

ON THE SPECTRUM OF THE DISTRIBUTIONAL KERNEL RELATED TO THE RESIDUE

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ABSTRACT. We study the spectrum of the distributional kernel $K_{\alpha,\beta}(x)$, where α and β are complex numbers and x is a point in the space \mathbb{R}^n of the n -dimensional Euclidean space. We found that for any nonzero point ξ that belongs to such a spectrum, there exists the residue of the Fourier transform $(-1)^k \widehat{K_{2k,2k}}(\xi)$, where $\alpha = \beta = 2k$, k is a nonnegative integer and $\xi \in \mathbb{R}^n$.

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1. Introduction. Gel'fand and Shilov [2, pages 253–256] have studied the generalized function P^λ , where

$$P = \sum_{i=1}^p x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \quad (1.1)$$

is a quadratic form, λ is a complex number, and $p+q = n$ is the dimension of \mathbb{R}^n . They found that P^λ has two sets of singularities, namely $\lambda = -1, -2, \dots, -k, \dots$ and $\lambda = -n/2, -n/2 - 1, \dots, -n/2 - k, \dots$, where k is a positive integer. For the singular point $\lambda = -k$, the generalized function P^λ has a simple pole with residue

$$\frac{(-1)^k}{(k-1)!} \delta_1^{(k-1)}(P) \quad \text{or} \quad \text{res}_{\lambda=-k} P^\lambda = \frac{(-1)^k}{(k-1)!} \delta_1^{(k-1)}(P) \quad (1.2)$$

for $p+q = n$ is odd with p odd and q even. Also, for the singular point $\lambda = -n/2 - k$ they obtained

$$\text{res}_{\lambda=-n/2-k} P^\lambda = \frac{(-1)^{q/2} L^k \delta(x)}{2^{2k} k! \Gamma((n/2) + k)} \quad (1.3)$$

for $p+q = n$ is odd with p odd and q even.

Now, let $K_{\alpha,\beta}(x)$ be the convolution of the functions $R_\alpha^H(u)$ and $R_\beta^\ell(v)$, that is,

$$K_{\alpha,\beta}(x) = R_\alpha^H(u) * R_\beta^\ell(v), \quad (1.4)$$

where $R_\alpha^H(u)$ and $R_\beta^\ell(v)$ are defined by (2.1) and (2.3), respectively. Since $R_\alpha^H(u)$ and $R_\beta^\ell(v)$ are tempered distributions, see [4, pages 30–31], thus $K_{\alpha,\beta}(x)$ is also a tempered distribution and is called the distributional kernel.

In this paper, we use the idea of Gel'fand and Shilov to find the residue of the Fourier transform $(-1)^k \widehat{K_{2k,2k}}(\xi)$, where $K_{2k,2k}$ is defined by (1.4) with $\alpha = \beta = 2k$ and k is a nonnegative integer. We found that for any nonzero point ξ that belongs to the spectrum of $(-1)^k K_{2k,2k}(x)$, there exists the residue of the Fourier transform

$(-1)^k \widehat{K_{2k,2k}}(\xi)$. Actually $(-1)^k K_{2k,2k}(x)$ is an elementary solution of the operator \diamond^k iterated k times, that is, $\diamond^k[(-1)^k K_{2k,2k}(x)] = \delta$, where δ is the Dirac-delta distribution.

The operator \diamond^k was first introduced by Kananthai [4] and named as the Diamond operator defined by

$$\diamond^k = \left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^2 \right]^k, \quad (1.5)$$

where $p+q = n$ is the dimension of \mathbb{R}^n .

Moreover, the operator \diamond^k can be expressed as the product of the operators \square^k and Δ^k , that is,

$$\diamond^k = \square^k \Delta^k = \Delta^k \square^k, \quad (1.6)$$

where \square^k is an ultra-hyperbolic operator iterated k times defined by

$$\square^k = \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, \quad (1.7)$$

where $p+q = n$. The operator Δ^k is an elliptic operator or Laplacian iterated k times defined by

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k. \quad (1.8)$$

Trione [7, page 11] has shown that the function $R_{2k}^H(u)$ defined by (2.1) with $\alpha = 2k$ is an elementary solution of the operator \square^k . Also, Aguirre Téllez [1, pages 147-148] has proved that the solution $R_{2k}^H(u)$ exists only for odd n with p odd and q even ($p+q = n$). Moreover, we can show that the function $(-1)^k R_{2k}^\ell(v)$ is an elementary solution of the operator Δ^k , where $R_{2k}^\ell(v)$ is defined by (2.3) with $\beta = 2k$.

2. Preliminaries

DEFINITION 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathbb{R}^n , and write $u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$, $p+q = n$. Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0, u > 0\}$ the set of an interior of the forward cone, and $\overline{\Gamma_+}$ denotes the closure of Γ_+ . For any complex number α , define

$$R_\alpha^H(u) = \begin{cases} \frac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \quad (2.1)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma((2+\alpha-n)/2) \Gamma((1-\alpha)/2) \Gamma(\alpha)}{\Gamma((2+\alpha-p)/2) \Gamma((p-\alpha)/2)}. \quad (2.2)$$

The function $R_\alpha^H(u)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Nozaki [6, page 72]. The function R_α^H is an ordinary function or classical function if $\operatorname{Re}(\alpha) \geq n$ and is a distribution of α if $\operatorname{Re}(\alpha) < n$. Let $\operatorname{supp} R_\alpha^H(u) \subset \overline{\Gamma_+}$, where $\operatorname{supp} R_\alpha^H(u)$ denotes the support of $R_\alpha^H(u)$.

DEFINITION 2.2. Let $x = (x_1, x_2, \dots, x_n)$ be a point of \mathbb{R}^n , and write $v = x_1^2 + x_2^2 + \dots + x_n^2$. For any complex number β , define

$$R_\beta^\ell(v) = \frac{2^{-\beta} \pi^{-n/2} \Gamma((n-\beta)/2) v^{(\beta-n)/2}}{\Gamma(\beta/2)}. \quad (2.3)$$

The function $R_\beta^\ell(v)$ is called the elliptic kernel of Marcel Riesz and is an ordinary function for $\operatorname{Re}(\beta) \geq n$ and is a distribution of β for $\operatorname{Re}(\beta) < n$.

DEFINITION 2.3. Let f be a continuous function, then the Fourier transform of f , denoted by $\mathfrak{F}f$ or $\hat{f}(\xi)$, is defined by

$$\mathfrak{F}f = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx, \quad (2.4)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, and $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$. From (2.4), the inverse Fourier transform of $\hat{f}(\xi)$ is defined by

$$f(x) = \mathfrak{F}^{-1}\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \hat{f}(\xi) dx. \quad (2.5)$$

If f is a distribution with compact support, by [8, Theorem 7.4.3, page 187] (2.5) can be written as

$$\mathfrak{F}f = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i(\xi, x)} \rangle. \quad (2.6)$$

LEMMA 2.4. *Given the equation*

$$\diamond^k u(x) = \delta, \quad (2.7)$$

where \diamond^k is the operator defined by (1.5), and δ is the Dirac-delta distribution, $u(x)$ is an unknown, k is a nonnegative integer and $x \in \mathbb{R}^n$, where n is odd with p odd, q even ($n = p + q$). Then $u(x) = (-1)^k K_{2k,2k}(x)$ is an elementary solution of the operator \diamond^k . Here $K_{2k,2k}(x) = R_{2k}^H(u) * R_{2k}^\ell(v)$ from (1.4) with $\alpha = \beta = 2k$.

PROOF. See [4, page 33]. □

In this paper, we study the spectrum of $(-1)^k K_{2k,2k}(x)$, relate to the residue of the Fourier transform $\widehat{(-1)^k K_{2k,2k}(x)}$.

LEMMA 2.5. *The Fourier transform*

$$\begin{aligned} \widehat{K_{\alpha,\beta}(x)} &= (2\pi)^{n/2} \mathfrak{F}R_\alpha^H(u) \mathfrak{F}R_\beta^\ell(v) \\ &= \frac{(i)^q 2^{\alpha+\beta} \pi^n}{(2\pi)^{n/2} K_n(\alpha) H_n(\beta)} \cdot \frac{\Gamma(\alpha/2) \Gamma(\beta/2)}{\Gamma((n-\alpha)/2) \Gamma((n-\beta)/2)} \\ &\quad \times \left(\sqrt{\sum_{i=1}^p \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2} \right)^{-\alpha} \left(\sqrt{\sum_{i=1}^n \xi_i^2} \right)^{-\beta}, \quad i = \sqrt{-1}. \end{aligned} \quad (2.8)$$

In particular, if $\alpha = \beta = 2k$, k is a nonnegative integer,

$$(-1)^k \widehat{K_{2k,2k}(\xi)} = \frac{1}{(2\pi)^{n/2}} \frac{1}{((\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2)^2)^k}, \quad (2.9)$$

where $R_\alpha^H(u)$ and $R_\beta^\ell(v)$ are defined by (2.1) and (2.3), respectively.

PROOF. See [2, page 194] and [5, pages 156-157]. \square

DEFINITION 2.6. The spectrum of the distributional kernel $K_{\alpha,\beta}(x)$ is the support of the Fourier transform $\widehat{K_{\alpha,\beta}(\xi)}$ or the spectrum of $K_{\alpha,\beta}(x) = \text{supp } \widehat{K_{\alpha,\beta}(\xi)}$. Now, from Lemma 2.5 we obtain

$$\text{supp } \widehat{K_{\alpha,\beta}(\xi)} = (\text{supp } \mathfrak{I}R_\alpha^H(u)) \cap (\text{supp } \mathfrak{I}R_\beta^\ell(v)). \quad (2.10)$$

In particular, from (2.9) the spectrum of

$$(-1)^k K_{2k,2k}(x) = \text{supp} \left[\frac{1}{(2\pi)^{n/2} ((\sum_{i=1}^p \xi_i^2)^2 - (\sum_{j=p+1}^{p+q} \xi_j^2)^2)^k} \right]. \quad (2.11)$$

LEMMA 2.7. Let $P(x_1, x_2, \dots, x_n)$ be a quadratic form of positive definite, and is defined by

$$P = P(x_1, x_2, \dots, x_n) = \left(\sum_{i=1}^p x_i^2 \right)^2 - \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^2, \quad (2.12)$$

then for any testing function $\varphi(x) \in D$, the space of infinitely differentiable function with compact support,

$$\langle \delta^{(k)}(P), \varphi \rangle = \int_0^\infty \left[\left(\frac{\partial}{4s^3 \partial s} \right)^k \left(s^{q-4} \frac{\psi(r, s)}{4} \right) \right]_{s=r} r^{p-1} dr, \quad (2.13)$$

$$\langle \delta^{(k)}(P), \varphi \rangle = (-1)^k \int_0^\infty \left[\left(\frac{\partial}{4r^3 \partial r} \right)^k \left(r^{p-4} \frac{\psi(r, s)}{4} \right) \right]_{r=s} s^{q-1} ds, \quad (2.14)$$

where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$, $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2$, and

$$\psi(r, s) = \int \varphi d\Omega^p d\Omega^q, \quad (2.15)$$

where $d\Omega^p$ and $d\Omega^q$ are the elements of surface area on the unit sphere in \mathbb{R}^p and \mathbb{R}^q , respectively. Both integrals (2.13) and (2.14) converge if $k < (1/4)(p+q-4)$ for any $\varphi(x) \in D$. If $k \geq (1/4)(p+q-4)$, these integrals must be understood in the sense of their regularization and (2.13) defined as $\langle \delta_1^{(k)}(p), \varphi \rangle$ and (2.14) defined as $\langle \delta_2^{(k)}(p), \varphi \rangle$. Moreover, if we put $u = r^2$, $v = s^2$, thus (2.13) and (2.14) become

$$\langle \delta^{(k)}(P), \varphi \rangle = \frac{1}{16} \int_0^\infty \left[\frac{\partial^k}{\partial v^k} (v^{(q-4)/4} \psi_1(u, v)) \right]_{v=u} u^{(1/4)(p-4)} du, \quad (2.16)$$

$$\langle \delta^{(k)}(P), \varphi \rangle = \frac{(-1)^k}{16} \int_0^\infty \left[\frac{\partial^k}{\partial u^k} (u^{(p-4)/4} \psi_1(u, v)) \right]_{u=v} v^{(1/4)(q-4)} dv, \quad (2.17)$$

where $\psi_1(u, v) = \psi(r, s)$.

PROOF. See [2, pages 247-251]. \square

LEMMA 2.8. Let $G_b = \{\xi \in \mathbb{R}^n : |\xi_1| \leq b_1, |\xi_2| \leq b_2, \dots, |\xi_n| \leq b_n\}$ be a parallelepiped in \mathbb{R}^n and b_i ($1 \leq i \leq n$) is a real constant and the inverse Fourier transform of $\widehat{K_{\alpha,\beta}(\xi)}$ is defined by

$$K_{\alpha,\beta}(x) = \mathfrak{I}^{-1} \widehat{K_{\alpha,\beta}(\xi)} = \frac{1}{(2\pi)^{n/2}} \int_{G_b} e^{i(\xi, x)} \widehat{K_{\alpha,\beta}(\xi)} d\xi, \quad (2.18)$$

where $K_{\alpha,\beta}$ is defined by (1.4) and $x, \xi \in \mathbb{R}^n$, then $K_{\alpha,\beta}(x)$ can be extended to the entire function $K_{\alpha,\beta}(z)$ and be analytic for all $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, where \mathbb{C}^n is the n -tuple space of complex number and

$$|K_{\alpha,\beta}(z)| \leq C \exp(b|\operatorname{Im}(z)|), \quad (2.19)$$

where $\exp(b|\operatorname{Im}(z)|) = \exp[b_1|\operatorname{Im}(z_1)| + b_2|\operatorname{Im}(z_2)| + \dots + b_n|\operatorname{Im}(z_n)|]$ and $C = (1/(2\pi)^{n/2}) \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| d\xi$ is a constant. Moreover, $K_{\alpha,\beta}(x)$ has a spectrum contained in G_b .

PROOF. Since the integral of (2.18) converges for all $\xi \in G_b$, thus $K_{\alpha,\beta}(x)$ can be extended to the entire function $K_{\alpha,\beta}(z)$ and be analytic for all $z \in \mathbb{C}^n$. Thus (2.18) can be written as

$$K_{\alpha,\beta}(z) = \frac{1}{(2\pi)^{n/2}} \int_{G_b} e^{i(\xi, z)} \widehat{K_{\alpha,\beta}(\xi)} d\xi. \quad (2.20)$$

Now,

$$\begin{aligned} |K_{\alpha,\beta}(z)| &\leq \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| |\exp(i\xi_1 z_1 + i\xi_2 z_2 + \dots + i\xi_n z_n)| d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| |\exp(i\xi_1 \sigma_1 + i\xi_2 \sigma_2 + \dots + i\xi_n \sigma_n \\ &\quad - \xi_1 \mu_1 - \xi_2 \mu_2 - \dots - \xi_n \mu_n)| d\xi, \end{aligned} \quad (2.21)$$

where

$$z_j = \sigma + i\mu_j \quad (j = 1, 2, \dots, n), \quad (2.22)$$

thus

$$|K_{\alpha,\beta}(z)| \leq \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| d\xi \exp(b_1|\mu_1| + b_2|\mu_2| + \dots + b_n|\mu_n|) \quad (2.23)$$

for $|\xi_j| \leq b_j$, or $|K_{\alpha,\beta}(z)| \leq C \exp(b_1|\operatorname{Im}(z_1)| + b_2|\operatorname{Im}(z_2)| + \dots + b_n|\operatorname{Im}(z_n)|)$, or $|K_{\alpha,\beta}(z)| \leq C \exp(b|\operatorname{Im}(z)|)$, where $C = (1/(2\pi)^{n/2}) \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| d\xi$ is a constant. \square

We must show that the support of $\widehat{K_{\alpha,\beta}(\xi)}$ is contained in G_b . Since $K_{\alpha,\beta}(z)$ is an analytic function that satisfies the inequality (2.19) and is called an entire function of order of growth ≤ 1 and of type $\leq b$, then by Paley-Wiener-Schartz theorem, see [3, page 162], $\widehat{K_{\alpha,\beta}(\xi)}$ has a support contained in G_b , that is the spectrum of $K_{\alpha,\beta}(x)$ is contained in G_b .

In particular, for $\alpha = \beta = 2k$, the spectrum of $(-1)^k K_{2k,2k}(x)$ is also contained in G_b , that is $\text{supp}[\widehat{(-1)^k K_{2k,2k}(\xi)}] \subset G_b$, where $(-1)^k K_{2k,2k}(x)$ is an elementary solution of the Diamond operator \diamond^k by Lemma 2.4, and the Fourier transform $\widehat{(-1)^k K_{2k,2k}(\xi)}$ given by (2.9) can be defined as follows.

DEFINITION 2.9. The Fourier transform

$$\widehat{(-1)^k K_{2k,2k}(\xi)} = \begin{cases} \frac{1}{(2\pi)^{n/2} [(\sum_{i=1}^p \xi_i^2)^2 - (\sum_{j=p+1}^{p+q} \xi_j^2)^2]^k}, & \text{for } \xi \in G_b, \\ 0, & \text{for } \xi \in CG_b, \end{cases} \quad (2.24)$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ and CG_b is the complement of G_b .

3. Main results

THEOREM 3.1. For any nonzero point $\xi \in M$ where M is a spectrum of $(-1)^k K_{2k,2k}(x)$, and $(-1)^k K_{2k,2k}(x)$ is an elementary solution of the operator \diamond^k by Lemma 2.4. Then there exists the residue of the Fourier transform $\widehat{(-1)^k K_{2k,2k}(\xi)}$ at the singular point $\lambda = -k$ and such a residue is

$$\frac{(-1)^{k-1}}{(2\pi)^{n/2} (k-1)!} \delta_1^{(k-1)(p)} \quad \text{or} \quad \text{res}_{\lambda=-k} (-1)^k \widehat{K_{2k,2k}(\xi)} = \frac{(-1)^{k-1}}{(2\pi)^{n/2} (k-1)!} \delta_1^{(k-1)(p)}, \quad (3.1)$$

where

$$P = (\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2), \quad (3.2)$$

$p+q = n$ and $\delta_1^{(k-1)}(P)$ is defined by (2.16) with $\delta^{(k-1)}(P) = \delta_1^{(k-1)}(P)$ and n is odd with p odd, q even.

PROOF. We define the generalized function P^λ , where P is given by (3.2) and λ is a complex number, by

$$\langle P^\lambda, \varphi \rangle = \int_{P>0} P^\lambda(\xi) \varphi(\xi) d\xi, \quad (3.3)$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ and $d\xi = d\xi_1 d\xi_2 \cdots d\xi_n$ and $\varphi(\xi) \in D$, the space of continuous infinitely differentiable function with compact support. Now,

$$\langle P^\lambda, \varphi \rangle = \int_{P>0} [(\xi_1^2 + \xi_2^2 + \dots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \dots + \xi_{p+q}^2)]^\lambda \varphi(\xi) d\xi. \quad (3.4)$$

We transform to bipolar coordinates defined by

$$\begin{aligned} \xi_1 &= rw_1, \quad \xi_2 = rw_2, \dots, \quad \xi_p = rw_p, \\ \xi_{p+1} &= sw_{p+1}, \quad \xi_{p+2} = sw_{p+2}, \dots, \quad \xi_{p+q} = sw_{p+q}, \quad p+q = n, \end{aligned} \quad (3.5)$$

where $\sum_{i=1}^p w_i^2 = 1$ and $\sum_{j=p+1}^{p+q} w_j^2 = 1$. Thus

$$r = \sqrt{\sum_{i=1}^p \xi_i^2}, \quad s = \sqrt{\sum_{j=p+1}^{p+q} \xi_j^2}. \quad (3.6)$$

We have $\langle P^\lambda, \varphi \rangle = \int [r^4 - s^4]^\lambda \varphi(\xi) d\xi$. Since the volume $d\xi = r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q$ where $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in \mathbb{R}^p and \mathbb{R}^q , respectively. Thus

$$\begin{aligned} \langle P^\lambda, \varphi \rangle &= \int_{p>0} (r^4 - s^4)^\lambda \varphi r^{p-1} s^{q-1} dr ds d\Omega^p d\Omega^q \\ &= \int_0^\infty \int_0^r (r^4 - s^4)^\lambda \psi(r, s) r^{p-1} s^{q-1} ds dr, \end{aligned} \quad (3.7)$$

where $\psi(r, s) = \int \varphi d\Omega_p d\Omega_q$.

Since $\varphi(\xi)$ is in D , then $\psi(r, s)$ is an infinitely differentiable function of r^4 and s^4 with bounded support. We now make the change of variable $u = r^4$, $v = s^4$, and writing $\psi(r, s) = \psi_1(u, v)$. Thus we obtain

$$\langle P^\lambda, \varphi \rangle = \frac{1}{16} \int_{u=0}^\infty \int_{v=0}^u (u - v)^\lambda \psi_1(u, v) u^{(p-4)/4} v^{(q-4)/4} dv du. \quad (3.8)$$

Write $v = ut$. We obtain

$$\langle P^\lambda, \varphi \rangle = \frac{1}{16} \int_0^\infty u^{\lambda+(1/4)(p+q)-1} du \int_0^1 (1-t)^\lambda t^{(q-4)/4} \psi_1(u, ut) dt. \quad (3.9)$$

Let the function

$$\Phi(\lambda, u) = \frac{1}{16} \int_0^1 (1-t)^\lambda t^{(q-4)/4} \psi_1(u, ut) dt. \quad (3.10)$$

Thus $\Phi(\lambda, u)$ has singularity at $\lambda = -k$ where it has simple poles. By Gel'fand and Shilov [2, page 254, equation (12)] we obtain the residue of $\Phi(\lambda, u)$ at $\lambda = -k$, that is,

$$\text{res}_{\lambda=-k} \Phi(\lambda, u) = \frac{1}{16} \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \{t^{(q-4)/4} \psi_1(u, ut)\} \right]_{t=1}. \quad (3.11)$$

Thus, $\text{res}_{\lambda=-k} \Phi(\lambda, u)$ is a functional concentrated on the surface $P = 0$ ($t = 1$, $u = v$, $p = u - v = 0$). On the other hand, from (3.9) and (3.10) we have

$$\langle P^\lambda, \varphi \rangle = \int_0^\infty u^{\lambda+(1/4)(p+q)-1} \Phi(\lambda, u) du. \quad (3.12)$$

Thus $\langle P^\lambda, \varphi \rangle$ in (3.12) has singularities at $\lambda = -n/4, -n/4 - 1, \dots, -n/4 - k$. At these points,

$$\text{res}_{\lambda=-n/4-k} \langle P^\lambda, \varphi \rangle = \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \Phi \left(-\frac{n}{4} - k, u \right) \right]_{u=0}. \quad (3.13)$$

Thus the residue of $\langle P^\lambda, \varphi \rangle$ at $\lambda = (-1/2)n - k$ is a functional concentrated on the vertex of the surface P . Now consider the case when the singular point $\lambda = -k$. Write (3.10) in the neighborhood of $\lambda = -k$ in the form $\Phi(\lambda, u) = \Phi_0(u)/(\lambda + k) + \Phi_1(\lambda, u)$ where $\Phi_0(u) = \text{res}_{\lambda=-k} \Phi(\lambda, u)$ and $\Phi_1(\lambda, u)$ is regular at $\lambda = -k$. Substitute $\Phi(\lambda, u)$ into (3.12) we obtain

$$\langle P^\lambda, \varphi \rangle = \frac{1}{\lambda + k} \int_0^\infty u^{\lambda+(1/4)(p+q)-1} \Phi_0(u) du + \int_0^\infty u^{\lambda+(1/4)(p+q)-1} \Phi_1(\lambda, u) du. \quad (3.14)$$

Thus $\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \int_0^\infty u^{-k+(1/4)(p+q)-1} \Phi_0(u) du$. By substituting $\Phi_0(u)$ and (3.11), we obtain

$$\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{(-1)^k}{16(k-1)!} \int_0^\infty \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \{ t^{1(q-4)/4} \psi_1(u, ut) \} \right]_{t=1} u^{-k+(1/4)(p+q)-1} du \quad (3.15)$$

since, we put $v = ut$. Thus $\partial^{k-1}/\partial t^{k-1} = u^{k-1} (\partial^{k-1}/\partial v^{k-1})$, by substituting $\partial^{k-1}/\partial t^{k-1}$ we obtain

$$\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{(-1)^k}{16(k-1)!} \int_0^\infty \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \{ v^{1(q-4)/4} \psi_1(u, v) \} \right]_{u=v} v^{(1/4)p-1} du. \quad (3.16)$$

Now, by (2.16)

$$\text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{(k-1)}(P). \quad (3.17)$$

Since, by [Definition 2.9](#) we have

$$(-1)^k \widehat{K_{2k,2k}(\xi)} = \frac{1}{(2\pi)^{n/2}} P^\lambda \quad \text{for } \lambda = -k, \quad (3.18)$$

and $\xi \in G_b$. Let M be a spectrum of $(-1)^k K_{2k,2k}(x)$ and $M \subset G_b$ by [Lemma 2.8](#). Thus for any nonzero $\xi \in M$ we can find the residue of $(-1)^k \widehat{K_{2k,2k}(\xi)}$, that is,

$$\begin{aligned} \text{res}_{\lambda=-k} \langle (-1)^k \widehat{K_{2k,2k}(\xi)}, \varphi(\xi) \rangle &= \frac{1}{(2\pi)^{n/2}} \text{res}_{\lambda=-k} \langle P^\lambda, \varphi \rangle \\ &= \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!} \langle \delta_1^{(k-1)}(P), \varphi \rangle \end{aligned} \quad (3.19)$$

or $\text{res}_{\lambda=-k} (-1)^k \widehat{K_{2k,2k}(\xi)} = ((-1)^{k-1}/(2\pi)^{n/2}(k-1)!) \delta_1^{(k-1)}(P)$ for $\xi \in M$ and $\xi \neq 0$.

Now consider the case $\xi = 0$. We have from (3.13) that, the residue of $\langle P^\lambda, \varphi \rangle$ occurs at the point $\lambda = (-1/2)n - k$ that is $\text{res}_{\lambda=-(1/2)n-k} \langle P^\lambda, \varphi \rangle$ is a functional concentrated on the vertex of surface P . Since $u = 0$ and $v = ut$, then $u = v = 0$, that implies

$$\sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2} = \sqrt{\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2} = 0. \quad (3.20)$$

It follows that $\xi_1 = \xi_2 = \cdots = \xi_{p+q} = 0$, $p+q = n$. Thus, the residue of $\langle P^\lambda, \varphi \rangle$ is concentrated on the point $\xi = 0$.

Since, from [Definition 2.9](#), $(1/(2\pi)^{n/2}) P^\lambda = (-1)^k \widehat{K_{2k,2k}(\xi)}$ if $\lambda = -k$. Thus we only consider the residue of $(-1)^k \widehat{K_{2k,2k}(\xi)}$ at $\lambda = -k$. From (3.12), we consider the residue of $\langle P^\lambda, \varphi \rangle$ only at $\lambda = -k$. That implies $(1/4)(p+q) - 1 = 0$ or $n = 4$ ($p+q = n$). Since $n = 4$ is an even dimension which contradicts [Lemma 2.4](#), the existence of the elementary solution $(-1)^k K_{2k,2k}(x)$ that exists for odd n . Thus cases (3.12) and (3.13) do not occur. This implies that the case $\xi = 0$ does not happen. It follows that

$$\text{res}_{\lambda=-k} (-1)^k \widehat{K_{2k,2k}(\xi)} = \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!} \delta_1^{(k-1)}(P) \quad (3.21)$$

for nonzero point $\xi \in M$ concentrated on the surface $P = 0$, where M is a spectrum of $(-1)^k K_{2k,2k}(x)$. \square

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