ON THE EXTENDED HARDY'S INEQUALITY

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ABSTRACT. We generalize a strengthened version of Hardy's inequality and give a new simpler proof.

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In the recent paper [4], Hardy's inequality was generalized. In this note, the results given in [4] are further generalized and a new much simpler proof is given. The following Hardy's inequality is well known [1, Theorem 349].

THEOREM 1 (Hardy's inequality). *Let* $\lambda_n > 0$, $A_n = \sum_{k=1}^{n} \lambda_k$, $a_n \geq 0$ $(n \in \mathbb{N})$, $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1}$$

Recently, [4] gave an improvement of Theorem 1, and the following result was proved.

THEOREM 2. *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $A_n = \sum_{k=1}^{n} \lambda_k$, $a_n \geq 0$ $(n \in \mathbb{N})$, $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{2(A_n + \lambda_n)} \right) \lambda_n a_n. \tag{2}$$

In this note, we will prove the following theorem.

THEOREM 3. *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $A_n = \sum_{k=1}^{n} \lambda_k$, $a_n \geq 0$ $(n \in \mathbb{N})$, $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \left(1 + \frac{5\lambda_n}{5A_n + \lambda_n} \right)^{-1/2} \lambda_n a_n. \tag{3}$$

To prove Theorem 3, we introduce some lemmas.

LEMMA 4. *For* $x > 0$, *then*

$$e \left(1 - \frac{1}{2x+1} \right) < \left(1 + \frac{1}{x} \right)^x < e \left(1 + \frac{5}{5x+1} \right)^{-1/2}. \tag{4}$$

PROOF. (i) Define $f(x)$ as

$$f(x) = x\ln\left(1 + \frac{1}{x}\right) + \frac{1}{2}\ln\left(1 + \frac{5}{5x+1}\right), \quad x \in (0, +\infty). \tag{5}$$

It is obvious that when $x > 0$, the inequality

$$\left(1 + \frac{1}{x}\right)^x < e\left(1 + \frac{5}{5x+1}\right)^{-1/2} \tag{6}$$

is equivalent to $f(x) < 1$. It is easy to see that

$$f'(x) = -\frac{1}{x+1} + \ln\left(1 + \frac{1}{x}\right) - \frac{25}{2(5x+6)(5x+1)} \tag{7}$$

and for $x \in (0, +\infty)$, it can be shown that

$$\begin{aligned}
f''(x) &= \frac{1}{(x+1)^2} - \frac{1}{x(x+1)} + \frac{25}{2(5x+1)^2} - \frac{25}{2(5x+6)^2} \\
&= \frac{-125x^3 - 50x^2 + 35x - 72}{2x(x+1)^2(5x+1)^2(5x+6)^2} < 0.
\end{aligned} \tag{8}$$

Hence $f'(x)$ is decreasing on $(0, +\infty)$. Then for any $x \in (0, +\infty)$, we have $f'(x) > \lim_{x \to +\infty} f'(x) = 0$, thus, $f(x)$ is increasing on $(0, +\infty)$, and $f(x) < \lim_{x \to +\infty} f(x) = 1$ for $x \in (0, +\infty)$. The inequality (6) is valid.

(ii) Define $g(x)$ as

$$g(x) = x\ln\left(1 + \frac{1}{x}\right) - \ln\left(1 - \frac{1}{2x+1}\right), \quad x \in (0, +\infty). \tag{9}$$

When $x > 0$, the inequality

$$e\left(1 - \frac{1}{2x+1}\right) < \left(1 + \frac{1}{x}\right)^x \tag{10}$$

is equivalent to $g(x) > 1$. For $x \in (0, +\infty)$, it can be shown that

$$\begin{aligned}
g'(x) &= -\frac{1}{x+1} + \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x(2x+1)}, \\
g''(x) &= \frac{5x^2 + 5x + 1}{x^2(x+1)^2(2x+1)^2} > 0.
\end{aligned} \tag{11}$$

Hence, $g'(x)$ is increasing on $(0, +\infty)$. Then for any $x \in (0, +\infty)$, we have $g'(x) < \lim_{x \to +\infty} g'(x) = 0$, therefore, $g(x)$ is decreasing on $(0, +\infty)$ and $g(x) > \lim_{x \to +\infty} g(x) = 1$ for $x \in (0, +\infty)$. Inequality (10) is valid.

By virtue of (6) and (10), inequalities (4) are valid. This proves Lemma 4. □

REMARK 5. By a direct calculation, we have

$$\left(1 + \frac{5}{5x+1}\right)^{-1/2} < 1 - \frac{1}{2(x+19/20)} \quad (x > 0). \tag{12}$$

Then by (4) and (12), we have

$$e\left(1 - \frac{1}{2x+1}\right) < \left(1 + \frac{1}{x}\right)^x < e\left[1 - \frac{1}{2(x+19/20)}\right] \quad (x > 0). \tag{13}$$

Inequality (13) is equivalent to

$$\frac{e}{2(x+19/20)} < e - \left(1 + \frac{1}{x}\right)^x < \frac{e}{2x+1} \quad (x > 0). \tag{14}$$

Thus, [1, Lemma 2] is contained in Lemma 4. Inequalities (4) and (14) are the new inequalities on the constant e (cf. [3, Theorem 3.8.26]; and [2, page 358]).

LEMMA 6 (see [1, Theorem 9]). *Let $g_m > 0$, $\alpha_m \geq 0$ $(m = 1, 2, \ldots, n)$, $\sum_{m=1}^{n} g_m = 1$, then*

$$\alpha_1^{g_1} \alpha_2^{g_2} \cdots \alpha_n^{g_n} \leq \sum_{m=1}^{n} g_m \alpha_m. \tag{15}$$

PROOF OF THEOREM 3. Setting $c_m > 0$, $g_m = \lambda_m/A_n$, $\alpha_m = c_m a_m$ $(m = 1, 2, \ldots, n)$, by Lemma 6, we have

$$(c_1 a_1)^{\lambda_1/A_1} (c_2 a_2)^{\lambda_2/A_2} \cdots (c_n a_n)^{\lambda_n/A_n} \leq \frac{1}{A_n} \sum_{m=1}^{n} \lambda_m c_m a_m. \tag{16}$$

Then we find that

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} = \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1/A_1} (c_2 a_2)^{\lambda_2/A_2} \cdots (c_n a_n)^{\lambda_n/A_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}}$$

$$\leq \sum_{n=1}^{\infty} \left[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}} \right] \frac{1}{A_n} \sum_{m=1}^{n} c_m \lambda_m a_m \tag{17}$$

$$= \sum_{m=1}^{\infty} \lambda_m a_m c_m \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{A_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}}.$$

Define $c_m = ((A_{m+1})/A_m)^{A_m/\lambda_m} A_m$ $(m = 1, 2, \ldots)$ and $A_0 = 0$. Because $0 < \lambda_{n+1} \leq \lambda_n$ $(n = 1, 2, \ldots)$, we have

$$c_m^{\lambda_m} = \frac{(A_{m+1})^{A_m}}{A_m^{A_{m-1}}}; \qquad \left(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}\right)^{1/A_n} = A_{n+1} \quad (n \in \mathbb{N});$$

$$c_m \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{A_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}} = \left(\frac{A_{m+1}}{A_m}\right)^{A_m/\lambda_m} A_m \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{A_n A_{n+1}}$$

$$= \left(1 + \frac{\lambda_{m+1}}{A_m}\right)^{A_m/\lambda_m} A_m \sum_{n=m}^{\infty} \left(\frac{1}{A_n} - \frac{1}{A_{n+1}}\right) \tag{18}$$

$$\leq \left(1 + \frac{\lambda_m}{A_m}\right)^{A_m/\lambda_m}.$$

Then by (4) and (17), we obtain that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/A_n} &\le \sum_{m=1}^{\infty} \left(1 + \frac{\lambda_m}{A_m}\right)^{A_m/\lambda_m} \lambda_m a_m \\
&\le e \sum_{m=1}^{\infty} \left(1 + \frac{5\lambda_m}{5A_m + \lambda_m}\right)^{-1/2} \lambda_m a_m.
\end{aligned}
\tag{19}
$$

Hence inequality (3) is valid, and Theorem 3 is proved. $\square$

REMARK 7. With inequality (12), Theorem 3 is obviously an improvement and an extension of [4, Theorem 1].

Setting $\lambda_n \equiv 1$, (3) changes into

$$
\sum_{n=1}^{\infty} \left(a_1 a_2 \cdots a_n\right)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{5}{5n+1}\right)^{-1/2} a_n.
\tag{20}
$$

By inequality (12), we have

$$
\sum_{n=1}^{\infty} \left(a_1 a_2 \cdots a_n\right)^{1/n} < e \sum_{n=1}^{\infty} \left[1 - \frac{1}{2(n+19/20)}\right] a_n.
\tag{21}
$$

Thus, inequalities (20) and (21) are obviously an improvement and extension of [5, Theorem 3.1].

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