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DOUBLE SEQUENCE CORE THEOREMS

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Abstract. In 1900, Pringsheim gave a definition of the convergence of double sequences. In this paper, that notion is extended by presenting definitions for the limit inferior and limit superior of double sequences. Also the core of a double sequence is defined. By using these definitions and the notion of regularity for 4-dimensional matrices, extensions, and variations of the Knopp Core theorem are proved.

Keywords and phrases. Core of a sequence, double sequence, regular matrix, P-convergent.

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1. Introduction. The notion of convergence for double sequences was presented by Pringsheim. Also, in [2, 3, 4, 5, 10] the 4-dimensional matrix transformation $(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the 4-dimensional matrices and double sequences have complex-valued entries unless specified otherwise. In this paper, we extend the notion of convergence by defining new double sequence spaces and consider the behavior of 4-dimensional matrix transformations on our new spaces. We also present definitions for limit inferior/limit superior of a double sequence, regularity of a 4-dimensional matrix, and the core of a double sequence. Using these definitions and the notion of regularity for a 4-dimensional matrix, we present multidimensional analogues to the Knopp Core theorem. We also present extensions and variations of this theorem.

2. Definitions and preliminary results

Definition 2.1 [Pringsheim, 1900]. A double sequence $[x]$ has *Pringsheim limit* L (denoted by P-$\lim[x] = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k,l > N$. We shall describe such an $[x]$ more briefly as "P-convergent."

A double sequence $[x]$ is *bounded* if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l (which shall be denoted by $[|x|] < M$). Note that a convergent double sequence need not be bounded. In 1900, Pringsheim gave the following definition: a double sequence $[x]$ is called *definite divergent* if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1, k \geq n_2$. This definition is clearly equivalent to P-lim $[|x|] = \infty$.

Definition 2.2. The sequence $[y]$ is a *subsequence* of the double sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j^i\}$ and $\{k_j^i\}$ such that $n_0^1 = k_0^1 = n_{-1}^0 = k_{-1}^0 = 0$ and

n_1^i & k_1^i are both chosen such that $\max\{n_{2i-3}^{i-1}, k_{2i-3}^{i-1}\} < n_1^i$ & k_1^i,
n_2^i & k_2^i are both chosen such that $\max\{n_1^i, k_1^i\} < n_2^i$ & k_2^i,
n_3^i & k_3^i are both chosen such that $\max\{n_2^i, k_2^i\} < n_3^i$ & k_3^i,
$\vdots$
n_{2i-1}^i & k_{2i-1}^i are both chosen such that $\max\{n_{2(i-1)}^i, k_{2(i-1)}^i\} < n_{2i-1}^i$ & k_{2i-1}^i,
with

$$\begin{aligned} y_{1,i} &= x_{n_1^i,k_1^i}, \\ y_{2,i} &= x_{n_2^i,k_2^i}, \\ y_{3,i} &= x_{n_3^i,k_3^i}, \\ &\vdots \\ y_{i,i} &= x_{n_i^i,k_i^i}, \\ y_{i,i+1} &= x_{n_{i+1}^i,k_{i+1}^i}, \\ &\vdots \\ y_{i,2i-1} &= x_{n_{2i-1}^i,k_{2i-1}^i} \end{aligned} \tag{2.1}$$

for $i = 1, 2, 3, \ldots$.

Example 2.1. The double sequences whose n, k-terms are $y_{n,k} = 1$ and $z_{n,k} = -1$ for each n and k are both subsequences of the double sequence whose n, kth term is $x_{n,k} = (-1)^{n+k}$. Indeed, every double sequence of 1's and -1's is a subsequence of this $[x]$.

A two dimensional matrix transformation is said to be *regular* if it maps every convergent sequence into a convergent sequence with the same limit. In 1926, Robison presented a 4-dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness: a 4-dimensional matrix A is said to be RH-*regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The following is a 4-dimensional analogue of the well-known Silverman-Toeplitz theorem [6].

Theorem 2.1 (Hamilton [2], Robison [10]). *The 4-dimensional matrix A is* RH-*regular if and only if*

(RH_1) $\text{P-lim}_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
(RH_2) $\text{P-lim}_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$;
(RH_3) $\text{P-lim}_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*
(RH_4) $\text{P-lim}_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*
(RH_5) $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ *is* P-*convergent; and*
(RH_6) *there exist positive numbers A and B such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

Definition 2.3. A number β is called a *Pringsheim limit point* of the double sequence $[x]$ provided that there exists a subsequence $[y]$ of $[x]$ that has Pringsheim limit β: $\text{P-lim}[y] = \beta$.

Remark 2.1. The definition of a Pringsheim limit point is equivalent to the following statement: β is a Pringsheim limit point of $[x]$ if and only if there exist two increasing index sequences $\{n_i\}$ and $\{k_i\}$ such that $\lim_i x_{n_i,k_i} = \beta$. A double sequence $[x]$ is *divergent* in the Pringsheim sense (P-divergent) provided that $[x]$ is not P-convergent. This is equivalent to the following: a double sequence $[x]$ is P-divergent if and only if either $[x]$ contains two subsequences with distinct finite limit points or $[x]$ contains an unbounded subsequence. Also note that, if $[x]$ contains an unbounded subsequence then $[x]$ also contains a definite divergent subsequence.

In [7] Knopp introduced the concept of the core a complex number sequence. We follow that idea in defining the core of a double sequence.

Definition 2.4. Let P-$C_n\{x\}$ be the least closed convex set that includes all points $x_{k,l}$ for $k,l > n$; then the *Pringsheim core* of the double sequence $[x]$ is the set P-$C\{x\} = \bigcap_{n=1}^{\infty}[\text{P-}C_n\{x\}]$.

Theorem 2.2 [Knopp, 1930]**.** *If A is a nonnegative regular matrix then the core of $[Ax]$ is contained in core of $[x]$, provided that $[Ax]$ exists.*

3. Main results.

In a manner similar to the classical definitions of the limit superior and the limit inferior of a sequence, we present definitions for the limit superior and the limit inferior of a double sequence. Using these definitions one can characterize the Pringsheim core of a real-valued double sequence as the closed interval $[\text{P-}\liminf x, \text{P-}\limsup x]$.

Definition 3.1. Let $[x] = \{x_{k,l}\}$ be a double sequence of real numbers and for each n, let $\alpha_n = \sup_n\{x_{k,l} : k,l \geq n\}$. The *Pringsheim limit superior* of $[x]$ is defined as follows:

(1) if $\alpha = +\infty$ for each n, then P-$\limsup[x] := +\infty$;

(2) if $\alpha < \infty$ for some n, then P-$\limsup[x] := \inf_n\{\alpha_n\}$.

Similarly, let $\beta_n = \inf_n\{x_{k,l} : k,l \geq n\}$ then the *Pringsheim limit inferior* of $[x]$ is defined as follows:

(1) if $\beta_n = -\infty$ for each n, then P-$\liminf[x] := -\infty$;

(2) if $\beta_n > -\infty$ for some n, then P-$\liminf[x] := \sup_n\{\beta_n\}$.

Example 3.1. The following is an example of an $[x]$ which is neither bounded above nor bounded below; however, the Pringsheim limit superior and inferior are both finite numbers

$$x_{k,l} := \begin{cases} k, & \text{if } l = 0, \\ -l, & \text{if } k = 0, \\ (-1)^k, & \text{if } l = k > 0, \\ 0, & \text{otherwise}; \end{cases} \tag{3.1}$$

thus P-$\liminf[x] = -1$ and P-$\limsup[x] = 1$.

The proof of the following proposition is the same as the proof for single dimensional sequences and is therefore left to the reader.

Proposition 3.1. *If $[x]$ is a real-valued double sequence then*

(1) P-$\liminf[x] \leq$ P-$\limsup[x]$;

(2) P-lim$[x] = L$ *if and only if* P-lim sup$[x]$ = P-lim inf$[x] = L$;
(3) P-lim sup$[-x] = -$(P-lim inf$[x]$);
(4) P-lim sup$([x]+[y]) \leq$ (P-lim sup$[x]$) + (P-lim sup$[y]$);
(5) P-lim inf$([x]+[y]) \geq$ (P-lim inf$[x]$) + (P-lim inf$[y]$);
(6) *if* $[y]$ *is a subsequence of the double sequence* $[x]$ *then*

$$\text{P-}\liminf[x] \leq \text{P-}\liminf[y] \leq \text{P-}\limsup[y] \leq \text{P-}\limsup[x]. \tag{3.2}$$

Theorem 3.1. *If A is a nonnegative* RH-*regular summability matrix, then* P-$C\{Ax\} \subseteq$ P-$C\{x\}$ *for any bounded sequence* $[x]$ *for which* $[Ax]$ *exists.*

Proof. Note that if P-$C\{x\}$ is the complex plane then the result is trivial. We shall establish our theorem by considering separately the cases where $[x]$ is bounded or unbounded. In both cases the result will be established by proving the following: if there exists a q such that for $\omega \notin$ P-$C_q\{x\}$, then there exists a p such that $\omega \notin$ P-$C_p\{Ax\}$. When $[x]$ is bounded, P-$C\{x\}$ is not the complex plane, thus there exists an $\omega \notin$ P-$C\{x\}$. This implies that there exists a q for which $\omega \notin$ P-$C_n\{x\}$. Since ω is finite, we may assume that $\omega = 0$ by the linearity of A. Since we are also given that P-$C_q\{x\}$ is a convex set, we can rotate P-$C_q\{x\}$ so that the distance from zero to P-$C_q\{x\}$ is the minimum of $\{|y| : y \in \text{P-}C_q\{x\}\}$ and is on the positive real axis; say that this minimum is $3d$. Since P-$C_q\{x\}$ is convex, all points of P-$C_q\{x\}$ have a real part which is at least $3d$. Let $M = \max\{|x_{k,l}|\}$. By the regularity conditions (RH_1)–(RH_4) and the assumption $a_{m,n,k,l} \geq 0$, there exists an N such that for $m,n > N$ the following holds:

$$\begin{aligned} &\sum_{k,l\in I_1} a_{m,n,k,l} < \frac{d}{3M}, \qquad \sum_{k,l\in I_2} a_{m,n,k,l} < \frac{d}{3M}, \\ &\sum_{k,l\in I_3} a_{m,n,k,l} < \frac{d}{3M}, \qquad \sum_{k,l\in I_4} a_{m,n,k,l} > \frac{2}{3}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} I_1 &= \{(k,l) : 0 \leq k \leq k_0 \;\&\; 0 \leq l \leq l_0\}, \\ I_2 &= \{(k,l) : k_0 < k < \infty \;\&\; 0 \leq l < l_0\}, \\ I_3 &= \{(k,l) : 0 < k \leq k_0 \;\&\; l_0 < l < \infty\}, \\ I_4 &= \{(k,l) : k_0 < k < \infty \;\&\; l_0 < l < \infty\}. \end{aligned} \tag{3.4}$$

Therefore for $m,n > N$

$$\begin{aligned} \Re\left\{\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}\right\} &= \Re\left\{\sum_{k,l\in I_1} a_{m,n,k,l}x_{k,l}\right\} + \Re\left\{\sum_{k,l\in I_2} a_{m,n,k,l}x_{k,l}\right\} \\ &\quad + \Re\left\{\sum_{k,l\in I_3} a_{m,n,k,l}x_{k,l}\right\} + \Re\left\{\sum_{k,l\in I_4} a_{m,n,k,l}x_{k,l}\right\} \\ &> -M\sum_{k,l\in I_1} a_{m,n,k,l} - M\sum_{k,l\in I_2} a_{m,n,k,l} \\ &\quad - M\sum_{k,l\in I_3} a_{m,n,k,l} + 3d\sum_{k,l\in I_4} a_{m,n,k,l} > -M\frac{3d}{3M} + 3d\frac{2}{3} = d. \end{aligned} \tag{3.5}$$

Therefore, $\Re\{Ax\} > d$ which implies that there exists a p for which $\omega = 0$ is also outside of P-$C_p\{Ax\}$. Now suppose that $[x]$ is unbounded; the ω may be the point at infinity or not. If ω is not the point at infinity then choose N such that for $m,n > N$ the following holds:

$$\sum_{k,l\in I_1} a_{m,n,k,l} < \frac{d}{3M}, \qquad \sum_{k,l\in I_2\cup I_3\cup I_4} a_{m,n,k,l} > \frac{2}{3}. \tag{3.6}$$

In a manner similar to the first part we obtain $\Re\{Ax\} > d$. In the case when ω is the point at infinity, P-$C_q\{x\}$ is bounded for all q, which implies that $x_{k,l}$ is bounded for $k,l > q$. We may assume that $[|x|] < B$ for some positive number B without loss of generality. Thus for m and n large we obtain the following:

$$\left|\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}\right| \le \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}|x_{k,l}| \le B\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty. \tag{3.7}$$

Hence, there exists a p such that the point at infinity is outside of P-$C_p\{Ax\}$. This completes the proof of our theorem. □

The following lemma is a multidimensional analogue of a lemma of Agnew in [1]. We use this lemma to prove Theorem 3.2, below.

Lemma 3.1. *If $\{a_{m,n,k,l}\}_{k,l=0,0}^{\infty,\infty}$ is a real or complex-valued 4-dimensional matrix such that* (RH_1), (RH_3), (RH_4), *and* P-$\limsup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}|a_{m,n,k,l}| = M$ *hold, then for any bounded double sequence $[x]$ we obtain the following:*

$$\text{P-}\limsup[|y|] \le M(\text{P-}\limsup[|x|]), \tag{3.8}$$

where

$$y_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}. \tag{3.9}$$

In addition, there exists a real-valued double sequence $[x]$ such that if $a_{m,n,k,l}$ is real with $0 <$ P-$\limsup[|x|] < \infty$ *then*

$$\text{P-}\limsup[|y|] = M(\text{P-}\limsup[|x|]). \tag{3.10}$$

Proof. Let $[x]$ be bounded and define

$$B := \text{P-}\limsup[|x|] < \infty. \tag{3.11}$$

Given $\epsilon > 0$ we can choose an N such that $|x_{k,l}| < (B+\epsilon)/3$ for each k, and/or $l > N$. Thus,

$$\begin{aligned}|y_{m,n}| &\le \sum_{k,l=0,0}^{N,N}|a_{m,n,k,l}||x_{k,l}| + \sum_{\substack{0\le l\le N,\\ N<k<\infty}}|a_{m,n,k,l}||x_{k,l}| \\ &\quad + \sum_{\substack{N<l\le\infty,\\ 0\le k\le N}}|a_{m,n,k,l}||x_{k,l}| + \sum_{k,l=N+1,N+1}^{\infty,\infty}|a_{m,n,k,l}||x_{k,l}| \\ &\le \sum_{k,l=0,0}^{N,N}|a_{m,n,k,l}||x_{k,l}| + \sum_{\substack{0\le l\le N,\\ N<k<\infty}}|a_{m,n,k,l}|\left(\frac{B+\epsilon}{3}\right) \\ &\quad + \sum_{\substack{N<l\le\infty,\\ 0\le k\le N}}|a_{m,n,k,l}|\left(\frac{B+\epsilon}{3}\right) + \sum_{k,l=N+1,N+1}^{\infty,\infty}|a_{m,n,k,l}|\left(\frac{B+\epsilon}{3}\right),\end{aligned} \tag{3.12}$$

which yields

$$\text{P-}\limsup[|y|] \le M(B+\epsilon). \tag{3.13}$$

Therefore the following holds:

$$\text{P-}\limsup[|y|] \le M(\text{P-}\limsup[|x|]). \tag{3.14}$$

Since

$$\text{P-}\limsup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| = M, \tag{3.15}$$

we may assume that $M > 0$ without loss of generality. Using the RH-regularity conditions we choose m_0, n_0, l_0, and k_0, so large that

$$\begin{aligned} &\sum_{k,l=0,0}^{\infty,\infty} |a_{m_0,n_0,k,l}| > M - \frac{1}{4}, && \sum_{\substack{0<l<l_0,\\ k_0\le k\le\infty}} |a_{m_0,n_0,k,l}| \le \frac{1}{4}, \\ &\sum_{\substack{l_0\le l\le\infty,\\ 0<k<k_0}} |a_{m_0,n_0,k,l}| \le \frac{1}{4}, && \sum_{k,l=l_0,k_0}^{\infty,\infty} |a_{m_0,n_0,k,l}| \le \frac{1}{4}. \end{aligned} \tag{3.16}$$

Let $[m_{p-1}], [n_{q-1}], [k_{p-1}]$, and $[l_{q-1}]$ be four chosen strictly increasing index sequences with $p,q = 1\cdots i-1, j-1$ with $k_0 = l_0 > 0$. Using the RH-regularity conditions we now choose $m_i > m_{i-1}$ and $n_j > n_{j-1}$ such that

$$\begin{gathered} \sum_{\substack{0\le k\le k_{i-1},\\ 0\le l\le\infty}} \left|a_{m_i,n_j,k,l}\right| < \frac{1}{2^{i+j}}, \qquad \sum_{\substack{0\le l\le l_{j-1},\\ k_{i-1}<k\le\infty}} \left|a_{m_i,n_j,k,l}\right| < \frac{1}{2^{i+j}}, \\ \sum_{k,l=0,0}^{\infty,\infty} |a_{m_i,n_j,k,l}| > M - \frac{1}{2^{i+j}}. \end{gathered} \tag{3.17}$$

In addition, we also choose $k_i > k_{i-1}$ and $l_j > l_{j-1}$ such that

$$\sum_{\substack{k_{i-1}<k<k_i,\\ l_j\le l\le\infty}} \left|a_{m_i,n_j,k,l}\right| < \frac{1}{2^{i+j}} \quad \text{and} \quad \sum_{\substack{l_{j-1}<l<\infty,\\ k_i\le k\le\infty}} \left|a_{m_i,n_j,k,l}\right| < \frac{1}{2^{i+j}}. \tag{3.18}$$

Let us define $[x]$ as follows:

$$x_{k,l} := \begin{cases} \dfrac{\bar{a}_{m_i,n_j,k,l}}{\left|a_{m_i,n_j,k,l}\right|}, & \text{if } k_{i-1} < k < k_i,\ l_{j-1} < l < l_j, \text{ and } a_{m_i,n_j,k,l} \ne 0; \\ 0, & \text{otherwise}. \end{cases} \tag{3.19}$$

Consider the following:

$$\begin{aligned} \left|y_{m_i,n_j}\right| = \left|\sum_{k,l=0,0}^{\infty,\infty} a_{m_i,n_j,k,l}x_{k,l}\right| &\ge -\sum_{\substack{0\le k\le k_{i-1},\\ 0\le l\le\infty}} \left|a_{m_i,n_j,k,l}\right| \\ &\quad - \sum_{\substack{0\le l\le l_{j-1},\\ k_{i-1}<k\le\infty}} \left|a_{m_i,n_j,k,l}\right| - \sum_{\substack{k_{i-1}<k<k_i,\\ l_j\le l\le\infty}} \left|a_{m_i,n_j,k,l}\right| \\ &\quad - \sum_{\substack{l_{j-1}<l<\infty,\\ k_i\le k\le\infty}} \left|a_{m_i,n_j,k,l}\right| + \sum_{\substack{l_{j-1}<l<l_j,\\ k_{i-1}<k<k_i}} a_{m_i,n_j,k,l}\,\text{sgn}\,(a_{m_i,n_j,k,l}) \end{aligned} \tag{3.20}$$

$$\geq -\frac{1}{2^{i+j}} - \frac{1}{2^{i+j}} - \frac{1}{2^{i+j}} - \frac{1}{2^{i+j}} + M - 5\left(\frac{1}{2^{i+j}}\right)$$
$$= M - 9\frac{1}{2^{i+j}}.$$

This implies that

$$\text{P-}\limsup[|y|] \geq M = M(\text{P-}\limsup[|x|]). \tag{3.21}$$

Thus, if A is real-valued then so is $[x]$ with $0 < \limsup[x] < \infty$

$$\text{P-}\limsup[|y|] = M(\text{P-}\limsup[|x|]). \tag{3.22}$$

□

Theorem 3.2. *If A is a 4-dimensional matrix, then the following are equivalent*

(1) *For all real-valued double sequences $[x]$*

$$\text{P-}\limsup[Ax] \leq \text{P-}\limsup[x]; \tag{3.23}$$

(2) *A is an* RH-*regular summability matrix with*

$$\text{P-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| = 1. \tag{3.24}$$

Proof. To show that (1) implies (2), let $[x]$ be a bounded P-convergent double sequence, thus

$$\text{P-}\liminf[x] = \text{P-}\limsup[x] = \text{P-}\lim[x], \tag{3.25}$$

and also

$$\text{P-}\limsup[A(-x)] \leq -(\text{P-}\liminf[x]). \tag{3.26}$$

These imply that $\text{P-}\liminf[x] \leq \text{P-}\liminf[Ax]$; thus

$$\text{P-}\liminf[x] \leq \text{P-}\liminf[Ax] \leq \text{P-}\limsup[Ax] \leq \text{P-}\limsup[x]. \tag{3.27}$$

Hence $[Ax]$ is P-convergent and $\text{P-}\lim[Ax] = \text{P-}\lim[x]$. Therefore A is an RH-regular summability matrix.

By Lemma 3.1 and its proof, there exists a bounded double sequence $[x]$ such that $\limsup[|x|] = 1$ and $\text{P-}\limsup[y] = A$, where A is defined by (RH_6). This implies that

$$1 \leq \text{P-}\liminf_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| \leq \text{P-}\limsup_{m,n} \sum_{k,l,=0,0}^{\infty,\infty} |a_{m,n,k,l}| \leq 1, \tag{3.28}$$

whence

$$\text{P-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| = 1. \tag{3.29}$$

To prove that (2) implies (1) we show that if $[x]$ is a bounded P-convergent sequence and A is an RH-regular matrix with

$$\text{P-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| = 1, \tag{3.30}$$

then

$$\text{P-}\limsup[Ax] \le \text{P-}\limsup[x]. \tag{3.31}$$

For $p,q > 1$ we obtain the following:

$$\begin{aligned}
&\Big|\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}x_{k,l}\Big| \\
&\quad = \Big|\sum_{k,l=0,0}^{\infty,\infty} \frac{|a_{m,n,k,l}x_{k,l}| - a_{m,n,k,l}x_{k,l}}{2} + \sum_{k,l=0,0}^{\infty,\infty} \frac{|a_{m,n,k,l}x_{k,l}| + a_{m,n,k,l}x_{k,l}}{2}\Big| \\
&\quad \le \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}||x_{k,l}| + \sum_{k,l=0,0}^{\infty,\infty} \left(|a_{m,n,k,l}| - a_{m,n,k,l}\right)|x_{k,l}| \\
&\quad \le \|x\| \sum_{k,l=0,0}^{p,q} |a_{m,n,k,l}| + \|x\| \sum_{\substack{p<k<\infty,\\ 0\le l\le q}} |a_{m,n,k,l}| \\
&\qquad + \|x\| \sum_{\substack{0\le k<p,\\ q<l<\infty}} |a_{m,n,k,l}| + \sup_{k,l>p,q} |x| \sum_{k,l>p,q} |a_{m,n,k,l}| + \|x\| \sum_{k,l=0,0}^{\infty,\infty} \left(|a_{m,n,k,l}| - a_{m,n,k,l}\right).
\end{aligned} \tag{3.32}$$

Using (RH_1)–(RH_4) and

$$\text{P-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| = 1, \tag{3.33}$$

we take Pringsheim limits and get the desired result. □

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