

q -SERIES, ELLIPTIC CURVES, AND ODD VALUES OF THE PARTITION FUNCTION

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ABSTRACT. Let $p(n)$ be the number of partitions of an integer n . Euler proved the following recurrence for $p(n)$:

$$p(n) = \sum_{k=1}^{\infty} (-1)^{k+1} \left(p(n - \omega(k)) + p(n - \omega(-k)) \right), \quad (*)$$

where $\omega(k) = (3k^2 + k)/2$. In view of Euler's result, one sees that it is fairly easy to compute $p(n)$ very quickly. However, many questions remain open even regarding the parity of $p(n)$. In this paper, we use various facts about elliptic curves and q -series to construct, for every $i \geq 1$, finite sets M_i for which $p(n)$ is odd for an odd number of $n \in M_i$.

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1. The partition function. A partition of a nonnegative integer n is any non-increasing sequence of positive integers whose sum is n . Let $p(n)$ denote the number of partitions of n . Even though Euler's recurrence (*) gives a method for computing $p(n)$, there are many open problems and conjectures regarding the overall behavior of the partition function. For instance, the following questions regard the parity of $p(n)$.

CONJECTURE 1.1 (Parkin-Shanks [5]). *The number of $n \leq x$ for which $p(n)$ is even is $\sim (1/2)x$.*

CONJECTURE 1.2 (Subbarao [7]). *In any arithmetic progression $r \pmod{t}$, there are infinitely many integers $N \equiv r \pmod{t}$ for which $p(N)$ is even, and there are infinitely many integers $M \equiv r \pmod{t}$ for which $p(M)$ is odd.*

K. Ono [3] has recently proven most of this conjecture.

NEWMAN'S PROBLEM (Newman [2]). Exhibit an infinite sequence of integers $n_1 < n_2 < \cdots$ for which $p(n_i)$ is odd (resp., even).

Euler proved that the *generating function* for $p(n)$ was given by the infinite product

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \quad (1.1)$$

Euler also discovered the identity

$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2}. \quad (1.2)$$

2. Elliptic curves. An elliptic curve over the rationals is a non-singular curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad (2.1)$$

where the coefficients a_i are integers. Any curve of the above form is isomorphic to one, say E , of the form

$$E: y^2 = x^3 + ax^2 + bx + c, \quad (2.2)$$

with integers a, b , and c . The *discriminant* of E , denoted by $\Delta(E)$, is given by

$$\Delta(E) = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2. \quad (2.3)$$

If p is prime, then let $GF(p)$ denote the finite field with p elements. If p is prime, then \bar{E}_p is the *reduction* of E to $GF(p)$. If the reduction is smooth, then we say E has *good* reduction at p . Otherwise, E has *bad* reduction at p . If $p \nmid \Delta(E)$, then E has good reduction at p .

The Hasse-Weil L -function of E , denoted by $L(E, s)$, is obtained by examining the reductions \bar{E}_p . If p is a prime of good reduction, then define the integer $a(p)$ as

$$a(p) = p + 1 - N_p, \quad (2.4)$$

where N_p is the number of points of \bar{E}_p rational over $GF(p)$, including the point at infinity. There are similar rules for those p with bad reduction. If p is prime and $k \geq 2$, then

$$a(p^k) = \begin{cases} a(p)a(p^{k-1}) - pa(p^{k-2}) & p \text{ good reduction,} \\ a(p)a(p^{k-1}) & p \text{ bad reduction.} \end{cases} \quad (2.5)$$

Furthermore, if $\gcd(n, m) = 1$, then

$$a(nm) = a(n)a(m). \quad (2.6)$$

The L -function is then given by

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \quad (2.7)$$

As a consequence of (2.5) and (2.6), we obtain:

PROPOSITION 2.1. *Let E be an elliptic curve and let $L(E, s) = \sum_{n=1}^{\infty} (a(n)/n^s)$ be its Hasse-Weil function. Suppose that $n > 1$ is relatively prime to $2 \cdot \Delta(E)$ with prime factorization*

$$n = \prod_i p_i^{a_i} \prod_j q_j^{b_j}, \quad (2.8)$$

where

$$a(p_i) \equiv 0 \pmod{2} \quad \text{and} \quad a(q_j) \equiv 1 \pmod{2}. \quad (2.9)$$

Then $a(n)$ is odd if and only if every $a_i \equiv 0 \pmod{2}$ and every $b_j \not\equiv 2 \pmod{3}$.

PROOF. By hypothesis, every p_i and q_j are odd primes all with good reduction. Then by (4), we find that for every $k \geq 2$,

$$\begin{aligned} a(p_i^k) &\equiv a(p_i^{k-2}) \pmod{2}, \\ a(q_j^k) &\equiv a(q_j^{k-1}) + a(q_j^{k-2}) \pmod{2}. \end{aligned} \quad (2.10)$$

It is easy to verify then that $a(p_i^k)$ is odd if and only if $k \equiv 0 \pmod{2}$, and that $a(q_j^k)$ is odd if and only if $k \not\equiv 2 \pmod{3}$. The result now follows easily from (2.6). \square

EXAMPLE 2.1. In this example, let E denote the curve

$$E: y^2 = x^3 - x. \quad (2.11)$$

Since $\Delta(E) = 4$, E has good reduction at every prime $p \neq 2$. If $p = 5$, then $\bar{E}_p = \bar{E}_5$ is the collection of points $(x, y) \in GF(5) \times GF(5)$ satisfying the congruence

$$y^2 \equiv x^3 - x \pmod{5}. \quad (2.12)$$

An easy computation verifies that the only such points are

$$(0, 0), (1, 0), (2, 1), (2, 4), (3, 2), (3, 3), (4, 0), \infty. \quad (2.13)$$

So in this case $N_5 = 8$, and so $a(5) = 5 + 1 - 8 = -2$. In fact, the first few terms of $L(E, s)$ are

$$L(E, s) = 1 - \frac{2}{5^s} - \frac{3}{9^s} + \frac{6}{13^s} + \cdots. \quad (2.14)$$

The Taniyama-Shimura-Weil conjecture states that all elliptic curves over the rationals are modular. A curve is modular if its L -function corresponds to the Fourier expansion at infinity of a modular form. Specifically, if E is modular and $L(E, s) = \sum_{n=1}^{\infty} (a(n)/n^s)$, then

$$F_E(z) = \sum_{n=1}^{\infty} a(n)q^n \quad (q = e^{2\pi iz}) \quad (2.15)$$

is a *modular* form. For a number of explicit examples (see [1]), the form $F_E(z)$ is given as a product of Dedekind's η -function defined by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (2.16)$$

For example, take the η -product

$$F_E(z) = \eta^4(6z) = q \prod_{n=1}^{\infty} (1 - q^{6n})^4. \quad (2.17)$$

The coefficients of the L -function $L(E, s)$ of the elliptic curve $E: y^2 = x^3 + 1$ are the same as those in the Fourier expansion of $F_E(z)$.

3. q -series results. In this section, we give two theorems which do not depend on elliptic curves. They simply depend on q -series manipulations.

THEOREM 3.1. *If $n = (2m+1)^2$, then an odd number of the values*

$$p\left(\frac{n-1}{4} - \left(\frac{a^2+a}{2} + 6b^2 + 2b\right)\right) \quad (3.1)$$

are odd, where $a \geq 0$ and b are integers.

PROOF. Consider the η -product

$$\eta^2(4z)\eta^2(8z) = \sum_{n=1}^{\infty} a(n)q^n = q \prod_{n=1}^{\infty} (1-q^{4n})^2 (1-q^{8n})^2. \quad (3.2)$$

Factor this as

$$\eta^2(4z)\eta^2(8z) = \eta^3(8z) \frac{\eta^2(4z)}{\eta(8z)}. \quad (3.3)$$

Recall the following identity due to Jacobi.

$$\prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{a=0}^{\infty} (-1)^a (2a+1) q^{(a^2+a)/2}. \quad (3.4)$$

Using this identity and another well known identity, we obtain

$$\eta^3(8z) = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(2n+1)^2} \quad (3.5)$$

and

$$\frac{\eta^2(4z)}{\eta(8z)} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{4n^2}, \quad (3.6)$$

so,

$$\begin{aligned} \eta^3(8z) \frac{\eta^2(4z)}{\eta(8z)} &= \left(\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(2n+1)^2} \right) \cdot \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{4n^2} \right) \\ &\equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod{2}. \end{aligned} \quad (3.7)$$

So,

$$\eta^2(4z)\eta^2(8z) \equiv q + q^9 + q^{25} + q^{49} + \dots \pmod{2}. \quad (3.8)$$

Because $\prod_{n=1}^{\infty} (1/(1-q^n))$ is the generating function for the partition function, we find that

$$q \left(\sum_{n=0}^{\infty} p(n) q^{4n} \right) \cdot \prod_{n=1}^{\infty} (1-q^{4n})^3 \cdot \prod_{n=1}^{\infty} (1-q^{8n})^2 = \eta^2(4z)\eta^2(8z). \quad (3.9)$$

Using Jacobi's identity, (1.2) and the fact that $(1 - q^{8n})^2 \equiv (1 - q^{16n}) \pmod{2}$, this becomes

$$\left(\sum_{n=0}^{\infty} p(n) q^{4n+1} \right) \cdot \left(\sum_{a=0}^{\infty} q^{2a^2+2a} \right) \cdot \left(\sum_{b=-\infty}^{\infty} q^{24b^2+8b} \right) \equiv \eta^2(4z) \eta^2(8z) \pmod{2}. \quad (3.10)$$

Therefore, we find that

$$\sum_{n=1}^{\infty} a(n) q^n \equiv \left(\sum_{n=0}^{\infty} p(n) q^{4n+1} \right) \cdot \left(\sum_{a \geq 0, b \in \mathbb{Z}} q^{2a^2+2a+24b^2+8b} \right) \pmod{2}. \quad (3.11)$$

Therefore, it is easy to check that

$$a(n) \equiv \sum_{a \geq 0, b \in \mathbb{Z}} p \left(\frac{n-1}{4} - \left(\frac{a^2+a}{2} + 6b^2 + 2b \right) \right) \pmod{2}. \quad (3.12)$$

The theorem now follows immediately. \square

THEOREM 3.2. *If $n = (6m+1)^2$, then an odd number of the values*

$$p \left(\frac{n-1}{6} - \left(\frac{a^2+a}{2} + 3b^2 + b \right) \right) \quad (3.13)$$

are odd, where $a \geq 0$ and b are integers.

PROOF. Consider the η -product

$$\eta^4(6z) = \prod_{n=0}^{\infty} (1 - q^{6n})^4. \quad (3.14)$$

Since $\eta^4(6z) \equiv \eta(24z) \pmod{2}$, we can use (1.2) to give us

$$\eta(24z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{36n^2+12n+1} \equiv \sum_{n=-\infty}^{\infty} q^{(6n+1)^2} \pmod{2}. \quad (3.15)$$

Thus, $\eta^4(6z) \equiv 1 + q^{25} + q^{49} + q^{121} + q^{169} + \dots \pmod{2}$. Because $\prod_{n=1}^{\infty} (1/(1 - q^n))$ is the generating function for the partition function, we find that

$$q \left(\sum_{n=0}^{\infty} p(n) q^{6n} \right) \cdot \prod_{n=1}^{\infty} (1 - q^{6n})^3 \cdot \prod_{n=1}^{\infty} (1 - q^{6n})^2 = \eta^4(6z). \quad (3.16)$$

Since $(1 - q^{6n})^2 \equiv (1 - q^{12n}) \pmod{2}$, we can use (3.4) and (1.2) to get

$$\left(\sum_{n=0}^{\infty} p(n) q^{6n+1} \right) \cdot \left(\sum_{a=0}^{\infty} q^{3a^2+3a} \right) \cdot \left(\sum_{b=-\infty}^{\infty} q^{18b^2+6b} \right) \equiv \eta^4(6z) \pmod{2}. \quad (3.17)$$

Therefore, we find that

$$\sum_{n=1}^{\infty} a(n) q^n \equiv \left(\sum_{n=0}^{\infty} p(n) q^{6n+1} \right) \cdot \left(\sum_{a \geq 0, b \in \mathbb{Z}} q^{3a^2+3a+18b^2+6b} \right) \pmod{2}. \quad (3.18)$$

Therefore, it is easy to check that

$$a(n) \equiv \sum_{a \geq 0, b \in \mathbb{Z}} p \left(\frac{n-1}{6} - \left(\frac{a^2+a}{2} + 3b^2 + b \right) \right) \pmod{2}. \quad (3.19)$$

The theorem now follows immediately. \square

EXAMPLE 3.1. Here, we illustrate an example of Theorem 3.2. If $m = 1$, then $n = (6m+1)^2 = 49$. We must find pairs (a, b) with $a \geq 0$ and b integers such that

$$\frac{n-1}{6} = 8 \geq \left(\frac{a^2+a}{2} + 3b^2 + b \right). \quad (3.20)$$

These pairs are: $(0, 0)$ $(0, -1)$ $(0, 1)$ $(1, 0)$ $(1, -1)$ $(1, 1)$ $(2, 0)$ $(2, -1)$ $(2, 1)$ $(3, 0)$ $(3, -1)$. Theorem 3.2 tells us that an odd number of the following values are odd:

$$\begin{aligned} p(8) = 22, \quad p(6) = 11, \quad p(4) = 5, \quad p(7) = 15, \quad p(5) = 7, \quad p(3) = 3, \\ p(5) = 7, \quad p(3) = 3, \quad p(1) = 1, \quad p(2) = 2, \quad p(0) = 1. \end{aligned} \quad (3.21)$$

Nine of the eleven are indeed odd.

GROUP LAW FOR ELLIPTIC CURVES. If E is an elliptic curve, $E: y^2 = x^3 + ax^2 + bx + c$, the point at infinity is taken to be its identity element O , and $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are points on E , then $P + Q := (x_3, y_3)$, where

$$x_3 = \lambda^2 - a - x_1 - x_2 \quad (3.22)$$

and

$$y_3 = \lambda x_3 + y_1 - \lambda x_1. \quad (3.23)$$

If $P = Q$, then

$$\lambda = \frac{3x^2 + 2ax + b}{2y}, \quad (3.24)$$

otherwise

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}. \quad (3.25)$$

The question of finding points of order two on a curve is the same as that of finding all the points such that $P + P = O$ but $P \neq O$. It is easily seen from the above that this is satisfied only when $y = 0$.

FUNDAMENTAL THEOREM. *If E is an elliptic curve and p is a prime of good reduction, then \tilde{E}_p with the point at infinity is a finite abelian group.*

THEOREM 3.3. *Let E be the elliptic curve*

$$E: y^2 = x^3 + ax^2 + bx + c, \quad (3.26)$$

and $L(E, s) = \sum_{n=1}^{\infty} (a(n)/n^s)$ its Hasse-Weil function. If the odd prime p has good reduction, then $a(p)$ is odd if and only if $x^3 + ax^2 + bx + c \equiv 0 \pmod{p}$ has no solution.

PROOF. By definition, $a(p) = p + 1 - N_p$, where N_p is the number of rational points of E over $GF(p)$. Since p is an odd prime, we find that $a(p)$ is odd if and only if

$$N_p \equiv 1 \pmod{2}. \quad (3.27)$$

The elliptic curve \bar{E}_p is a finite abelian group with N_p elements, so Lagrange's theorem states that N_p is a multiple of the order of each of the individual points. Thus, asking when N_p is odd is the same as asking for which \bar{E}_p are there no points of order two. A point of order 2 on an elliptic curve is one whose y -coordinate is zero. Thus, N_p and, consequently, $a(p)$ is odd if the equation $y^2 \equiv x^3 + ax^2 + bx + c \equiv 0 \pmod{p}$ has no solution for which $y = 0$. \square

THEOREM 3.4. *Let $p_1 < p_2 < \dots$ be the primes for which*

$$x^3 - 4x^2 - 160x - 1264 \equiv 0 \pmod{p_i} \quad (3.28)$$

have solutions in $GF(p_i)$ and let $q_1 < q_2 < \dots$ be the primes for which

$$x^3 - 4x^2 - 160x - 1264 \equiv 0 \pmod{q_j} \quad (3.29)$$

has no solutions in $GF(q_j)$. Suppose that $n > 1$ is relatively prime to 2378 and that it has the factorization

$$n = \prod_i p_i^{a_i} \prod_j q_j^{b_j}. \quad (3.30)$$

If every $a_i \equiv 0 \pmod{2}$ and every $b_j \not\equiv 2 \pmod{3}$, then an odd number of the values

$$p \left(n - 1 - \left(\frac{a^2 + a}{2} + 33b^2 + 11b \right) \right) \quad (3.31)$$

are odd, where $a \geq 0$ and b are integers.

PROOF. In [1], it is proved that if E is the curve

$$E: y^2 = x^3 - 4x^2 - 160x - 1264, \quad (3.32)$$

then its Hasse-Weil function $L(E, s) = \sum_{n=1}^{\infty} (a(n)/n^s)$ has the property that its coefficients $a(n)$ are given by

$$\eta^2(z)\eta^2(11z) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2. \quad (3.33)$$

However, since $(1 - q^{11n})^2 \equiv (1 - q^{22n}) \pmod{2}$, we find that

$$\sum_{n=1}^{\infty} a(n)q^n \equiv q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{22n}) \pmod{2}. \quad (3.34)$$

Therefore, we find by (1.1) that

$$\left(\sum_{n=0}^{\infty} p(n)q^{n+1} \right) \prod_{n=1}^{\infty} (1 - q^n)^3 \prod_{n=1}^{\infty} (1 - q^{22n}) \equiv \sum_{n=1}^{\infty} a(n)q^n \pmod{2}. \quad (3.35)$$

But by Jacobi's identity (3.4) and Euler's identity (1.2), this reduces to

$$\left(\sum_{n=0}^{\infty} p(n)q^{n+1} \right) \cdot \left(\sum_{a=0}^{\infty} q^{(a^2+a)/2} \right) \cdot \left(\sum_{n=-\infty}^{\infty} q^{33b^2+11b} \right) \equiv \sum_{n=1}^{\infty} a(n)q^n \pmod{2}. \quad (3.36)$$

Therefore, it turns out that

$$a(n) \equiv \sum_{a \geq 0, b \in \mathbb{Z}} p \left(n - 1 - \left(\frac{a^2 + a}{2} + 33b^2 + 11b \right) \right) \pmod{2}. \quad (3.37)$$

The result now follows immediately from Theorem 3.3 and Proposition 2.1. \square

EXAMPLE 3.2. It is easy to show that there is no solution to the equation

$$x^3 - 4x^2 - 160x - 1264 \equiv 0 \pmod{p_i} \quad (3.38)$$

for the primes 3 and 5. So by (2.6), $n = 15$ is a suitable choice to illustrate Theorem 3.4. We must, therefore, find all pairs (a, b) with $a \geq 0$ and $b \in \mathbb{Z}$ such that $14 \geq (\frac{a^2+a}{2} + 33b^2 + 11b)$. These pairs are: $(0, 0)$ $(1, 0)$ $(2, 0)$ $(3, 0)$ $(4, 0)$. So by Theorem 3.4, an odd number of the following

$$p(14) = 135, \quad p(13) = 101, \quad p(11) = 56, \quad p(8) = 22, \quad p(4) = 5 \quad (3.39)$$

are odd.

THEOREM 3.5. Let $p_1 < p_2 < \dots$ be the primes for which

$$x^3 + x^2 + 72x - 368 \equiv 0 \pmod{p_i} \quad (3.40)$$

have solutions in $GF(p_i)$ and $q_1 < q_2 < \dots$ the primes for which

$$x^3 + x^2 + 72x - 368 \equiv 0 \pmod{q_j} \quad (3.41)$$

has no solutions in $GF(q_j)$. Suppose that $n > 1$ is relatively prime to 14 and that its prime factorization is

$$n = \prod_i p_i^{a_i} \prod_j q_j^{b_j}. \quad (3.42)$$

If every $a_i \equiv 0 \pmod{2}$ and every $b_j \not\equiv 2 \pmod{3}$, then an odd number of the values

$$p \left(n - 1 - \left(\frac{7a^2 + 7a}{2} + 6b^2 + 2b \right) \right) \quad (3.43)$$

are odd, where $a \geq 0$ and b are integers.

PROOF. In [1], it is proved that if E is the curve

$$E: y^2 = x^3 + x^2 + 72x - 368, \quad (3.44)$$

then its Hasse-Weil function $L(E, s) = \sum_{n=1}^{\infty} (a(n)/n^s)$ has the property that its coefficients $a(n)$ are given by

$$\eta(z)\eta(2z)\eta(7z)\eta(14z) = q \prod_{n=1}^{\infty} (1-q^n)(1-q^{2n})(1-q^{7n})(1-q^{14n}). \quad (3.45)$$

Using Euler's identity (1.2), Jacobi's identity (3.4), and the fact that $(1-q^{2n}) \equiv (1-q^n)^2 \pmod{2}$, the theorem follows in a manner similar to that of Theorem 3.4. \square

THEOREM 3.6. *Let $p_1 < p_2 < \dots$ be the primes for which*

$$x^3 + x^2 + 4x + 4 \equiv 0 \pmod{p_i} \quad (3.46)$$

have solutions in $GF(p_i)$ and $q_1 < q_2 < \dots$ the primes for which

$$x^3 + x^2 + 4x + 4 \equiv 0 \pmod{q_j} \quad (3.47)$$

has no solutions in $GF(q_j)$. Suppose that $n > 1$ is relatively prime to 10 and that its prime factorization is

$$n = \prod_i p_i^{a_i} \prod_j q_j^{b_j}. \quad (3.48)$$

If every $a_i \equiv 0 \pmod{2}$ and every $b_j \not\equiv 2 \pmod{3}$, then an odd number of the values

$$p \left(\frac{n-1}{2} - (a^2 + a + 30b^2 + 10b) \right), \quad (3.49)$$

are odd, where $a \geq 0$ and b are integers.

PROOF. If E is the curve

$$E: y^2 = x^3 + x^2 + 4x + 4, \quad (3.50)$$

then in [1], it was proved that the coefficients $a(n)$ of its Hasse-Weil function $L(E, s) = \sum_{n=1}^{\infty} (a(n)/n^s)$ are given by

$$\eta^2(2z)\eta^2(10z) = q \prod_{n=1}^{\infty} (1-q^{2n})^2 (1-q^{10n})^2. \quad (3.51)$$

The proof follows in a manner similar to Theorem 3.4. \square

THEOREM 3.7. *Let $p_1 < p_2 < \dots$ be the primes for which*

$$x^3 - x^2 - 4x + 4 \equiv 0 \pmod{p_i} \quad (3.52)$$

have solutions in $GF(p_i)$ and $q_1 < q_2 < \dots$ the primes for which

$$x^3 - x^2 - 4x + 4 \equiv 0 \pmod{q_j} \quad (3.53)$$

has no solutions in $GF(q_j)$. Suppose that $n > 1$ is relatively prime to 6 and that its prime factorization is

$$n = \prod_i p_i^{a_i} \prod_j q_j^{b_j}. \quad (3.54)$$

If every $a_i \equiv 0 \pmod{2}$ and every $b_j \not\equiv 2 \pmod{3}$, then an odd number of the values

$$p \left(\frac{n-1}{2} - (3a^2 + 3a + 12b^2 + 4b) \right) \quad (3.55)$$

are odd, where $a \geq 0$ and b are integers.

PROOF. If E is the curve

$$E: y^2 = x^3 - x^2 - 4x + 4, \quad (3.56)$$

then in [1], it was proved that the coefficients $a(n)$ of its Hasse-Weil function $L(E, s) = \sum_{n=1}^{\infty} (a(n)/n^s)$ are given by

$$\eta(2)\eta(4z)\eta(6z)\eta(12z) = q \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{4n})(1 - q^{6n})(1 - q^{12n}). \quad (3.57)$$

The proof follows in a manner similar to Theorem 3.4. \square

THEOREM 3.8. Let $p_1 < p_2 < \dots$ be the primes for which

$$x^3 - 432 \equiv 0 \pmod{p_i} \quad (3.58)$$

have solutions in $GF(p_i)$ and q_j are the primes for which

$$x^3 - 432 \equiv 0 \pmod{q_j} \quad (3.59)$$

has no solutions in $GF(q_j)$. Suppose that $n > 1$ is relatively prime to 6 and that its prime factorization is

$$n = \prod_i p_i^{a_i} \prod_j q_j^{b_j}. \quad (3.60)$$

If every $a_i \equiv 0 \pmod{2}$ and every $b_j \not\equiv 2 \pmod{3}$, then an odd number of the values

$$p \left(\frac{n-1}{3} - \left(\frac{3a^2 + 3a}{2} + 27b^2 + 9b \right) \right) \quad (3.61)$$

are odd, where $a \geq 0$ and b are integers.

PROOF. If E is the curve

$$E: y^2 = x^3 - 432, \quad (3.62)$$

then in [1], it was proved that the coefficients $a(n)$ of its Hasse-Weil function $L(E, s) = \sum_{n=1}^{\infty} (a(n)/n^s)$ are given by

$$\eta^2(3z)\eta^2(9z) = q \prod_{n=1}^{\infty} (1 - q^{3n})^2 (1 - q^{9n})^2. \quad (3.63)$$

The proof follows in a manner similar to Theorem 3.4. \square

Also, realize that the curves in Theorems 3.4, 3.5, and 3.8 were all changed from the form they are normally shown into the form $y^2 = x^3 + ax^2 + bx + c$ by a simple change of variables to ease the job of finding points of order two.

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REFERENCES

- [1] Y. Martin and K. Ono, *Eta-quotients and elliptic curves*, Proc. Amer. Math. Soc. **125** (1997), no. 11, 3169–3176. MR 97m:11057. Zbl 970.60471.
- [2] M. Newman, *Periodicity modulo m and divisibility properties of the partition function*, Trans. Amer. Math. Soc. **97** (1960), 225–236. MR 22#6778. Zbl 106.03903.
- [3] K. Ono, *Parity of the partition function in arithmetic progressions*, J. Reine Angew. Math. **472** (1996), 1–15. MR 97e:11131. Zbl 835.11038.
- [4] ———, *Odd values of the partition function*, Discrete Math. **169** (1997), no. 1-3, 263–268. CMP 97 13. Zbl 970.40475.
- [5] T. R. Parkin and D. Shanks, *On the distribution of parity in the partition function*, Math. Comp. **21** (1967), 466–480. MR 37#2711. Zbl 149.28501.
- [6] J. H. Silverman and J. Tate, *Rational points on elliptic curves*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992. MR 93g:11003. Zbl 752.14034.
- [7] M. V. Subbarao, *Some remarks on the partition function*, Amer. Math. Monthly **73** (1966), 851–854. MR 34#1293. Zbl 173.01803.

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