda$ and $y \in D_\lambda$. Then either $x \in D_\lambda$ or $x \notin D_\lambda$. If $x \in D_\lambda$, then $\mu(x) = n \geq \min\{\mu(x * y), \mu(y)\}$. If $x \notin D_\lambda$, then $x \in D_\delta - D_\lambda$ for some $\delta < \lambda$, and so $\mu(x) > m = \min\{\mu(x * y), \mu(y)\}$. Finally, case (3°) implies $x * y \notin D_\lambda$ and $y \notin D_\lambda$. Thus we have that either $x \in D_\lambda$ or $x \notin D_\lambda$. If $x \in D_\lambda$ then obviously $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. If $x \notin D_\lambda$ then $x \in D_\epsilon - D_\lambda$ for some $\epsilon < \lambda$, and thus $\mu(x) \geq m = \min\{\mu(x * y), \mu(y)\}$. This completes the proof. $\square$

Let D be a subset of X . The least closed ideal of X containing D is called the closed ideal *generated* by D , denoted by $\langle D \rangle$. Note that if C and D are subsets of X and $C \subseteq D$, then $\langle C \rangle \subseteq \langle D \rangle$. Let μ be a fuzzy set in X . The least fuzzy closed ideal of X containing μ is called a fuzzy closed ideal of X *generated* by μ , denoted by $\langle \mu \rangle$.

LEMMA 3.6. *For a fuzzy set μ in X , then*

$$\mu(x) = \sup\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}, \quad \forall x \in X. \quad (3.10)$$

PROOF. Let $\delta := \sup\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0, 1]$ such that $x \in U(\mu; \alpha)$, and so $\delta - \varepsilon < \mu(x)$. Since ε is arbitrary, it

follows that $\mu(x) \geq \delta$. Now let $\mu(x) = \beta$. Then $x \in U(\mu; \beta)$ and hence $\beta \in \{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\}$. Therefore

$$\mu(x) = \beta \leq \sup \{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\} = \delta, \quad (3.11)$$

and consequently $\mu(x) = \delta$, as desired. $\square$

THEOREM 3.7. *Let μ be a fuzzy set in X . Then the fuzzy set μ^* in X defined by*

$$\mu^*(x) = \sup \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle\} \quad (3.12)$$

for all $x \in X$ is the fuzzy closed ideal $\langle \mu \rangle$ generated by μ .

PROOF. We first show that μ^* is a fuzzy closed ideal of X . For any $y \in \text{Im}(\mu^*)$, let $y_n = y - 1/n$ for any $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers, and let $x \in U(\mu^*; y)$. Then $\mu^*(x) \geq y$, and so

$$\sup \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle\} \geq y > y_n, \quad (3.13)$$

for all $n \in \mathbf{N}$. Hence there exists $\beta \in [0, 1]$ such that $\beta > y_n$ and $x \in \langle U(\mu; \beta) \rangle$. It follows that $U(\mu; \beta) \subseteq U(\mu; y_n)$ so that $x \in \langle U(\mu; \beta) \rangle \subseteq \langle U(\mu; y_n) \rangle$ for all $n \in \mathbf{N}$. Consequently, $x \in \cap_{n \in \mathbf{N}} \langle U(\mu; y_n) \rangle$. On the other hand, if $x \in \cap_{n \in \mathbf{N}} \langle U(\mu; y_n) \rangle$, then $y_n \in \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle\}$ for any $n \in \mathbf{N}$. Therefore

$$y - \frac{1}{n} = y_n \leq \sup \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle\} = \mu^*(x), \quad (3.14)$$

for all $n \in \mathbf{N}$. Since n is an arbitrary positive integer, it follows that $y \leq \mu^*(x)$ so that $x \in U(\mu^*; y)$. Hence $U(\mu^*; y) = \cap_{n \in \mathbf{N}} \langle U(\mu; y_n) \rangle$, which is a closed ideal of X . Using [Theorem 3.3](#), we know that μ^* is a fuzzy closed ideal of X . We now prove that μ^* contains μ . For any $x \in X$, let $\beta \in \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle\}$. Then $x \in U(\mu; \beta)$ and so $x \in \langle U(\mu; \beta) \rangle$. Thus we get $\beta \in \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle\}$, and so

$$\{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\} \subseteq \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle\}. \quad (3.15)$$

It follows from [Lemma 3.6](#) that

$$\begin{aligned} \mu(x) &= \sup \{\alpha \in [0, 1] \mid x \in U(\mu; \alpha)\} \\ &\leq \sup \{\alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle\} \\ &= \mu^*(x). \end{aligned} \quad (3.16)$$

Hence $\mu \subseteq \mu^*$. Finally let ν be a fuzzy closed ideal of X containing μ and let $x \in X$. If $\mu^*(x) = 0$, then clearly $\mu^*(x) \leq \nu(x)$. Assume that $\mu^*(x) = y \neq 0$. Then $x \in U(\mu^*; y) = \cap_{n \in \mathbf{N}} \langle U(\mu; y_n) \rangle$, that is, $x \in U(\mu; y_n)$ for all $n \in \mathbf{N}$. It follows that $\nu(x) \geq \mu(x) \geq y_n = y - 1/n$ for all $n \in \mathbf{N}$ so that $\nu(x) \geq y = \mu^*(x)$ since n is arbitrary. This shows that $\mu^* \subseteq \mu$, completing the proof. $\square$

DEFINITION 3.8. A fuzzy closed ideal μ of X is said to be n -valued if $\text{Im}(\mu)$ is a finite set of n elements. When no specific n is intended, we call μ a *finite-valued fuzzy closed ideal*.

THEOREM 3.9. *Let μ be a fuzzy closed ideal of X . Then μ is finite valued if and only if there exists a finite-valued fuzzy set ν in X which generates μ . In this case, the range sets of μ and ν are identical.*

PROOF. If $\mu : X \rightarrow [0, 1]$ is a finite-valued fuzzy closed ideal of X , then we may choose $\nu = \mu$. Conversely, assume that $\nu : X \rightarrow [0, 1]$ is a finite-valued fuzzy set. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be distinct elements of $\nu(X)$ such that $\alpha_1 > \alpha_2 > \dots > \alpha_n$, and let $C_i = \nu^{-1}(\alpha_i)$ for $i = 1, 2, \dots, n$. Clearly, $\cup_{i=1}^j C_i \subseteq \cup_{i=1}^k C_i$ whenever $j < k \leq n$. Hence if we let $D_j = \langle \cup_{i=1}^j C_i \rangle$, then we have the following chain:

$$D_1 \subseteq D_2 \subseteq \dots \subseteq D_n = X. \quad (3.17)$$

Define a fuzzy set $\mu : X \rightarrow [0, 1]$ as follows:

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x \in D_1, \\ \alpha_j & \text{if } x \in D_j \setminus D_{j-1}. \end{cases} \quad (3.18)$$

We claim that μ is a fuzzy closed ideal of X generated by ν . Clearly $\mu(0 * x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Then there exist i and j in $\{1, 2, \dots, n\}$ such that $x * y \in D_i$ and $y \in D_j$. Without loss of generality, we may assume that i and j are the smallest integers such that $i \geq j$, $x * y \in D_i$, and $y \in D_j$. Since D_i is a closed ideal of X , it follows from $D_j \subseteq D_i$ that $x \in D_i$. Hence $\mu(x) \geq \alpha_i = \min\{\mu(x * y), \mu(y)\}$, and so μ is a fuzzy closed ideal of X . If $\nu(x) = \alpha_j$ for every $x \in X$, then $x \in C_j$ and thus $x \in D_j$. But we have $\mu(x) \geq \alpha_j = \nu(x)$. Therefore μ contains ν . Let $\delta : X \rightarrow [0, 1]$ be a fuzzy closed ideal of X containing ν . Then $U(\nu; \alpha_j) \subseteq U(\delta; \alpha_j)$ for every j . Hence $U(\delta; \alpha_j)$, being a closed ideal, contains the closed ideal generated by $U(\nu; \alpha_j) = \cup_{i=1}^j C_i$. Consequently, $D_j \subseteq U(\delta; \alpha_j)$. It follows that μ is contained in δ and that μ is generated by ν . Finally, note that $|\text{Im}(\mu)| = n = |\text{Im}(\nu)|$. This completes the proof. $\square$

THEOREM 3.10. *Let $D_1 \supseteq D_2 \supseteq \dots$ be a descending chain of closed ideals of X which terminates at finite step. For a fuzzy closed ideal μ of X , if a sequence of elements of $\text{Im}(\mu)$ is strictly increasing, then μ is finite valued.*

PROOF. Suppose that μ is infinite valued. Let $\{\alpha_n\}$ be a strictly increasing sequence of elements of $\text{Im}(\mu)$. Then $0 \leq \alpha_1 < \alpha_2 < \dots \leq 1$. Note that $U(\mu; \alpha_t)$ is a closed ideal of X for $t = 1, 2, 3, \dots$. Let $x \in U(\mu; \alpha_t)$ for $t = 2, 3, \dots$. Then $\mu(x) \geq \alpha_t > \alpha_{t-1}$, which implies that $x \in U(\mu; \alpha_{t-1})$. Hence $U(\mu; \alpha_t) \subseteq U(\mu; \alpha_{t-1})$ for $t = 2, 3, \dots$. Since $\alpha_{t-1} \in \text{Im}(\mu)$, there exists $x_{t-1} \in X$ such that $\mu(x_{t-1}) = \alpha_{t-1}$. It follows that $x_{t-1} \in U(\mu; \alpha_{t-1})$, but $x_{t-1} \notin U(\mu; \alpha_t)$. Thus $U(\mu; \alpha_t) \subsetneq U(\mu; \alpha_{t-1})$, and so we obtain a strictly descending chain $U(\mu; \alpha_1) \supsetneq U(\mu; \alpha_2) \supsetneq \dots$ of closed ideals of X which is not terminating. This is impossible and the proof is complete. $\square$

Now we consider the converse of [Theorem 3.10](#).

THEOREM 3.11. *Let μ be a finite-valued fuzzy closed ideal of X . Then every descending chain of closed ideals of X terminates at finite step.*

PROOF. Suppose there exists a strictly descending chain $D_0 \supsetneq D_1 \supsetneq D_2 \supsetneq \dots$ of closed ideals of X which does not terminate at finite step. Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in D_n \setminus D_{n+1}, \quad n = 0, 1, 2, \dots, \\ 1 & \text{if } x \in \cap_{n=0}^{\infty} D_n, \end{cases} \quad (3.19)$$

where D_0 stands for X . Clearly, $\mu(0 * x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Assume that $x * y \in D_n \setminus D_{n+1}$ and $y \in D_k \setminus D_{k+1}$ for $n = 0, 1, 2, \dots; k = 0, 1, 2, \dots$. Without loss of generality, we may assume that $n \leq k$. Then clearly $y \in D_n$, and so $x \in D_n$ because D_n is a closed ideal of X . Hence

$$\mu(x) \geq \frac{n}{n+1} = \min \{ \mu(x * y), \mu(y) \}. \quad (3.20)$$

If $x * y \in \cap_{n=0}^{\infty} D_n$ and $y \in \cap_{n=0}^{\infty} D_n$, then $x \in \cap_{n=0}^{\infty} D_n$. Thus $\mu(x) = 1 = \min \{ \mu(x * y), \mu(y) \}$. If $x * y \notin \cap_{n=0}^{\infty} D_n$ and $y \in \cap_{n=0}^{\infty} D_n$, then there exists a positive integer k such that $x * y \in D_k \setminus D_{k+1}$. It follows that $x \in D_k$ so that

$$\mu(x) \geq \frac{k}{k+1} = \min \{ \mu(x * y), \mu(y) \}. \quad (3.21)$$

Finally suppose that $x * y \in \cap_{n=0}^{\infty} D_n$ and $y \notin \cap_{n=0}^{\infty} D_n$. Then $y \in D_r \setminus D_{r+1}$ for some positive integer r . It follows that $x \in D_r$, and hence

$$\mu(x) \geq \frac{r}{r+1} = \min \{ \mu(x * y), \mu(y) \}. \quad (3.22)$$

Consequently, we conclude that μ is a fuzzy closed ideal of X and μ has an infinite number of different values. This is a contradiction, and the proof is complete. $\square$

THEOREM 3.12. *The following are equivalent:*

- (i) *Every ascending chain of closed ideals of X terminates at finite step.*
- (ii) *The set of values of any fuzzy closed ideal of X is a well-ordered subset of $[0, 1]$.*

PROOF. (i) $\Rightarrow$ (ii). Let μ be a fuzzy closed ideal of X . Suppose that the set of values of μ is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{\alpha_n\}$ such that $\mu(x_n) = \alpha_n$. It follows that

$$U(\mu; \alpha_1) \subsetneq U(\mu; \alpha_2) \subsetneq U(\mu; \alpha_3) \subsetneq \dots \quad (3.23)$$

is a strictly ascending chain of closed ideals of X . This is impossible.

(ii) $\Rightarrow$ (i). Assume that there exists a strictly ascending chain

$$D_1 \subsetneq D_2 \subsetneq D_3 \subsetneq \dots \quad (3.24)$$

of closed ideals of X . Note that $D := \cup_{n \in \mathbb{N}} D_n$ is a closed ideal of X . Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin D_n, \\ \frac{1}{k} & \text{where } k = \min \{ n \in \mathbb{N} \mid x \in D_n \}. \end{cases} \quad (3.25)$$

We claim that μ is a fuzzy closed ideal of X . Let $x \in X$. If $x \notin D_n$, then obviously $\mu(0 * x) \geq 0 = \mu(x)$. If $x \in D_n \setminus D_{n-1}$ for $n = 2, 3, \dots$, then $0 * x \in D_n$. Hence $\mu(0 * x) \geq 1/n = \mu(x)$. Let $x, y \in X$. If $x * y \in D_n \setminus D_{n-1}$ and $y \in D_n \setminus D_{n-1}$ for $n = 2, 3, \dots$, then $x \in D_n$. It follows that

$$\mu(x) \geq \frac{1}{n} = \min\{\mu(x * y), \mu(y)\}. \quad (3.26)$$

Suppose that $x * y \in D_n$ and $y \in D_n \setminus D_m$ for all $m < n$. Then $x \in D_n$, and so $\mu(x) \geq 1/n \geq 1/m + 1 \geq \mu(y)$. Hence $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Similarly for the case $x * y \in D_n \setminus D_m$ and $y \in D_n$, we get $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. Therefore μ is a fuzzy closed ideal of X . Since the chain (3.24) is not terminating, μ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy closed ideal is well ordered. This completes the proof. $\square$

4. T -fuzzy subalgebras and T -fuzzy closed ideals

DEFINITION 4.1. A fuzzy set μ in X is said to satisfy *imaginable property* if $\text{Im}(\mu) \subseteq \Delta_T$.

DEFINITION 4.2. A fuzzy set μ in X is called a *fuzzy subalgebra* of X with respect to a t -norm T (briefly, *T -fuzzy subalgebra* of X) if $\mu(x * y) \geq T(\mu(x), \mu(y))$ for all $x, y \in X$. A T -fuzzy subalgebra of X is said to be *imaginable* if it satisfies the imaginable property.

EXAMPLE 4.3. Let T_m be a t -norm defined by $T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0, 1]$ and let $X = \{0, a, b, c, d\}$ be a BCH-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	0	d
c	c	c	c	0	d
d	d	d	d	d	0

(1) Define a fuzzy set $\mu : X \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} 0.9 & \text{if } x \in \{0, d\}, \\ 0.09 & \text{otherwise.} \end{cases} \quad (4.1)$$

Then μ is a T_m -fuzzy subalgebra of X , which is not imaginable.

(2) Let ν be a fuzzy set in X defined by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in \{0, d\}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

Then ν is an imaginable T_m -fuzzy subalgebra of X .

PROPOSITION 4.4. *Let A be a subalgebra of X and let μ be a fuzzy set in X defined by*

$$\mu(x) := \begin{cases} \alpha_1 & \text{if } x \in A, \\ \alpha_2 & \text{otherwise,} \end{cases} \quad (4.3)$$

for all $x \in X$, where $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 > \alpha_2$. Then μ is a T_m -fuzzy subalgebra of X . In particular, if $\alpha_1 = 1$ and $\alpha_2 = 0$ then μ is an imaginable T_m -fuzzy subalgebra of X , where T_m is the t -norm in [Example 4.3](#).

PROOF. Let $x, y \in X$. If $x \in A$ and $y \in A$ then

$$\begin{aligned} T_m(\mu(x), \mu(y)) &= T_m(\alpha_1, \alpha_1) = \max(2\alpha_1 - 1, 0) \\ &= \begin{cases} 2\alpha_1 - 1 & \text{if } \alpha_1 \geq \frac{1}{2} \\ 0 & \text{if } \alpha_1 < \frac{1}{2} \end{cases} \\ &\leq \alpha_1 = \mu(x * y). \end{aligned} \quad (4.4)$$

If $x \in A$ and $y \notin A$ (or, $x \notin A$ and $y \in A$) then

$$\begin{aligned} T_m(\mu(x), \mu(y)) &= T_m(\alpha_1, \alpha_2) = \max(\alpha_1 + \alpha_2 - 1, 0) \\ &= \begin{cases} \alpha_1 + \alpha_2 - 1 & \text{if } \alpha_1 + \alpha_2 \geq 1 \\ 0 & \text{otherwise} \end{cases} \\ &\leq \alpha_2 \leq \mu(x * y). \end{aligned} \quad (4.5)$$

If $x, y \notin A$ then

$$\begin{aligned} T_m(\mu(x), \mu(y)) &= T_m(2\alpha_2 - 1, 0) \\ &= \begin{cases} 2\alpha_2 - 1 & \text{if } \alpha_2 \geq \frac{1}{2} \\ 0 & \text{if } \alpha_2 < \frac{1}{2} \end{cases} \\ &\leq \alpha_2 \leq \mu(x * y). \end{aligned} \quad (4.6)$$

Hence μ is a T_m -fuzzy subalgebra of X . Assume that $\alpha_1 = 1$ and $\alpha_2 = 0$. Then

$$\begin{aligned} T_m(\alpha_1, \alpha_1) &= \max(\alpha_1 + \alpha_1 - 1, 0) = 1 = \alpha_1, \\ T_m(\alpha_2, \alpha_2) &= \max(\alpha_2 + \alpha_2 - 1, 0) = 0 = \alpha_2. \end{aligned} \quad (4.7)$$

Thus $\alpha_1, \alpha_2 \in \Delta_{T_m}$, that is, $\text{Im}(\mu) \subseteq \Delta_{T_m}$ and so μ is imaginable. This completes the proof. $\square$

PROPOSITION 4.5. *If μ is an imaginable T -fuzzy subalgebra of X , then $\mu(0 * x) \geq \mu(x)$ for all $x \in X$.*

PROOF. For any $x \in X$ we have

$$\begin{aligned}
 \mu(0 * x) &\geq T(\mu(0), \mu(x)) \\
 &= T(\mu(x * x), \mu(x)) \quad [\text{by (H1)}] \\
 &\geq T(T(\mu(x), \mu(x)), \mu(x)) \quad [\text{by (T2) and (T3)}] \\
 &= \mu(x), \quad [\text{since } \mu \text{ satisfies the imaginable property}].
 \end{aligned} \tag{4.8}$$

This completes the proof. $\square$

THEOREM 4.6. *Let μ be a T -fuzzy subalgebra of X and let $\alpha \in [0, 1]$ be such that $T(\alpha, \alpha) = \alpha$. Then $U(\mu; \alpha)$ is either empty or a subalgebra of X , and moreover $\mu(0) \geq \mu(x)$ for all $x \in X$.*

PROOF. Let $x, y \in U(\mu; \alpha)$. Then

$$\mu(x * y) \geq T(\mu(x), \mu(y)) \geq T(\alpha, \alpha) = \alpha, \tag{4.9}$$

which implies that $x * y \in U(\mu; \alpha)$. Hence $U(\mu; \alpha)$ is a subalgebra of X . Since $x * x = 0$ for all $x \in X$, we have $\mu(0) = \mu(x * x) \geq T(\mu(x), \mu(x)) = \mu(x)$ for all $x \in X$. $\square$

Since $T(1, 1) = 1$, we have the following corollary.

COROLLARY 4.7. *If μ is a T -fuzzy subalgebra of X , then $U(\mu; 1)$ is either empty or a subalgebra of X .*

THEOREM 4.8. *Let μ be a T -fuzzy subalgebra of X . If there is a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} T(\mu(x_n), \mu(x_n)) = 1$, then $\mu(0) = 1$.*

PROOF. Let $x \in X$. Then $\mu(0) = \mu(x * x) \geq T(\mu(x), \mu(x))$. Therefore $\mu(0) \geq T(\mu(x_n), \mu(x_n))$ for each $n \in \mathbb{N}$. Since $1 \geq \mu(0) \geq \lim_{n \rightarrow \infty} T(\mu(x_n), \mu(x_n)) = 1$, it follows that $\mu(0) = 1$, this completes the proof. $\square$

Let $f : X \rightarrow Y$ be a mapping of BCH-algebras. For a fuzzy set μ in Y , the *inverse image* of μ under f , denoted by $f^{-1}(\mu)$, is defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in X$.

THEOREM 4.9. *Let $f : X \rightarrow Y$ be a homomorphism of BCH-algebras. If μ is a T -fuzzy subalgebra of Y , then $f^{-1}(\mu)$ is a T -fuzzy subalgebra of X .*

PROOF. For any $x, y \in X$, we have

$$\begin{aligned}
 f^{-1}(\mu)(x * y) &= \mu(f(x * y)) = \mu(f(x) * f(y)) \\
 &\geq T(\mu(f(x)), \mu(f(y))) \\
 &= T(f^{-1}(\mu)(x), f^{-1}(\mu)(y)).
 \end{aligned} \tag{4.10}$$

This completes the proof. $\square$

If μ is a fuzzy set in X and f is a mapping defined on X . The fuzzy set $f(\mu)$ in $f(X)$ defined by $f(\mu)(y) = \sup\{\mu(x) \mid x \in f^{-1}(y)\}$ for all $y \in f(X)$ is called the *image* of μ under f . A fuzzy set μ in X is said to have *sup property* if, for every subset $T \subseteq X$, there exists $t_0 \in T$ such that $\mu(t_0) = \sup\{\mu(t) \mid t \in T\}$.

THEOREM 4.10. *An onto homomorphic image of a fuzzy subalgebra with sup property is a fuzzy subalgebra.*

PROOF. Let $f : X \rightarrow Y$ be an onto homomorphism of BCH-algebras and let μ be a fuzzy subalgebra of X with sup property. Given $u, v \in Y$, let $x_0 \in f^{-1}(u)$ and $y_0 \in f^{-1}(v)$ be such that

$$\mu(x_0) = \sup \{\mu(t) \mid t \in f^{-1}(u)\}, \quad \mu(y_0) = \sup \{\mu(t) \mid t \in f^{-1}(v)\}, \quad (4.11)$$

respectively. Then

$$\begin{aligned} f(\mu)(u * v) &= \sup \{\mu(z) \mid z \in f^{-1}(u * v)\} \\ &\geq \min \{\mu(x_0), \mu(y_0)\} \\ &= \min \{\sup \{\mu(t) \mid t \in f^{-1}(u)\}, \sup \{\mu(t) \mid t \in f^{-1}(v)\}\} \\ &= \min \{f(\mu)(u), f(\mu)(v)\}. \end{aligned} \quad (4.12)$$

Hence $f(\mu)$ is a fuzzy subalgebra of Y . $\square$

[Theorem 4.10](#) can be strengthened in the following way. To do this we need the following definition.

DEFINITION 4.11. A t -norm T on $[0, 1]$ is called a *continuous t -norm* if T is a continuous function from $[0, 1] \times [0, 1]$ to $[0, 1]$ with respect to the usual topology.

Note that the function “min” is a continuous t -norm.

THEOREM 4.12. *Let T be a continuous t -norm and let $f : X \rightarrow Y$ be an onto homomorphism of BCH-algebras. If μ is a T -fuzzy subalgebra of X , then $f(\mu)$ is a T -fuzzy subalgebra of Y .*

PROOF. Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$, and $A_{12} = f^{-1}(y_1 * y_2)$, where $y_1, y_2 \in Y$. Consider the set

$$A_1 * A_2 := \{x \in X \mid x = a_1 * a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2\}. \quad (4.13)$$

If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ and so

$$f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2, \quad (4.14)$$

that is, $x \in f^{-1}(y_1 * y_2) = A_{12}$. Thus $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} f(\mu)(y_1 * y_2) &= \sup \{\mu(x) \mid x \in f^{-1}(y_1 * y_2)\} = \sup \{\mu(x) \mid x \in A_{12}\} \\ &\geq \sup \{\mu(x) \mid x \in A_1 * A_2\} \\ &\geq \sup \{\mu(x_1 * x_2) \mid x_1 \in A_1, x_2 \in A_2\} \\ &\geq \sup \{T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2\}. \end{aligned} \quad (4.15)$$

Since T is continuous, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $\sup \{\mu(x_1) \mid x_1 \in A_1\} - x_1^* \leq \delta$ and $\sup \{\mu(x_2) \mid x_2 \in A_2\} - x_2^* \leq \delta$ then

$$T(\sup \{\mu(x_1) \mid x_1 \in A_1\}, \sup \{\mu(x_2) \mid x_2 \in A_2\}) - T(x_1^*, x_2^*) \leq \varepsilon. \quad (4.16)$$

Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\sup\{\mu(x_1) \mid x_1 \in A_1\} - \mu(a_1) \leq \delta$ and $\sup\{\mu(x_2) \mid x_2 \in A_2\} - \mu(a_2) \leq \delta$. Then

$$T(\sup\{\mu(x_1) \mid x_1 \in A_1\}, \sup\{\mu(x_2) \mid x_2 \in A_2\}) - T(\mu(a_1), \mu(a_2)) \leq \varepsilon. \quad (4.17)$$

Consequently

$$\begin{aligned} f(\mu)(y_1 * y_2) &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2\} \\ &\geq T(\sup\{\mu(x_1) \mid x_1 \in A_1\}, \sup\{\mu(x_2) \mid x_2 \in A_2\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)), \end{aligned} \quad (4.18)$$

which shows that $f(\mu)$ is a T -fuzzy subalgebra of Y . $\square$

LEMMA 4.13 (see [1]). For all $\alpha, \beta, \gamma, \delta \in [0, 1]$,

$$T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma), T(\beta, \delta)). \quad (4.19)$$

THEOREM 4.14. Let $X = X_1 \times X_2$ be the direct product BCH-algebra of BCH-algebras X_1 and X_2 . If μ_1 (resp., μ_2) is a T -fuzzy subalgebra of X_1 (resp., X_2), then $\mu = \mu_1 \times \mu_2$ is a T -fuzzy subalgebra of X defined by

$$\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)), \quad (4.20)$$

for all $(x_1, x_2) \in X_1 \times X_2$.

PROOF. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of $X = X_1 \times X_2$. Then

$$\begin{aligned} \mu(x * y) &= \mu((x_1, x_2) * (y_1, y_2)) = \mu(x_1 * y_1, x_2 * y_2) \\ &= T(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2)) \\ &\geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2))) \\ &= T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2))) \\ &= T(\mu(x_1, x_2), \mu(x_2, y_2)) \\ &= T(\mu(x), \mu(y)). \end{aligned} \quad (4.21)$$

Hence μ is a T -fuzzy subalgebra of X . $\square$

We will generalize the idea to the product of n T -fuzzy subalgebras. We first need to generalize the domain of T to $\prod_{i=1}^n [0, 1]$ as follows:

DEFINITION 4.15 (see [1]). The function $T_n : \prod_{i=1}^n [0, 1] \rightarrow [0, 1]$ is defined by

$$T_n(\alpha_1, \alpha_2, \dots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)), \quad (4.22)$$

for all $1 \leq i \leq n$, where $n \geq 2$, $T_2 = T$, and $T_1 = \text{id}$ (identity).

LEMMA 4.16 (see [1]). For every $\alpha_i, \beta_i \in [0, 1]$ where $1 \leq i \leq n$ and $n \geq 2$,

$$T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \dots, T(\alpha_n, \beta_n)) = T(T_n(\alpha_1, \alpha_2, \dots, \alpha_n), T_n(\beta_1, \beta_2, \dots, \beta_n)). \quad (4.23)$$

THEOREM 4.17. Let $\{X_i\}_{i=1}^n$ be the finite collection of BCH-algebras and $X = \prod_{i=1}^n X_i$ the direct product BCH-algebra of $\{X_i\}$. Let μ_i be a T -fuzzy subalgebra of X_i , where $1 \leq i \leq n$. Then $\mu = \prod_{i=1}^n \mu_i$ defined by

$$\begin{aligned} \mu(x_1, x_2, \dots, x_n) &= \left(\prod_{i=1}^n \mu_i \right) (x_1, x_2, \dots, x_n) \\ &= T_n(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)), \end{aligned} \quad (4.24)$$

is a T -fuzzy subalgebra of the BCH-algebra X .

PROOF. Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be any elements of $X = \prod_{i=1}^n X_i$. Then

$$\begin{aligned} \mu(x * y) &= \mu(x_1 * y_1, x_2 * y_2, \dots, x_n * y_n) \\ &= T_n(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2), \dots, \mu_n(x_n * y_n)) \\ &\geq T_n(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2)), \dots, T(\mu_n(x_n), \mu_n(y_n))) \\ &= T(T_n(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)), T_n(\mu_1(y_1), \mu_2(y_2), \dots, \mu_n(y_n))) \\ &= T(\mu(x_1, x_2, \dots, x_n), \mu(y_1, y_2, \dots, y_n)) \\ &= T(\mu(x), \mu(y)). \end{aligned} \quad (4.25)$$

Hence μ is a T -fuzzy subalgebra of X . □

DEFINITION 4.18. Let μ and ν be fuzzy sets in X . Then the T -product of μ and ν , written $[\mu \cdot \nu]_T$, is defined by $[\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x))$ for all $x \in X$.

THEOREM 4.19. Let μ and ν be T -fuzzy subalgebras of X . If T^* is a t -norm which dominates T , that is,

$$T^*(T(\alpha, \beta), T(\gamma, \delta)) \geq T(T^*(\alpha, \gamma), T^*(\beta, \delta)), \quad (4.26)$$

for all $\alpha, \beta, \gamma, \delta \in [0, 1]$, then the T^* -product of μ and ν , $[\mu \cdot \nu]_{T^*}$, is a T -fuzzy subalgebra of X .

PROOF. For any $x, y \in X$ we have

$$\begin{aligned} [\mu \cdot \nu]_{T^*}(x * y) &= T^*(\mu(x * y), \nu(x * y)) \\ &\geq T^*(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \\ &\geq T(T^*(\mu(x), \nu(x)), T^*(\mu(y), \nu(y))) \\ &= T([\mu \cdot \nu]_{T^*}(x), [\mu \cdot \nu]_{T^*}(y)). \end{aligned} \quad (4.27)$$

Hence $[\mu \cdot \nu]_{T^*}$ is a T -fuzzy subalgebra of X . □

Let $f : X \rightarrow Y$ be an onto homomorphism of BCH-algebras. Let T and T^* be t -norms such that T^* dominates T . If μ and ν are T -fuzzy subalgebras of Y , then the T^* -product of μ and ν , $[\mu \cdot \nu]_{T^*}$, is a T -fuzzy subalgebra of Y . Since every onto homomorphic inverse image of a T -fuzzy subalgebra is a T -fuzzy subalgebra, the

inverse images $f^{-1}(\mu)$, $f^{-1}(\nu)$, and $f^{-1}([\mu \cdot \nu]_{T^*})$ are T -fuzzy subalgebras of X . The next theorem provides that the relation between $f^{-1}([\mu \cdot \nu]_{T^*})$ and the T^* -product $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

THEOREM 4.20. *Let $f : X \rightarrow Y$ be an onto homomorphism of BCH-algebras. Let T^* be a t -norm such that T^* dominates T . Let μ and ν be T -fuzzy subalgebras of Y . If $[\mu \cdot \nu]_{T^*}$ is the T^* -product of μ and ν and $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ is the T^* -product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then*

$$f^{-1}([\mu \cdot \nu]_{T^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}. \quad (4.28)$$

PROOF. For any $x \in X$ we get

$$\begin{aligned} f^{-1}([\mu \cdot \nu]_{T^*})(x) &= [\mu \cdot \nu]_{T^*}(f(x)) \\ &= T^*(\mu(f(x)), \nu(f(x))) \\ &= T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x)) \\ &= [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}(x), \end{aligned} \quad (4.29)$$

This completes the proof. $\square$

DEFINITION 4.21. A fuzzy set μ in X is called a *fuzzy closed ideal* of X under a t -norm T (briefly, *T -fuzzy closed ideal* of X) if

- (F1) $\mu(0 * x) \geq \mu(x)$ for all $x \in X$,
- (F3) $\mu(x) \geq T(\mu(x * y), \mu(y))$ for all $x, y \in X$.

A T -fuzzy closed ideal of X is said to be *imaginable* if it satisfies the imaginable property.

EXAMPLE 4.22. Let T_m be a t -norm in [Example 4.3](#). Consider a BCH-algebra $X = \{0, a, b, c\}$ with Cayley table as follows:

$*$	0	a	b	c
0	0	c	0	c
a	a	0	c	b
b	b	c	0	a
c	c	0	c	0

(1) Define a fuzzy set $\mu : X \rightarrow [0, 1]$ by $\mu(0) = \mu(c) = 0.8$ and $\mu(a) = \mu(b) = 0.3$. Then μ is a T_m -fuzzy closed ideal of X which is not imaginable.

(2) Let ν be a fuzzy set in X defined by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in \{0, c\}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.30)$$

Then ν is an imaginable T_m -fuzzy closed ideal of X .

THEOREM 4.23. *Every imaginable T -fuzzy subalgebra satisfying (F3) is an imaginable T -fuzzy closed ideal.*

PROOF. Using [Proposition 4.5](#), it is straightforward. $\square$

PROPOSITION 4.24. *If μ is an imaginable T -fuzzy closed ideal of X , then $\mu(0) \geq \mu(x)$ for all $x \in X$.*

PROOF. Using (F1), (F3), and (T2), we have

$$\mu(0) \geq T(\mu(0 * x), \mu(x)) \geq T(\mu(x), \mu(x)) = \mu(x) \quad (4.31)$$

for all $x \in X$, completing the proof. $\square$

THEOREM 4.25. *Every T -fuzzy closed ideal is a T -fuzzy subalgebra.*

PROOF. Let μ be a T -fuzzy closed ideal of X and let $x, y \in X$. Then

$$\begin{aligned} \mu(x * y) &\geq T(\mu((x * y) * x), \mu(x)) \quad [\text{by (F3)}] \\ &= T(\mu((x * x) * y), \mu(x)) \quad [\text{by (H3)}] \\ &= T(\mu(0 * y), \mu(x)) \quad [\text{by (H1)}] \\ &\geq T(\mu(x), \mu(y)) \quad [\text{by (F1), (T2), and (T3)}]. \end{aligned} \quad (4.32)$$

Hence μ is a T -fuzzy subalgebra of X . $\square$

The converse of [Theorem 4.25](#) may not be true. For example, the T_m -fuzzy subalgebra μ in [Example 4.3\(1\)](#) is not a T_m -fuzzy closed ideal of X since

$$\mu(a) = 0.09 < 0.9 = T_m(\mu(a * d), \mu(d)). \quad (4.33)$$

We give a condition for a T -fuzzy subalgebra to be a T -fuzzy closed ideal.

THEOREM 4.26. *Let μ be a T -fuzzy subalgebra of X . If μ satisfies the imaginable property and the inequality*

$$\mu(x * y) \leq \mu(y * x) \quad \forall x, y \in X, \quad (4.34)$$

then μ is a T -fuzzy closed ideal of X .

PROOF. Let μ be an imaginable T -fuzzy subalgebra of X which satisfies the inequality

$$\mu(x * y) \leq \mu(y * x) \quad \forall x, y \in X. \quad (4.35)$$

It follows from [Proposition 4.5](#) that $\mu(0 * x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} \mu(x) &= \mu(x * 0) \geq \mu(0 * x) = \mu((y * y) * x) \\ &= \mu((y * x) * y) \geq T(\mu(y * x), \mu(y)) \geq T(\mu(x * y), \mu(y)). \end{aligned} \quad (4.36)$$

Hence μ is a T -fuzzy closed ideal of X . $\square$

PROPOSITION 4.27. *Let T_m be a t -norm in [Example 4.3](#). Let D be a closed ideal of X and let μ be a fuzzy set in X defined by*

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x \in D, \\ \alpha_2 & \text{otherwise,} \end{cases} \quad (4.37)$$

for all $x \in X$.

- (i) If $\alpha_1 = 1$ and $\alpha_2 = 0$, then μ is an imaginable T_m -fuzzy closed ideal of X .
(ii) If $\alpha_1, \alpha_2 \in (0, 1)$ and $\alpha_1 > \alpha_2$, then μ is a T_m -fuzzy closed ideal of X which is not imaginable.

PROOF. (i) If $x \in D$, then $0 * x \in D$ and so $\mu(0 * x) = 1 = \mu(x)$. If $x \notin D$, then clearly $\mu(x) = 0 \leq \mu(0 * x)$. Now obviously if $x \in D$, then

$$\mu(x) = 1 \geq T_m(\mu(x * y), \mu(y)), \quad (4.38)$$

for all $y \in X$. Assume that $x \notin D$. Then $x * y \notin D$ or $y \notin D$, that is, $\mu(x * y) = 0$ or $\mu(y) = 0$. It follows that

$$T_m(\mu(x * y), \mu(y)) = 0 = \mu(x). \quad (4.39)$$

Hence $\mu(x) \geq T_m(\mu(x * y), \mu(y))$ for all $x, y \in X$. Clearly $\text{Im}(\mu) \subseteq \Delta_{T_m}$.

(ii) Similar to (i), we know that μ is a T_m -fuzzy closed ideal of X . Taking $\alpha_1 = 0.7$, then

$$T_m(\alpha_1, \alpha_1) = T_m(0.7, 0.7) = \max(0.7 + 0.7 - 1, 0) = 0.4 \neq \alpha_1. \quad (4.40)$$

Hence $\alpha_1 \notin \Delta_{T_m}$, that is, $\text{Im}(\mu) \not\subseteq \Delta_{T_m}$, and so μ is not imaginable. $\square$

PROPOSITION 4.28. Let μ be an imaginable T -fuzzy closed ideal of X . If μ satisfies the inequality $\mu(x) \geq \mu(0 * x)$ for all $x \in X$, then it satisfies the equality $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$.

PROOF. Let μ be an imaginable T -fuzzy closed ideal of X satisfying the inequality $\mu(x) \geq \mu(0 * x)$ for all $x \in X$. For every $x, y \in X$, we have

$$\begin{aligned} \mu(y * x) &\geq \mu(0 * (y * x)) \quad [\text{by assumption}] \\ &\geq T(\mu((0 * (y * x)) * (x * y)), \mu(x * y)) \quad [\text{by (F3)}] \\ &= T(\mu(((0 * y) * (0 * x)) * (x * y)), \mu(x * y)) \quad [\text{by (P3)}] \\ &= T(\mu(((0 * y) * (x * y)) * (0 * x)), \mu(x * y)) \quad [\text{by (H3)}] \\ &= T(\mu(((0 * (x * y)) * y) * (0 * x)), \mu(x * y)) \quad [\text{by (H3)}] \\ &= T(\mu((((0 * x) * (0 * y)) * y) * (0 * x)), \mu(x * y)) \quad [\text{by (P3)}] \\ &= T(\mu((((0 * x) * (0 * y)) * (0 * x)) * y), \mu(x * y)) \quad [\text{by (H3)}] \\ &= T(\mu((((0 * x) * (0 * x)) * (0 * y)) * y), \mu(x * y)) \quad [\text{by (H3)}] \\ &= T(\mu((0 * (0 * y)) * y), \mu(x * y)) \quad [\text{by (H1)}] \\ &= T(\mu(0), \mu(x * y)) \quad [\text{by (H3) and (H1)}] \\ &= T(\mu((x * y) * (x * y)), \mu(x * y)) \quad [\text{by (H1)}] \\ &\geq T(T(\mu(x * y), \mu(x * y)), \mu(x * y)) \quad [\text{by Proposition 4.24 and (T2)}] \\ &= \mu(x * y) \quad [\text{since } \mu \text{ is imaginable}]. \end{aligned} \quad (4.41)$$

Similarly we have $\mu(x * y) \geq \mu(y * x)$ for all $x, y \in X$, completing the proof. $\square$

THEOREM 4.29. *Every imaginable T -fuzzy closed ideal is a fuzzy closed ideal.*

PROOF. Let μ be an imaginable T -fuzzy closed ideal of X . Then

$$\mu(x) \geq T(\mu(x * y), \mu(y)) \quad \forall x, y \in X. \quad (4.42)$$

Since μ is imaginable, we have

$$\begin{aligned} \min(\mu(x * y), \mu(y)) &= T(\min(\mu(x * y), \mu(y)), \min(\mu(x * y), \mu(y))) \\ &\leq T(\mu(x * y), \mu(y)) \\ &\leq \min(\mu(x * y), \mu(y)). \end{aligned} \quad (4.43)$$

It follows that $\mu(x) \geq T(\mu(x * y), \mu(y)) = \min(\mu(x * y), \mu(y))$ so that μ is a fuzzy closed ideal of X . $\square$

Combining Theorems 3.3, 4.29, we have the following corollary.

COROLLARY 4.30. *If μ is an imaginable T -fuzzy closed ideal of X , then the nonempty level set of μ is a closed ideal of X .*

Noticing that the fuzzy set μ in Example 4.22(1) is a fuzzy closed ideal of X , we know from Example 4.22(1) that there exists a t -norm such that the converse of Theorem 4.29 may not be true.

PROPOSITION 4.31. *Every imaginable T -fuzzy closed ideal is order reversing.*

PROOF. Let μ be an imaginable T -fuzzy closed ideal of X and let $x, y \in X$ be such that $x \leq y$. Using (P4), (T2), Theorem 4.29, Proposition 4.24, and the definition of a fuzzy closed ideal, we get

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x * y), \mu(y)\} \geq T(\mu(x * y), \mu(y)) \\ &= T(\mu(0), \mu(y)) \geq T(\mu(y), \mu(y)) = \mu(y). \end{aligned} \quad (4.44)$$

This completes the proof. $\square$

PROPOSITION 4.32. *Let μ be a T -fuzzy closed ideal of X , where T is a diagonal t -norm on $[0, 1]$, that is, $T(\alpha, \alpha) = \alpha$ for all $\alpha \in [0, 1]$. If $(x * a) * b = 0$ for all $a, b, x \in X$, then $\mu(x) \geq T(\mu(a), \mu(b))$.*

PROOF. Let $a, b, x \in X$ be such that $(x * a) * b = 0$. Then

$$\begin{aligned} \mu(x) &\geq T(\mu(x * a), \mu(a)) \\ &\geq T(T(\mu((x * a) * b), \mu(b)), \mu(a)) \\ &= T(T(\mu(0), \mu(b)), \mu(a)) \\ &\geq T(T(\mu(b), \mu(b)), \mu(a)) \\ &= T(\mu(a), \mu(b)), \end{aligned} \quad (4.45)$$

completing the proof. $\square$

COROLLARY 4.33. *Let μ be a T -fuzzy closed ideal of X , where T is a diagonal t -norm on $[0, 1]$. If $(\cdots ((x * a_1) * a_2) * \cdots) * a_n = 0$ for all $x, a_1, a_2, \dots, a_n \in X$, then*

$$\mu(x) \geq T_n(\mu(a_1), \mu(a_2), \dots, \mu(a_n)). \quad (4.46)$$

PROOF. Using induction on n , the proof is straightforward. $\square$

THEOREM 4.34. *There exists a t -norm T such that every closed ideal of X can be realized as a level closed ideal of a T -fuzzy closed ideal of X .*

PROOF. Let D be a closed ideal of X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases} \quad (4.47)$$

where $\alpha \in (0, 1)$ is fixed. It is clear that $U(\mu; \alpha) = D$. We will prove that μ is a T_m -fuzzy closed ideal of X , where T_m is a t -norm in [Example 4.3](#). If $x \in D$, then $0 * x \in D$ and so $\mu(0 * x) = \alpha = \mu(x)$. If $x \notin D$, then clearly $\mu(x) = 0 \leq \mu(0 * x)$. Let $x, y \in X$. If $x \in D$, then $\mu(x) = \alpha \geq T_m(\mu(x * y), \mu(y))$. If $x \notin D$, then $x * y \notin D$ or $y \notin D$. It follows that $\mu(x) = 0 = T_m(\mu(x * y), \mu(y))$. This completes the proof. $\square$

For a family $\{\mu_\alpha \mid \alpha \in \Lambda\}$ of fuzzy sets in X , define the join $\vee_{\alpha \in \Lambda} \mu_\alpha$ and the meet $\wedge_{\alpha \in \Lambda} \mu_\alpha$ as follows:

$$(\vee_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\}, \quad (\wedge_{\alpha \in \Lambda} \mu_\alpha)(x) = \inf \{\mu_\alpha(x) \mid \alpha \in \Lambda\}, \quad (4.48)$$

for all $x \in X$, where Λ is any index set.

THEOREM 4.35. *The family of T -fuzzy closed ideals in X is a completely distributive lattice with respect to meet “ $\wedge$ ” and the join “ $\vee$ ”.*

PROOF. Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$, it is sufficient to show that $\vee_{\alpha \in \Lambda} \mu_\alpha$ and $\wedge_{\alpha \in \Lambda} \mu_\alpha$ are T -fuzzy closed ideals of X for a family of T -fuzzy closed ideals $\{\mu_\alpha \mid \alpha \in \Lambda\}$. For any $x \in X$, we have

$$\begin{aligned} (\vee_{\alpha \in \Lambda} \mu_\alpha)(0 * x) &= \sup \{\mu_\alpha(0 * x) \mid \alpha \in \Lambda\} \\ &\geq \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\ &= (\vee_{\alpha \in \Lambda} \mu_\alpha)(x), \\ (\wedge_{\alpha \in \Lambda} \mu_\alpha)(0 * x) &= \inf \{\mu_\alpha(0 * x) \mid \alpha \in \Lambda\} \\ &\geq \inf \{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\ &= (\wedge_{\alpha \in \Lambda} \mu_\alpha)(x). \end{aligned} \quad (4.49)$$

Let $x, y \in X$. Then

$$\begin{aligned} (\vee_{\alpha \in \Lambda} \mu_\alpha)(x) &= \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\ &\geq \sup \{T(\mu_\alpha(x * y), \mu_\alpha(y)) \mid \alpha \in \Lambda\} \\ &\geq T(\sup \{\mu_\alpha(x * y) \mid \alpha \in \Lambda\}, \sup \{\mu_\alpha(y) \mid \alpha \in \Lambda\}) \\ &= T((\vee_{\alpha \in \Lambda} \mu_\alpha)(x * y), (\vee_{\alpha \in \Lambda} \mu_\alpha)(y)), \end{aligned}$$

$$\begin{aligned}
(\wedge_{\alpha \in \Lambda} \mu_{\alpha})(x) &= \inf \{ \mu_{\alpha}(x) \mid \alpha \in \Lambda \} \\
&\geq \inf \{ T(\mu_{\alpha}(x * y), \mu_{\alpha}(y)) \mid \alpha \in \Lambda \} \\
&\geq T(\inf \{ \mu_{\alpha}(x * y) \mid \alpha \in \Lambda \}, \inf \{ \mu_{\alpha}(y) \mid \alpha \in \Lambda \}) \\
&= T((\wedge_{\alpha \in \Lambda} \mu_{\alpha})(x * y), (\wedge_{\alpha \in \Lambda} \mu_{\alpha})(y)).
\end{aligned}
\tag{4.50}$$

Hence $\vee_{\alpha \in \Lambda} \mu_{\alpha}$ and $\wedge_{\alpha \in \Lambda} \mu_{\alpha}$ are T -fuzzy closed ideals of X , completing the proof. $\square$

5. Conclusions and future works. We inquired into further properties on fuzzy closed ideals in BCH-algebras, and using a t -norm T , we introduced the notion of (imaginable) T -fuzzy subalgebras and (imaginable) T -fuzzy closed ideals, and obtained some related results. Moreover, we discussed the direct product and T -product of T -fuzzy subalgebras. We finally showed that the family of T -fuzzy closed ideals is a completely distributive lattice. These ideas enable us to define the notion of (imaginable) T -fuzzy filters in BCH-algebras, and to discuss the direct products and T -products of T -fuzzy filters. It also gives us possible problems to discuss relations among T -fuzzy subalgebras, T -fuzzy closed ideals and T -fuzzy filters, and to construct the normalizations. We may also use these ideas to introduce the notion of interval-valued fuzzy subalgebras/closed ideals.

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