Internat. J. Math. & Math. Sci.
Vol. 22, No. 3 (1999) 569–571
S 0161-1712⟨99⟩22569-0

BOUNDED AND L^2-SOLUTIONS TO A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION WITH A SQUARE INTEGRABLE FORCING TERM

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(Received 12 May 1998)

ABSTRACT. This paper presents two theorems concerning the nonlinear differential equation $x'' + c(t)f(x)x' + a(t,x) = e(t)$, where $e(t)$ is a continuous square-integrable function. The first theorem gives sufficient conditions when all the solutions of this equation are bounded while the second theorem discusses when all the solutions are in $L^2[0,\infty)$.

Keywords and phrases. Bounded, L^2-solutions, square-integrable, absolutely integrable.

1991 Mathematics Subject Classification. 34C11.

In this paper, we discuss, using standard methods, the bounded properties of the following second order nonlinear differential equation with square-integrable forcing term $e(t)$. Without loss of generality, we restrict our discussion to the nonnegative real line $[0,\infty)$. Specifically, we study the equation

$$x'' + c(t)f(x)x' + a(t,x) = e(t). \tag{1}$$

Our purpose here is to extend some previous results where $e(t)$ was an absolutely integrable but not necessarily a square integrable function. This is a somewhat more general result since previous work considered functions such as $e(t) = 1/(1+t)^2$ but not functions like $e(t) = 1/(1+t)$ which is not absolutely integrable though it is square integrable. The theorems presented here cover that case. Also, we develop the conditions under which all the solutions are L^2-solutions. By an L^2-solution, we mean a solution of (1) such that $\int_0^\infty x(t)^2 dt < \infty$. For some previous work covering the absolutely integrable and homogeneous cases, see [1, 2, 7, 6, 5, 4], especially, [2] for its excellent bibliography. We now turn our attention to our main results.

We see under what conditions all the solutions of (1), as well as their derivatives, are bounded. In our proof, we do not need to resort to the use of the direct method of Liapunov which is often the case. We now state and prove our first theorem.

THEOREM 1. *Given the differential equation (1) where $e(t)$ is continuous on $[0,\infty)$ and $\int_0^\infty e(t)^2 dt < \infty$. Suppose that $c(\cdot)$ is continuous on $[0,\infty)$ with $c(t) > c_\circ > 0$ and $f(\cdot)$ is continuous on R with $f(x) > f_\circ > 0$ where $c_\circ$ and $f_\circ$ are positive constants. Furthermore, let $a(t,x)$ be continuous on $[0,\infty)xR$ with $\int_0^\infty a(t,x)dx = \infty$ uniformly in t, and $x(\partial/\partial t)a(t,x) \le 0$, then any solution x to (1), as well as its derivative, is bounded as $t \to \infty$ and $\int_0^\infty x'(t)^2 dt < \infty$.*

Proof. By standard existence theory, there is a solution of (1) which exists on $[0,T)$ for some $T > 0$. Multiply equation (1) by x' and perform an integration by parts from 0 to t on the last term of the LHS of (1) in order to obtain,

$$\begin{aligned}\frac{x'(t)^2}{2}+\int_0^t c(s)f(x(s))x'(s)^2ds+\int_{x(0)}^{x(t)}a(t,u)du\\ -\int_0^t\int_{x(0)}^{x(s)}\frac{\partial}{\partial s}a(s,u)du\,ds=\frac{x'(0)^2}{2}+\int_0^t e(s)x'(s)ds.\end{aligned}\tag{2}$$

Now, using the Cauchy-Schwarz inequality for integrals on the RHS of (2), we get

$$\begin{aligned}\frac{x'(t)^2}{2}+\int_0^t c(s)f(x(s))x'(s)^2ds+\int_{x(0)}^{x(t)}a(t,u)du\\ -\int_0^t\int_{x(0)}^{x(s)}\frac{\partial}{\partial s}a(s,u)du\,ds\le\frac{x'(0)^2}{2}+\left(\int_0^t e(s)^2ds\right)^{1/2}\left(\int_0^t x'(s)^2ds\right)^{1/2}.\end{aligned}\tag{3}$$

Next, let $H(t)=\left(\int_0^t x'(s)^2ds\right)^{1/2}$. Dividing both sides by $H(t)$ yields,

$$\begin{aligned}H(t)^{-1}\frac{x'(t)^2}{2}+\int_0^t c(s)f(x(s))x'(s)^2ds+\int_{x(0)}^{x(t)}a(t,u)du\\ -\int_0^t\int_{x(0)}^{x(s)}a(s,u)du\,ds\le H(t)^{-1}\frac{x'(0)^2}{2}+\left(\int_0^t e(s)^2ds\right)^{1/2}.\end{aligned}\tag{4}$$

We first need to show that $|x|$ remains bounded. If not, then should $|x|$ increase without bound, all terms of the LHS of equation (4) become positive by our hypotheses. Furthermore, $H(t)^{-1}c_\circ f_\circ\int_0^t x'(s)^2ds=c_\circ f_\circ(\int_0^t x'(s)^2ds)^{1/2}$ is bounded by the RHS of equation (4). This implies that x' is square integrable and is also bounded after we examine the first term of the LHS of (4). However, the above then implies that $|x|$ must be bounded. Otherwise, the LHS of (4) becomes infinite which is impossible. A standard argument now permits the solution to be extended on all of $[0,\infty)$, [3, p. 17–18]. Our proof is now complete. □

By imposing more stringent conditions on $a(t,x)$, all the solutions become L^2-solutions. We now state and prove under what conditions this is true.

Theorem 2. *Assume the hypotheses of Theorem 1 hold. In addition, suppose that* $a(t,x)x > a_\circ x^2$ *for some positive constant* $a_\circ$, *and* $c'(t)\le 0$, *then all the solutions of (1) are* L^2*-solutions.*

Proof. In order to see that x is in $L^2[0,\infty)$, we must first multiply equation (1) by x, then integrate from 0 to t, and integrate by parts the first term on the LHS in order to obtain

$$\begin{aligned}x(t)x'(t)-\int_0^t x'(s)^2ds+\int_0^t c(s)f(x(s))x(s)x'(s)ds\\ +\int_0^t x(s)a(s,x(s))ds=x(0)x'(0)+\int_0^t e(s)x(s)ds.\end{aligned}\tag{5}$$

Next, let $F(x) = \int_0^x uf(u)du$. Now, upon another integration by parts, the above may be rewritten as

$$x(t)x'(t) - \int_0^t x'(s)^2 ds + c(t)F(x(t)) \\ - \int_0^t F(x(s))c'(s)ds + \int_0^t x(s)a(s,x(s))ds \le K, \tag{6}$$

where $K = |x(0)x'(0)| + |\int_0^t e(s)x(s)ds| + |c(0)F(x(0))|$. Notice that the term $M(t) = \int_0^t e(s)x(s)ds$ is bounded by $(\int_0^t e(s)^2 ds)(\int_0^t x(s)^2)^{1/2}$ by using the Cauchy-Scwharz inequality. Dividing the LHS of (6) by $M(t)$ and using the hypotheses of Theorem 2 immediately yields,

$$M(t)^{-1}x(t)x'(t) - \int_0^t x'(s)^2 ds + c(t)F(x(t)) \\ - \int_0^t F(x(s))c'(s)ds + a_\circ \left(\int_0^t x(s)^2 ds\right)^{1/2} \le \frac{K}{M(t)}. \tag{7}$$

Since the RHS of (7) is bounded and all the terms on the LHS of (7) are either bounded or positive, the result follows because the LHS cannot be unbounded. Here, we need that x' is square integrable.

Example. Consider the second order linear differential equation,

$$x'' + K(t)x' + l(t)x = h(t), \tag{8}$$

where $K(\cdot)$ and $l(\cdot)$ are continuous functions defined for $t \ge 0$ with $K(\cdot)$ and $l(\cdot)$ having continuous nonpositive derivatives with $K(t) > K_\circ > 0, l(t) > l_\circ > 0$, and $\int_0^\infty h(t)^2 dt < \infty$, where $K_\circ$ and $l_\circ$ are positive constants. Given these conditions, the above theorems show that all the solutions of (8), as well as their derivatives, are bounded and in $L^2[0,\infty)$. □

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