Internat. J. Math. & Math. Sci.
Vol. 22, No. 2 (1999) 431–435
S 0161-1712(99)22431-3

A SHORT PROOF OF AN IDENTITY OF SYLVESTER

GAURAV BHATNAGAR

(Received 16 October 1996)

ABSTRACT. We present two short proofs of an identity found by Sylvester and rediscovered by Louck. The first proof is an elementary version of Knuth's proof and is analogous to Macdonald's proof of a related identity of Milne. The second is Sylvester's own proof of his identity.

Keywords and phrases. Sylvester's identity, Louck's identity, Milne's identity, complete homogeneous symmetric function, multiple basic hypergeometric series.

1991 Mathematics Subject Classification. Primary 01A55, 05E05; Secondary 33D70.

1. Sylvester's identity. Our purpose in this paper is to present two proofs of a fundamental identity found in Sylvester's work, which is in Sylvester's own words [17, p. 90], "a simple theorem for expressing, by means of partial fractions, the sum of the homogeneous powers and products of any number of quantities."

The identity in question is

$$h_{q-n+1}(\mathbf{x}) = \sum_{r=1}^{n} x_r^q \prod_{\substack{i=1\\ i\neq r}}^{n} \frac{1}{(x_r - x_i)}, \tag{1.1}$$

where q is a nonnegative integer and the complete homogeneous symmetric function $h_m(\mathbf{x})$ in the variables $\mathbf{x} \equiv (x_1,\dots,x_n)$ is defined by means of the generating function

$$\sum_{m\geq 0} h_m(\mathbf{x}) t^m = \prod_{i=1}^{n} \frac{1}{1 - x_i t}. \tag{1.2}$$

Further, if $m < 0$, then $h_m(\mathbf{x})$ is defined to be 0.

Sylvester [16, p. 42] uses the fact that the sum in (1.1) is 0 when $q = 0,\dots,n-2$, and is a polynomial when $q \geq n-1$. In his later work on partitions [17], he uses (1.1) again. But the identity is most clearly formulated only in the lectures he gave in 1859 [18, p. 156]. A little more than a hundred years later, (1.1) was rediscovered by Louck [8]. Chen and Louck [2] have pointed out that for $q = 0,1,\dots,n-1$, the identity was known to Waring [20] in 1779. The $q = n-1$ case of (1.1) was rediscovered by Good [4] in his elegant proof of Dyson's [3] conjecture.

In Section 2, we will present two short proofs of Sylvester's theorem. Both involve partial fraction expansions. The first proof succeeds in finding the left hand side of (1.1) by starting from the sum in the right hand side and is analogous to Macdonald's proof of a related identity. The second is Sylvester's own proof of his identity and, as suggested by his description above, transforms the left hand side of (1.1) into the sum on the right. Finally, in Section 3, we comment briefly upon the importance of these identities.

2. Partial fractions. We first consider Macdonald's clever proof of an identity found by Milne [12]:

$$\sum_{r=1}^{n}(1-y_r)\prod_{\substack{i=1\\i\neq r}}^{n}\left[\frac{1-x_iy_i/x_r}{1-x_i/x_r}\right]=1-y_1y_2\cdots y_n. \tag{2.1}$$

Macdonald (see [13]) proved (2.1) by setting $t=0$ in the partial fraction expansion

$$\prod_{i=1}^{n}\frac{(1-tx_iy_i)}{(1-tx_i)}=y_1\cdots y_n+\sum_{r=1}^{n}\frac{1-y_r}{1-tx_r}\prod_{\substack{i=1\\i\neq r}}^{n}\left[\frac{1-x_iy_i/x_r}{1-x_i/x_r}\right]. \tag{2.2}$$

Our first proof of Sylvester's identity is analogous to Macdonald's proof of (2.1). To prove (1.1), we consider the partial fraction expansion

$$z^q\prod_{i=1}^{n}\frac{1}{(z-x_i)}=\sum_{r=1}^{n}\frac{x_r^q}{z-x_r}\prod_{\substack{i=1\\i\neq r}}^{n}\frac{1}{(x_r-x_i)}+p_q(z), \tag{2.3}$$

where $p_q(z)=0$ if $q=0,1,\dots,n-1$. Further, if $q\geq n$, then it is a polynomial of degree $q-n$. Let $F_q(x_1,\dots,x_n)$ represent the sum on the right hand side of (1.1). Next, set $z=0$ in (2.3) to obtain

$$F_{q-1}(x_1,\dots,x_n)=p_q(0). \tag{2.4}$$

Our proof will be complete once we compute $p_q(0)$. But $p_q(0)$ is nothing but the constant term in the quotient obtained when z^q is divided by $\prod_{i=1}^{n}(z-x_i)$. That is,

$$\begin{aligned} p_q(0) &= \text{ the constant term in } z^q\prod_{i=1}^{n}\frac{1}{(z-x_i)} \\ &= \text{ the coefficient of } z^{n-q} \text{ in } \prod_{i=1}^{n}\frac{1}{(1-x_i/z)} \\ &= h_{q-n}(\mathbf{x}), \end{aligned} \tag{2.5}$$

by comparing with (1.2), it follows that

$$F_q(x_1,\dots,x_n)=p_{q+1}(0)=h_{q-n+1}(\mathbf{x}). \tag{2.6}$$

This completes the derivation of Sylvester's identity.

Macdonald's proof of (2.1) is very simple, but the choice of the particular rational function on the left hand side of (2.2) is unmotivated. A similar remark holds for (2.3). However, a simple observation remedies this situation.

Once again, consider the sum side of (1.1), where n is replaced by $n+1$, and x_{n+1} is renamed z. In this manner, we obtain

$$F_q(x_1,\dots,x_n,z)=\sum_{r=1}^{n}\frac{x_r^q}{x_r-z}\prod_{\substack{i=1\\i\neq r}}^{n}\frac{1}{(x_r-x_i)}+z^q\prod_{i=1}^{n}\frac{1}{(z-x_i)}. \tag{2.7}$$

It is clear that (2.7) is the same as (2.3), our starting point in the proof of Sylvester's identity. The particular choice of the rational function considered is now transparent. The same observation applies to Macdonald's proof of Milne's identity.

This observation is also relevant to Askey's proof of Milne's identity, which is reproduced by Milne [13]. Askey first proved that the sum side of (2.1) is independent of $x_1,\dots,x_n$. Suppressing even the dependence on $y_1,\dots,y_n$, we let f_n denote the left hand side of (2.1). To complete his proof, Askey found a simple recursion for f_n:

$$f_{n+1} = y_{n+1} f_n + (1 - y_{n+1}), \tag{2.8}$$

from which (2.1) follows quite easily.

Instead, we find another recursion for f_n by replacing n by $n+1$ in (2.1) and taking the limit as $x_{n+1} \to 0$. In this manner, we obtain

$$f_{n+1} - f_n = y_1 \cdots y_n - y_1 \cdots y_{n+1}. \tag{2.9}$$

We also have the initial condition $f_1 = 1 - y_1$. Milne's identity follows by noting that

$$\begin{aligned} f_1 + \sum_{r=1}^{n-1} (f_{r+1} - f_r) &= 1 - y_1 + \sum_{r=1}^{n-1} (y_1 \cdots y_r - y_1 \cdots y_{r+1}) \\ &= 1 - y_1 \cdots y_n, \end{aligned} \tag{2.10}$$

by telescoping. Recursion (2.9) is perhaps even simpler than Askey's recursion.

The proof of (1.1) presented above is also related to Sylvester's proof of his identity. In Sylvester's notes [18], where (1.1) appears explicitly, he does not include his proof. But based on his remarks reproduced above and some of his work in his previous paper [17], it seems likely that he obtained (1.1) by considering the partial fraction expansion

$$\prod_{i=1}^{n} \frac{1}{z - x_i} = \sum_{r=1}^{n} \frac{1}{z - x_r} \prod_{\substack{i=1 \\ i \neq r}}^{n} \frac{1}{(x_r - x_i)}. \tag{2.11}$$

By equating the coefficients of z^{-q-1} on both sides of the equation, we immediately obtain (1.1). Compare this with our computation of $p_q(0)$ above.

It is interesting to note that setting $z = 0$ in (2.11) and replacing x_i by x_i^{-1}, we obtain Good's identity, the $q = n-1$ case of (1.1).

3. Concluding remarks. Our first proof of Sylvester's identity is an elementary version of the proof given by Knuth [7, §1.2.3, problem 33], who found it necessary to use Cauchy's residue theorem. Variations of Sylvester's proof are given by Chen and Louck [2] and Strehl and Wilf [15], though these authors prefer to use the Lagrange interpolation formula rather than partial fractions. Knuth mentions that special cases of (1.1) are useful in the theory of divided differences. Indeed, (1.1) has been rediscovered by Verde-Star [19] in this context. It appears in the context of mathematical physics in the work of Louck and Biedenharn [9, 10]. Far reaching generalizations of (1.1) have been found by Gustafson and Milne [6] and by Chen and Louck [2].

Milne [12] first proved (2.1) using (1.1). Several other proofs of (2.1), including those of Macdonald and Askey, are compiled by Milne [13]. Yet another proof is given by

Strehl and Wilf [15]. Identity (2.1) is fundamental in the study of multiple basic hypergeometric series. See, for instance, Milne [11, 13] and Gustafson [5].

Finally, we note that Macdonald's proof of (2.1) is also relevant. Bhatnagar and Milne [1] and Schlosser [14] have used (2.2) to generalize the identities which Milne [11] found using (2.1).

Acknowledgements. We thank Steve Milne for passing on A. Lascoux's remark that Sylvester's work is peppered with the kind of identities which appear in this paper. It is likely that Milne's identity is also classical. It would be interesting to know about its history. We also thank Mary Scott, Ohio State University's mathematics librarian, for finding the correct reference for Waring [20].

References

[1] G. S. Bhatnagar and S. C. Milne, *Generalized bibasic hypergeometric series and their $U(n)$ extensions*, Adv. in Math. **131** (1997), no. 1, 188–252. Zbl 885.33011.

[2] W. Y. C. Chen and J. D. Louck, *Interpolation for symmetric functions*, Adv. in Math. **117** (1996), no. 1, 147–156. MR 97g:05165. Zbl 857.05092.

[3] F. J. Dyson, *Statistical theory of the energy levels of complex systems. I*, J. Mathematical Phys. **3** (1962), 140–156. MR 26#1111. Zbl 105.41604.

[4] I. J. Good, *Short proof of a conjecture by Dyson*, J. Mathematical Phys. **11** (1970), 1884. MR 41#3290.

[5] R. A. Gustafson, *The Macdonald identities for affine root systems of classical type and hypergeometric series very-well-poised on semisimple Lie algebras*, Ramanujan International Symposium on Analysis (Pune, 1987) (New Delhi), Macmillan of India, 1989, pp. 185–224. MR 92k:33015.

[6] R. A. Gustafson and S. C. Milne, *Schur functions, Good's identity, and hypergeometric series well poised in* $\mathrm{SU}(n)$, Adv. in Math. **48** (1983), no. 2, 177–188. MR 84m:05013. Zbl 516.33015.

[7] D. E. Knuth, *The art of computer programming vol 1: Fundamental algorithms*, second printing of the 2nd ed., vol. XXII, Addison-Wesley Series in Computer Science and Information Processing, no. 634, Addison-Wesley Publishing Co., London, Amsterdam, 1968. MR 51 14624. Zbl 191.17903.

[8] J. D. Louck, *Theory of angular momentum in N-dimensional space*, Tech. Report LA-2451, Los Alamos Scientific Laboratory, 1960.

[9] J. D. Louck and L. C. Biedenharn, *Canonical unit adjoint tensor operators in $U(n)$*, J. Mathematical Phys. **11** (1970), 2368–2414. MR 45 6295. Zbl 196.14501.

[10] ———, *On the structure of the canonical tensor operators in the unitary groups. III. Further developments of the boson polynomials and their implications*, J. Mathematical Phys. **14** (1973), 1336–1357. MR 49 6807.

[11] S. C. Milne, *An elementary proof of the Macdonald identities for $A_l^{(1)}$*, Adv. in Math. **57** (1985), no. 1, 34–70. MR 87e:17020. Zbl 586.33011.

[12] ———, *A q-analog of hypergeometric series well-poised in* $\mathrm{SU}(n)$ *and invariant G-functions*, Adv. in Math. **58** (1985), no. 1, 1–60. MR 87d:22028. Zbl 586.33014.

[13] ———, *A q-analog of the Gauss summation theorem for hypergeometric series in* $\mathrm{U}(n)$, Adv. in Math. **72** (1988), no. 1, 59–131. MR 90c:33006. Zbl 658.33005.

[14] M. Schlosser, *Multidimensional matrix inversions and A_r and D_r basic hypergeometric series*, Ramanujan J. **1** (1997), no. 3, 243–274. Zbl 980.45703.

[15] V. Strehl and H. S. Wilf, *Five surprisingly simple complexities*, J. Symbolic Comput. **20** (1995), no. 5-6, 725–729. MR 97j:05011. Zbl 851.68053.

[16] J. J. Sylvester, *On rational derivation from equations of coexistence, that is to say, a new and extended theory of elimination, Part I*, Philos. Mag. **15** (1839), 428–435,

reprinted in Collected Mathematical Papers I 40–46 reprinted by Chelsea; New york, 1973.

[17] ______, *On the partition of numbers*, Quart. J. Math. **1** (1857), 141–152, reprinted in Collected Mathematical Papers II 90–99 reprinted by Chelsea; New york, 1973.

[18] ______, *Outlines of seven lectures on the partition of numbers*, Proc. Lond. Math. Soc. **28** (1897), 33–96, reprinted in Collected Mathematical Papers II 119–175 reprinted by Chelsea; New york, 1973.

[19] L. Verde-Star, *Divided differences and combinatorial identities*, Stud. Appl. Math. **85** (1991), no. 3, 215–242. MR 92i:65027. Zbl 776.65008.

[20] E. Waring, *Problems concerning interpolations*, Philos. Trans. Roy. Soc. **69** (1779), 59–67.

Bahatnagar: Department of Mathematics,The Ohio State University, Columbus, OH 43210, USA

E-mail address: gaurav@math.ohio-state.edu

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk