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ON AZUMAYA GALOIS EXTENSIONS AND SKEW GROUP RINGS

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ABSTRACT. Two characterizations of an Azumaya Galois extension of a ring are given in terms of the Azumaya skew group ring of the Galois group over the extension and a Galois extension of a ring with a special Galois system is determined by the trace of the Galois group.

Keywords and phrases. Azumaya algebras, Galois extensions, H-separable extensions, skew group rings.

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1. Introduction. Let S be a ring with 1, G a finite automorphism group of S of order n for some integer n invertible in S, S^G the subring of the elements fixed under each element in G, C the center of S, and $S*G$ the skew group ring of G over S. In [3] and [2], S is called an Azumaya Galois extension of S^G if it is a G-Galois extension of S^G which is an Azumaya C^G-algebra. It was shown that S is an Azumaya Galois extension if and only if $S*G$ is an Azumaya C^G-algebra. The purpose of the present paper is to give two more characterizations of an Azumaya Galois extension in terms of the Azumaya skew group ring $S*G$. We show that S is an Azumaya G-Galois extension if and only if $S*G$ is an Azumaya algebra over its center Z, a G'-Galois extension with an inner Galois group G' induced by the elements of G, and ZG is a finitely generated projective C^G-module of rank n. Moreover, for the skew group ring $S*G$, where S is a separable C^G-algebra, an expression of the commutator subring of C in $S*G$ is obtained by using S and its commutator subring in $S*G$. Furthermore, let H be a normal subgroup of G, K the commutator subgroup of H in G, and H' the inner automorphism group of $S*G$ induced by the elements of H (K' and $(G/H)'$ are similarly defined). Then, it is shown that $(S*G)^{K'}$ is a $(G/K)'$-Galois extension with a Galois system $\{m^{-1}g_j, g_j^{-1}/g_j \text{ in } H\}$ if and only if $\mathrm{Tr}_{G'}(g_i) = 0$ for each g_i not in K, where m is the order of H for some integer m and $\mathrm{Tr}_{G'}(g_i)$ is the trace of G' at g_i.

2. Preliminaries. Throughout, let S be a ring with 1, $G = \{g_1,\dots,g_n\}$ for some integer n invertible in S, C the center of S, S^G the subring of the elements fixed under each element in G, and $S*G$ the skew group ring of G over S. Let B be a subring of a ring A. We call A a separable extension of B if there exist $\{a_i,b_i\}$ in A, $i = 1,\dots,m$ for some integer m, such that $\sum a_ib_i = 1$ and $\sum aa_i \otimes b_i = \sum a_i \otimes b_ia$ for all a in A, where $\otimes$ is over B and $\{a_i,b_i\}$ is called a separable system for A. An Azumaya algebra is a separable extension over its center. A ring A is called an H-separable extension of B if $A \otimes A$ is a direct summand of a finite direct sum of A as an A-bimodule, where $\otimes$ over B. Denote the commutator subring of B in A by

$V_A(B)$. An H-separable extension A over B is equivalent to the existence of an H-separable system $\{d_i$ in $V_A(B)$; $\sum(x_{ij}\otimes y_{ij})$ in $V_{A\otimes A}(A)\}$, $j=1,\dots,u$ and $i=1,\dots,v$ for some integers u and v such that $\sum d_i(\sum(x_{ij}\otimes y_{ij}))=1\otimes 1$, $i=1,\dots,v$ and $j=1,\dots,u$. The ring S is called a G-Galois extension of S^G if there exist $\{c_i,d_i$ in $S, i=1,\dots,k$ for some integer $k\}$ such that $\sum c_id_i=1$ and $\sum a_ig_j(b_i)=0$ for each $g_j\neq 1$, where $\{c_i,d_i\}$ is called a G-Galois system for S. It is well known that an Azumaya algebra is an H-separable extension and that an H-separable extension is a separable extension. A skew group ring $S*G$ is a ring with a free basis $\{g_i\}$ over S such that $g_is=(g_i(s))g_i$ for each g_i in G and s in S. We denote the center of $S*G$ by Z, the inner automorphism group of $S*G$ induced by the elements of the subgroup H of G by H' $(=\{g'/g'(x)=gxg^{-1}$ for g in H and all x in $S*G\})$, and the commutator subgroup of H in G by $V_G(H)$.

3. Skew group rings. In this section, keeping the notations of Section 2, we give two characterizations of an Azumaya Galois extension and an expression of the commutator subring of C in $S*G$ when S is a separable C^G-algebra.

Theorem 3.1. *The following statements are equivalent:*

(i) *S is an Azumaya Galois extension,*

(ii) *$S*G$ is an Azumaya Z-algebra and S satisfies the double centralizer property in $S*G$, and*

(iii) *$S*G$ is an Azumaya Z-algebra and a G'-Galois extension of $(S*G)^{G'}$, and ZG is a finitely generated and projective C^G-module of* rank n.

Proof. (i)⇒(ii). Since S is an Azumaya Galois extension, $S*G$ is an Azumaya C^G-algebra (that is, $Z=C^G$) and $S*G$ is an H-separable extension of S [3, Thm. 3.1]. Noting that S is a direct summand of $S*G$ as a left S-module, we conclude that $V_{S*G}(V_{S*G}(S))=S$ [6, Prop. 1.2].

(ii)⇒(i). Since $V_{S*G}(V_{S*G}(S))=S$, Z is contained in S; and so Z is contained in C. But then $Z=C^G$. This implies that $S*G$ is an Azumaya C^G-algebra by (ii). Thus, S is an Azumaya Galois extension [3, Thm. 3.1].

(i)⇒(iii). Since the restriction of G' to S is G, $S*G$ is a G'-Galois extension of $(S*G)^{G'}$ with the same Galois system as S (for S is G-Galois). Also, by hypothesis, S is an Azumaya Galois extension, so $S*G$ is an Azumaya C^G-algebra [3, Thm. 3.1]. Moreover, since $Z=C^G$, ZG is a free Z-module of rank n.

(iii)⇒(i). Since $S*G$ is a G'-Galois extension of $(S*G)^{G'}$ with an inner Galois group G', it is an H-separable extension of $(S*G)^{G'}$ [7, Cor. 3]. But n is a unit in S, so $V_{S*G}((S*G)^{G'})$ is a separable Z-algebra and a finitely generated and projective Z-module of rank n [7, Prop. 4]. Moreover, $S*G$ is a G'-Galois extension of $(S*G)^{G'}$, so it is finitely generated and projective $(S*G)^{G'}$-module. Since n is a unit in S, ZG is a separable Z-algebra. But then $V_{S*G}((S*G)^{G'})=V_{S*G}(V_{S*G}(ZG))=ZG$ by the commutator theorem for Azumaya algebras [4, Thm. 4.3]. Therefore, ZG is a finitely generated and projective Z-module of rank n [1, Prop. 4]. From the fact that there are n elements $\{g_i\}$ of G as generators of ZG, it is not difficult to show that $\{g_i\}$ are free over Z. Hence, Z is a finitely generated and projective C^G-module. Thus, the rank of ZG over C^G is a product of the rank of ZG over Z and the rank of Z over C^G; that is,

$n = n(\text{rank of } Z \text{ over } C^G)$. This implies that $Z = C^G$. Therefore, S^*G is an Azumaya C^G-algebra; and so S is an Azumaya Galois extension [3, Thm. 3.1]. □

Corollary 3.2. *Let S be a separable C^G-algebra. If $V_{S*G}(S)$ is a G''-Galois extension, where G'' is the inner automorphism group of $V_{S*G}(S)$ induced by and isomorphic with G, then S is an Azumaya Galois algebra.*

Proof. Since $V_{S*G}(S)$ is a G''-Galois extension, there exists a G''-Galois system $\{c_i, d_i \text{ in } V_{S*G}(S) \mid i = 1,\dots,k\}$ for $V_{S*G}(S)$. Then, it is straightforward to check that $\{c_j; \sum g_j d_i \otimes g_j^{-1}, i = 1,\dots,k \text{ and } j = 1,\dots,m$ for some integers k and $m\}$ is an H-separable system for S^*G over S [1, Thm. 1]. Hence, S satisfies the double centralizer property in S^*G [7, Prop. 1.2]. Moreover, n is a unit in S, so S^*G is a separable extension of S. By hypothesis, S is a separable C^G-algebra, so S^*G is a separable C^G-algebra by the transitivity of separable extensions. But then S^*G is an Azumaya Z-algebra. Therefore, S is an Azumaya Galois extension by Theorem 3.1. □

For the skew group ring S^*G of G over a separable C^G-algebra S, we next give an expression of $V_{S*G}(C)$ in terms of S and $V_{S*G}(S)$ (for more about $V_{S*G}(S)$, see [1]).

Theorem 3.3. *If S is a separable C^G-algebra, then*

(i) *CZ is a commutative separable subalgebra of S^*G and*

(ii) *SZ, $V_{S*G}(S)$, and $V_{S*G}(C)$ are Azumaya CZ-algebras contained in S^*G, such that $V_{S*G}(C) \cong SZ \otimes V_{S*G}(S)$, where $\otimes$ is over CZ.*

Proof. (i) Since S is a separable C^G-algebra, C is also a separable C^G-algebra. Hence, $C \otimes Z$ is a separable Z-algebra, where $\otimes$ is over C^G; and so the homomorphic image CZ of $C \otimes Z$ is also a separable Z-algebra. Clearly, CZ is commutative.

(ii) Since n is a unit in S, S^*G is a separable S-extension. Hence, S^*G is a separable C^G- algebra by the transitivity of separable extensions; and so S^*G is an Azumaya Z-algebra. But then $V_{S*G}(CZ)$ is a separable subalgebra of S^*G such that $V_{S*G}(V_{S*G}(CZ)) = CZ$ [4, Thm. 4.3] (for CZ is a separable subalgebra of S^*G by (i)). This implies that the center of $V_{S*G}(CZ)$ is CZ. Thus, $V_{S*G}(CZ)$ is an Azumaya CZ-algebra. By hypothesis again, S is a separable C^G-algebra, so it is an Azumaya C-algebra. Hence, $S \otimes CZ$ is an Azumaya CZ-algebra, where $\otimes$ is over C. Thus, SZ is also an Azumaya CZ-algebra. Noting that $SZ \subset V_{S*G}(CZ)$, we conclude that $V_{S*G}(CZ) \cong SZ \otimes V_{S*G}(SZ)$, where $\otimes$ is over CZ [7, Thm. 4.3]. Moreover, since $V_{S*G}(CZ) = V_{S*G}(C)$ and $V_{S*G}(SZ) = V_{S*G}(S)$, we conclude that $V_{S*G}(C) \cong SZ \otimes V_{S*G}(S)$, where $\otimes$ is over CZ. □

By [3, Thm. 3.1], if S is an Azumaya Galois extension, then S^*G is an Azumaya C^G-algebra (that is, $Z = C^G$) and S is a separable C^G-algebra. Thus, we have the following result.

Corollary 3.4. *If S is an Azumaya Galois extension, then $V_{S*G}(C) \cong S \otimes V_{S*G}(S)$ as Azumaya C-algebras, where $\otimes$ is over C such that $V_{S*G}(C)$ is a G'-Galois extension of $V_{S*G}(CG)$.*

Proof. By the above remark, it suffices to show that $V_{S*G}(C)$ is a G'-Galois extension of $V_{S*G}(CG)$. In fact, since S is a G-Galois extension and $S \subset V_{S*G}(C)$, $V_{S*G}(C)$ is a G'-Galois extension with the same Galois system as S by noting that $V_{S*G}(C)$ is

G'-invariant (for G is the restriction of G' to S). Moreover, it is clear that $(V_{S*G}(C))^{G'} = V_{S*G}(CG)$. □

4. A Galois system. It is well known that $\{n^{-1}g_i, g_i^{-1} \mid g_i \text{ in } G\}$ is a separable system for a separable group ring RG over a ring R with 1, where $G = \{g_i \mid i = 1,\dots,n\}$ for some integer n invertible in R, for a separable skew group ring $S*G$ over S and for a separable projective group ring RG_f over R as defined in [9]. In this section, we give an equivalent condition for $(S*G)^{K'}$ to have a $(G/K)'$-Galois system similar to the above separable system for a normal subgroup K of G.

Theorem 4.1. *Let H be a normal subgroup of G and $V_G(H) = K$. Then*
(i) *K is a normal subgroup of G and*
(ii) $\mathrm{Tr}_{H'}(g_i) = 0$ *for each g_i not in K if and only if $(S*G)^{K'}$ is a $(G/K)'$-Galois extension of $(S*G)^{G'}$ with a Galois system $\{m^{-1}g_j, g_j^{-1} \mid g_j \text{ in } H\}$, where m is the order of H.*

Proof. (i) We want to show that $g_iKg_i^{-1} \subset K$ for each g_i in G. For any x in K and y in H, $g_ixg_i^{-1}y = g_ixg_i^{-1}yg_ig_i^{-1} = g_ixzg_i^{-1}$, where $z = g_i^{-1}yg_i$. Since H is normal in G, z is in H. Hence, $xz = zx$. But then $g_ixg_i^{-1}y = g_ixzg_i^{-1} = g_izxg_i^{-1} = g_ig_i^{-1}yg_ixg_i^{-1} = yg_ixg_i^{-1}$. This implies that $g_ixg_i^{-1}$ is in K. Thus, K is normal in G.

(ii) Assume that $\mathrm{Tr}_{H'}(g_i) = 0$ for each g_i not in K. Then $\sum g_jg_ig_j^{-1} = 0$, where $H = \{g_j \mid j = 1,\dots,m$ for some integer $m\}$; that is, $\sum g_jg_ig_j^{-1}g_i^{-1}g_i = \sum g_j((g_i)'(g_j^{-1}))g_i = 0$, $j = 1,\dots,m$. Hence, $(m^{-1})\sum g_j((g_i)'(g_j^{-1})) = 0$ for each g_i not in K. Clearly, for each g_i in K, $(m^{-1})\sum g_j((g_i)'(g_j^{-1})) = 1$. Thus, $\{m^{-1}g_j, g_j^{-1} \mid g_j \text{ in } H\}$ is a $(G/K)'$-Galois system for $(S*G)^{K'}$ (for $H \subset (S*G)^{K'}$), where m is the order of H.

Conversely, $(m^{-1})\sum g_j((g_i)'(g_j^{-1})) = 0$ for each g_i not in K, so $\sum g_jg_ig_j^{-1}g_i^{-1} = 0$. Hence, $\sum g_jg_ig_j^{-1} = 0$; that is, $\mathrm{Tr}_{H'}(g_i) = 0$ for each g_i not in K. □

We derive the following corollaries.

Corollary 4.2. *$S*G$ has a $(G/K)'$-Galois system $\{n^{-1}g_i, g_i^{-1} \mid g_i \text{ in } G\}$, where K is the center of G, if and only if $\mathrm{Tr}_{G'}(g_i) = 0$ for each g_i not in K.*

Proof. Let H be G. Then K = the center of G; and so the corollary follows immediately from the theorem. □

Corollary 4.3. *$S*G$ has a G'-Galois system $\{n^{-1}g_i, g_i^{-1} \mid g_i \text{ in } G\}$ if and only if $\mathrm{Tr}_{G'}(g_i) = 0$ for each $g_i \neq 1$.*

Proof. This is the case of the theorem that the center of G is trivial. □

We derive an equivalent condition for a Galois subring of $S*G$ arising from a G'-invariant subring.

Corollary 4.4. *Let A be a G'-invariant subring of $S*G$ and $H = \{g_i \text{ in } G \mid g_i(a) = a$ for each a in $A\}$. Then*
(i) *H is normal in G and*
(ii) *Denoting $(S*G)^{H'}$ by B and $V_G(H)$ by K, $\mathrm{Tr}_{K'}(g_i) = 0$ for each g_i not in H if and only if $\{m^{-1}g_j, g_j^{-1} \mid g_j \text{ in } K\}$ is a $(G/H)'$-Galois system for B, where m is the*

order of K.

Proof. Part (i) is straightforward and part (ii) follows immediately from Theorem 4.1. □

We conclude the present paper with an example of an Azumaya skew group ring S^*G which is a G'-Galois extension such that the rank of ZG over C^G is not n (see Theorem 3.1(iii)). Hence, S is not an Azumaya Galois extension by Theorem 3.1.

Let R be the real field, $S = R[i,j,k]$ the quaternion algebra over R, and $G = \{1, g \mid g(x) = ix(i)^{-1}$ for each x in $S\}$. Then

(1) S is a G-Galois extension with a Galois system $\{2^{-1}, 2^{-1}j; 1, -j\}$. Hence, S^*G is a G'-Galois extension with the same Galois system.

(2) Since S^*G is a separable extension of S and S is an Azumaya R-algebra, S^*G is a separable R-algebra. Hence, S^*G is an Azumaya Z-algebra.

(3) The center Z of S^*G is $(R + Ri)$ by direct computation.

(4) ZG is free over Z by direct verification.

(5) $C = R$ and $C^G = C = R$.

(6) Z is a free R-module of rank 2 and ZG is a free C^G-module of rank 4 ($\neq 2$ = the order of G), so one of the three conditions in Theorem 3.1(iii) does not hold.

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