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NUMERICAL SOLUTION OF INTEGRAL EQUATIONS WITH FINITE PART INTEGRALS

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ABSTRACT. We obtain convergence rates for several algorithms that solve a class of Hadamard singular integral equations using the general theory of approximations for unbounded operators.

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1. Introduction. In several physical problems in aerodynamics, hydrodynamics, and elasticity, one encounters integral equations of the form

$$Ax = \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}\,x(\tau)}{(\tau-t)^2}\,d\tau + \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\tau^2}\,h(t,\tau)\,x(\tau)\,d\tau = y, \tag{1.1}$$

where the first integral in (1.1) is a finite part integral [4]. Under suitable conditions on the kernel and the right-hand side, the convergence of Galerkin's method and several collocation methods, proposed by Ioakimidis [5] and Williams [9], has been discussed by Golberg [2, 3]. This author, also, used a classical Fredholm theory to establish the existence of a solution and likewise the basic tools necessary to discuss convergence. In this paper, we discuss the convergence of the mechanical quadratures method for solving (1.1) and the convergence of the least squares method for solving the Hadamard singular integral equation

$$Kx = \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}\,x(\tau)}{(\tau-t)^2}\,d\tau + T(x,t) = y, \tag{1.2}$$

where T is the given continuous operator.

2. Least squares method. Let $X = L_{2,\rho}$ denote the space of square integrable functions with respect to $\rho = \sqrt{1-t^2}$. The inner product on $L_{2,\rho}$ is given by

$$(\phi,\psi)_\rho = \frac{2}{\pi}\int_{-1}^{1}\rho(t)\,\phi(t)\,\psi(t)\,dt \quad \text{and} \quad \|\phi\|_\rho = \sqrt{(\phi,\phi)_\rho}. \tag{2.1}$$

Let

$$U_m(t) = \frac{\sin[(m+1)\arccos t]}{\sqrt{1-t^2}}, \quad m = 0, 1, 2, \dots \tag{2.2}$$

denote the Chebyshev polynomials of the second kind. The solution x is, now, approximated by

$$x_n(t) = \sum_{k=1}^{n} \alpha_k U_{k-1}(t), \quad -1 \le t \le 1. \tag{2.3}$$

According to this method, we obtain a system of n linear algebraic equations in n unknowns

$$\sum_{i=1}^{n} \alpha_i (KU_{i-1}, KU_{j-1})_\rho = (y, KU_{j-1})_\rho, \quad 1 \le j \le n. \tag{2.4}$$

It is easy to prove that (2.4) is equivalent to

$$\sum_{k=1}^{n} \alpha_k \Big\{ (TU_{k-1}, TU_{j-1})_\rho - j(TU_{k-1}, U_{j-1})_\rho - k(TU_{j-1}, U_{k-1})_\rho \Big\} + j^2 \alpha_j = (y, KU_{j-1})_\rho, \quad 1 \le j \le n. \tag{2.5}$$

Theorem 2.1. *If the following conditions hold*

(i) $y \in L_{2,\rho}$, T *is a continuous operator in* $L_{2,\rho}$;

(ii) $\ker K = \{0\}$;

(iii) *equation (1.2) has a solution* $x^* \in L_{2,\rho}$ *for a given* $y \in L_{2,\rho}$, *then for all* $n \in \mathbb{N}$ *equation (2.5) has a unique solution* $\{\alpha_k^*\}_1^n$; *and if*

(iv) $\{KU_{i-1}\}$ *is closed in* $L_{2,\rho}$,

then

$$\|r_n\|_\rho = \|y - Kx_n^*\|_\rho \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \quad x_n^* = \sum_{k=1}^{n} \alpha_k^* U_{k-1}(t). \tag{2.6}$$

Proof. Since $\{U_{k-1}\}$ are linearly independent, then, from (ii), it follows that $\{KU_{k-1}\}$ are, also, linearly independent. Therefore, the system of equations (2.5) is non-singular and so, it has a unique solution for all n. Also, for $\beta_k \in \mathbb{R}$, we get

$$\|r_n\|_\rho = \|y - Kx_n^*\| \le \Big\| y - \sum_{i=1}^{n} \beta_i KU_{i-1} \Big\|_\rho. \tag{2.7}$$

If condition (iv) is satisfied, then $\|r_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. □

Now, we replace condition (ii) by the following condition:

(ii)′ $K : L_{2,\rho} \longrightarrow L_{2,\rho}$ has a left bounded inverse operator K_l^{-1}.

Theorem 2.2. *Assume that* (i), (ii)′, (iii), *and* (iv) *are satisfied, then* $\|x_n^* - x^*\|_\rho = O(\|r_n\|_\rho) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Proof. From (ii)′ and (iii), we have $x^* - x_n^* = K_l^{-1}K(x^* - x_n^*) = K_l^{-1}(y - Kx_n^*)$, then $\|x_n^* - x^*\|_\rho \longrightarrow 0$ as $n \longrightarrow \infty$. □

3. Mechanical quadratures methods. We introduce the following method for solving (1.1): Consider the approximation x_n of x given by (2.3). Due to this method, we get the following

$$\sum_{i=1}^{n}\alpha_i\left\{-iU_{i-1}(t_j)+\frac{1}{n+1}\sum_{r=1}^{n}(1-t_r^2)\,h(t_j,t_r)\,U_{i-1}(t_r)\right\}=y(t_j),\quad 1\le j\le n,\tag{3.1}$$

where $t_j=\cos(j\pi/n+1)$.

Theorem 3.1. *If the following conditions*
(i) $y\in C[-1,1]$, $h\in C[-1,1;-1,1]$
(ii) *equation (1.1) has a unique solution* $x^*\in X$ *for all* $y\in X$
are satisfied, then, for n *sufficiently large, equation (3.1) has a unique solution* α_k^*,

$$||x^*-x_n^*||_\rho=O\{E_n(y)_C+E_n^t(h)_C+E_n^\tau(h)_C\},\tag{3.2}$$

where $E_n^t(f)_C=\inf\{\|f-P_n\|_C, P_n$ *is a polynomial of degree* $\le n\}$.

Proof. Define the projection operator $P_n^t: C\longrightarrow X$ by

$$P_n^t(x)=\sum_{k=1}^{n}x(t_k)\frac{U_n(t)}{(t-t_k)U_n'(t_k)}.\tag{3.3}$$

Thus, equation (3.1) can be written in the equivalent form

$$A_nx_n=Gx_n+P_n^tHP_n^\tau(hx_n)=P_n^ty,\quad x_n,P_ny\in X_n,\tag{3.4}$$

where $X_n=\{x_n: x_n=\sum_{k=1}^{n}\alpha_kU_{k-1}(t),\, t\in[-1,1]\}$,

$$\begin{aligned}Gx&=\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}\,x(\tau)}{(\tau-t)^2}\,d\tau,\quad x\in X,\\ Hhx&=\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\tau^2}\,h(t,\tau)x(\tau)\,d\tau.\end{aligned}\tag{3.5}$$

Since $P_n^tHP_n^\tau(hx_n)=P_n^tH((P_n^\tau h)x_n)$, then, for all $x_n\in X_n$, we get

$$\begin{aligned}||Ax_n-A_nx_n||_\rho&=||Hhx_n-P_n^tHP_n^\tau(hx_n)||_\rho\\ &\le||(Hh-P_n^tHh)x_n||_\rho+||P_n^tH(h-P_n^\tau h)x_n||_\rho.\end{aligned}\tag{3.6}$$

According to [8], $\|P_n\|\le C_1$, $\|x-P_nx\|_\rho\le C_2E_n(x)_C$, $x\in C[-1,1]$, we get

$$\begin{aligned}||Ax_n-A_nx_n||_\rho&\le C_2E_n(Hhx_n)+C_1||H(h-P_n^\tau)x_n||_C\\ &=O\{E_n^t(h)_C+E_n^\tau(h)_C\}||x_n||_\rho,\end{aligned}\tag{3.7}$$

so that

$$||A-A_n||_{X_n\to X}=O\{E_n^t(h)_C+E_n^\tau(h)_C\}=\epsilon_n.\tag{3.8}$$

According to [1], for all n such that $\|A^{-1}\|\epsilon_n<1$, A_n has a bounded inverse and $\|A_n^{-1}\|=O(1)$, $A_n: X_n\longrightarrow X_n$. Since $\|y-P_ny\|_\rho=O\{E_n(y)_C\}=\delta_n$. Finally, we have

$$\begin{aligned}||x^*-x_n^*||_\rho&=||A^{-1}y-A_n^{-1}P_ny||_\rho\\ &=O(\epsilon_n+\delta_n)\\ &=O\{E_n^t(h)_C+E_n^\tau(h)_C+E_n(y)_C\}.\end{aligned}\tag{3.9}$$

□

Lemma 3.1. *If Q_n is an algebraic polynomial of degree $n-1$, then*

$$||Q_n(t)||_C \le n\sqrt{\frac{n}{2}}\,||Q_n||_\rho, \quad n \ge 2. \tag{3.10}$$

Proof. One may write Q_n as $Q_n(t) = \sum_{k=1}^{n} C_{k-1}(Q_n)U_{k-1}(t)$, $n \in \mathbb{N}$, where $C_j(Q_n) = (Q_n, U_j)_\rho$. Since $|U_{k-1}| \le k$ it follows that

$$\begin{aligned} ||Q_n(t)||_C &\le \left\{\sum_{k=1}^{n} C_{k-1}\,|\,(Q_n)^2\,|\right\}^{1/2}\left\{\sum_{k=1}^{n} k^2\right\}^{1/2} \\ &= ||Q_n||_\rho\sqrt{n(n+1)(2n+1)/6} \\ &\le n\sqrt{n/2}\,||Q_n||_\rho \\ &= ||Q_n||_\rho O(n^{3/2}). \end{aligned} \tag{3.11}$$

□

Define $W^rH_\alpha = \{x : x^{(r-1)}$ is absolutely continuous, $x^{(r)} \in H_\alpha\}$.

Theorem 3.2. *Assume that conditions* (i) *and* (ii) *of Theorem* 3.1 *are satisfied,*

$$y(t) \in W^rH_\alpha,\ h(t,\tau) \in W^rH_\alpha, \quad r \ge 0,\ 0 < \alpha \le 1, \tag{3.12}$$

then

$$||x^* - x_n^*||_\rho = O(n^{-r-\alpha}), \tag{3.13}$$

$$||x^* - x_n^*||_C = O(n^{3/2-r-\alpha}). \tag{3.14}$$

Proof. Since $y(t) \in W^rH_\alpha$, $h(t,\tau) \in W^rH_\alpha$, then, according to [7], one has $E_n(y) = O(n^{-r-\alpha})$, $E_n^t(h) = O(n^{-r-\alpha})$, $E_n^\tau(h) = O(n^{-r-\alpha})$, $r+\alpha > 0$. This proves (3.13). It is easy to show that

$$\begin{aligned} ||x^* - x_n^*||_C &= \sum_{k=1}^{\infty} ||x_{2^kn}^* - x_{2^{k-1}n}^*||_C \\ &\le \sum_{k=1}^{\infty}(2^kn)^{(3/2)}\left[||x^* - x_{2^kn}^*||_\rho + ||x^* - x_{2^{k-1}n}^*||_\rho\right] \\ &\le C_3\sum_{k=1}^{\infty}(2^kn)^{(3/2)}(2^kn)^{-r-\alpha} = C_4n^{-r-\alpha+3/2}, \end{aligned} \tag{3.15}$$

where C_3, C_4 are constants. This proves (3.14). □

Define $C_\rho[-1,1] = \{x : \sqrt{1-t^2}x \in C[-1,1]\}$ and $\|x\|_{C_\rho} = \max\{\sqrt{1-t^2}|x(t)|\}$.

Lemma 3.2. *If Q_n is a polynomial of degree $n-1$, then*

$$||Q_n(t)||_{C_\rho} \le \sqrt{n}\,||Q_n||_X. \tag{3.16}$$

Proof. Since $|U_m(t)| \le (1-t^2)^{-1/2}$, $-1 \le t \le 1$, $m = 0, 1, \dots,$ then

$$\begin{aligned} \sqrt{1-t^2}\,|\,Q_n(t)\,| &\le \sum_{k=1}^{n}|\,C_{k-1}(Q_n)\,| \\ &\le \left\{\sum_{k=1}^{n}|\,C_{k-1}(Q_n)\,|^2\right\}^{1/2}\sqrt{n} = \sqrt{n}||Q_n||_\rho. \end{aligned} \tag{3.17}$$

□

Theorem 3.3. *If* $\|x^* - x_n^*\|_X = O(n^{-m})$, $m \in \mathbb{R}$, *then*

$$||x^* - x_n^*||_{C_\rho} = O(n^{1/2-m}), \quad m > \frac{1}{2}, \tag{3.18}$$

Proof. Using Lemma 3.2 and the same technique as in Theorem 3.2, one obtains (3.18). □

Conclusion. For the Mechanical Quadratures methods, the rate of convergence in space $C_\rho[-1,1]$ is better than that given in space $C[-1,1]$.

4. Approximation by degenerate kernels.

We approximate $h(t,\tau)$ by

$$h_n(t,\tau) = \sum_{k=1}^{n} a_k(t) b_k(\tau), \quad -1 \le t,\ \tau \le 1, \tag{4.1}$$

where $\{a_k\}$, $\{b_k\}$ are two sets of linearly independent functions. By substituting back into (1.1), we obtain

$$A_n x = Gx + \int_{-1}^{1} \sqrt{1-\tau^2}\, h_n(t,\tau)\, x(\tau)\, d\tau = y. \tag{4.2}$$

(4.2) can be written as

$$Gx + \sum_{k=1}^{n} \alpha_k a_k = y, \tag{4.3}$$

where $\alpha_k = \int_{-1}^{1} \sqrt{1-\tau^2} x(\tau) b_k(\tau)\, d\tau$. Then the solution of (4.3) is given by

$$\begin{aligned} x &= G^{-1} y - \sum_{k=1}^{n} \alpha_k G^{-1} a_k, \\ G^{-1} y &= -\sum_{k=1}^{\infty} \frac{(y, U_{k-1})_\rho}{k} U_{k-1}(t). \end{aligned} \tag{4.4}$$

Multiplying (4.4) by $\sqrt{1-t^2} b_j(t)$ and integrating, we get the linear system of equations

$$\alpha_j + \sum_{k=1}^{n} \gamma_{jk} \alpha_k = y_j, \quad 1 \le j \le n, \tag{4.5}$$

where $\gamma_{jk} = \int_{-1}^{1} \sqrt{1-t^2} b_j G^{-1}(a_k)\, dt$, $y_j = \int_{-1}^{1} \sqrt{1-t^2} b_j G^{-1}(y)\, dt$. Define $q = \sqrt{1-t^2} \sqrt{1-\tau^2}$.

Theorem 4.1. *Suppose that*

(i) $y \in L_{2,\rho}$

(ii) $h \in L_{2,q}[-1,1;-1,1]$

(iii) $\epsilon_n^2 = \int_{-1}^{1} \int_{-1}^{1} q(t,\tau) \left| h(t,\tau) - h_n(t,\tau) \right|^2 dt\, d\tau \longrightarrow 0, n \longrightarrow \infty$

(iv) *equation (1.1) has a unique solution.*

Then for all n *such that* $q_n = \epsilon_n \|A^{-1}\| < 1, A : X \longrightarrow X$, *the linear system of equations (4.5) has a unique solution* $\{\alpha_k^*\}_1^n$ *and the approximate solution* $x_n^* = G^{-1}(y) - \sum_{k=1}^{n} \alpha_k^* G^{-1}(a_k)$ *converges to the exact solution* x^*, $\|x^* - x_n^*\|_\rho = O(\epsilon_n)$.

Proof. For $x \in X$, we have

$$\begin{aligned}\|Ax - A_n x\|_\rho &= \left\| \int_{-1}^{1} \sqrt{1-\tau^2}\,[h(t,\tau) - h_n(t,\tau)]\,x(\tau)\,d\tau \right\|_\rho \\ &\le \|x\|_\rho \|h - h_n\|_{L_{2,q}} = \epsilon_n \|x\|_\rho.\end{aligned} \tag{4.6}$$

Then $\|A - A_n\|_{X \to X} \le \epsilon_n$, according to [6] (4.5) has a unique solution $\{\alpha_k^*\}_1^n$, $\|x^* - x_n^*\| = O(\epsilon_n)$. □

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