

REGULARIZED SUM FOR EIGENFUNCTIONS OF MULTI-POINT PROBLEM IN THE COMMENSURABLE CASE

S.A. SALEH

Department of Mathematics
Faculty of Science
Tanta University
Tanta, Egypt

(Received January 3, 1995 and in revised form May 22, 1995)

ABSTRACT. Consider the eigenvalue problem which is given in the interval $[0, \pi]$ by the differential equation

$$-y''(x) + q(x)y(x) = \lambda y(x); \quad 0 \leq x \leq \pi \quad (0.1)$$

and multi-point conditions

$$\begin{aligned} U_1(y) &= \alpha_1 y(0) + \alpha_2 y(\pi) + \sum_{k=3}^n \alpha_k y(x_k, \pi) = 0, \\ U_2(y) &= \beta_1 y(0) + \beta_2 y(\pi) + \sum_{k=3}^n \beta_k y(x_k, \pi) = 0, \end{aligned} \quad (0.2)$$

where $q(x)$ is sufficiently smooth function defined in the interval $[0, \pi]$. We assume that the points x_3, x_4, \dots, x_n divide the interval $[0, 1]$ to commensurable parts and $\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0$. Let $\lambda_{k,s} = \rho_{k,s}^2$ be the eigenvalues of the problem (0.1)-(0.2) for which we shall assume that they are simple, where k, s , are positive integers and suppose that $H_{k,s}(x, \xi)$ are the residue of Green's function $G(x, \xi, \rho)$ for the problem (0.1)-(0.2) at the points $\rho_{k,s}$. The aim of this work is to calculate the regularized sum which is given by the form :

$$\sum_{(k)} \sum_{(s)} \left[\rho_{k,s}^{-\sigma} H_{k,s}(x, \xi) - R_{k,s}(\sigma, x, \xi, \rho) \right] = S_{\sigma}(x, \xi) \quad (0.3)$$

The above summation can be represented by the coefficients of the asymptotic expansion of the function $G(x, \xi, \rho)$ in negative powers of k . In series (0.3) σ is an integer, while $R_{k,s}(\sigma, x, \xi, \rho)$ is a function of variables x, ξ and defined in the square $[0, \pi] \times [0, \pi]$ which ensure the convergence of the series (0.3).

KEY WORDS AND PHRASES : Regularized sum for eigenfunctions, asymptotic formula, Green's function, differential operator.

1991 AMS SUBJECT CLASSIFICATION CODES : 47E05

1. INTRODUCTION.

It is well-known that the sum of the diagonal elements in a square matrix is equal to the sum of the eigenvalues of its operator in finite dimensional space. In other words the trace of a matrix is equal to the spectral trace in n -dimensional spaces.

It is worth mentioning that this theorem is satisfied also in the case of nuclear operators acting in Hilbert space. Sadovnichii [1] proved this theorem. Thus we might ask the following question. Is the last theorem applicable to the case of unbounded operators?, especially in the case of differential operators since in general case the trace of a matrix and spectral trace do not exist. Consider, for example, the boundary-value problem :

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad 0 \leq x \leq \pi \quad (1.1)$$

$$y(0) = y(\pi) = 0, \quad (1.2)$$

where $q(x)$ is sufficiently smooth function.

The eigenvalues λ_n of problem (1.1), (1.2) has the asymptotic expansion in the form :

$$\lambda_n \sim n^2 + c_0 + \frac{c_2}{n^2} + \frac{c_4}{n^4} + \dots, \quad (1.3)$$

where
$$c_0 = \frac{1}{2\pi} \int_0^\pi q(\xi) d\xi \quad (1.4)$$

From Equation (1.3) it is clear that $\sum_{n=1}^{\infty} \lambda_n$ diverges, while $\sum_{n=1}^{\infty} (\lambda_n - n^2 - c_0)$ converges, and is called the regular trace for the problem (1.1), (1.2).

The study of regular trace for differential operators plays an important role in several fields such as mathematical analysis, theoretical physics and quantum mechanics, where the regular traces give the asymptotic expansion for the eigenvalues of operators. We can also use the regular trace in the inverse spectral problems in functional analysis.

A good number of works has been devoted to the deduction of the formulae of regularized traces of differential operators Gelfand, Levitan [2], Charles, Halberg and Kramer [3], Lidsky, Sadovnichii [4, 5, 6], Sadovnichii, Lyubishkin and Belabbasy [7, 8], Saleh [9] and many other authors

The concept of the regularized trace with a weight for the differential operators was introduced by Sadovnichii [10].

The main goal now is to derive asymptotic formulae for the solutions of (0.1) when $|\lambda| \rightarrow \infty$ and then use them to obtain the asymptotic formulae for the eigenvalues of the problem (0.1), (0.2). The concluding part of this paper is devoted to the derivation of the regularized sums of eigenfunctions of the second order, and we shall give some examples to illustrate the mentioned concept of regularized sums of eigenfunctions.

2. ASYMPTOTIC FORMULAE FOR THE SOLUTION OF THE STURM-LIOUVILLE EQUATION

The solution of the differential equation (0.1) admits asymptotic expansions in powers of ρ^{-1} which become more precise as the number of derivatives that the function $q(x)$ has increases " Marchenko [11], Naimark [12]. Let $y_1(x, \rho)$ and $y_2(x, \rho)$ be linearly independent solutions of (0.1), then

$$\begin{aligned} y_1(x, \rho) &:: e^{i\rho x} \left[1 + \sum_{v=1}^N \frac{u_v(x)}{\rho^v} + O\left(\frac{1}{\rho^{N+1}}\right) \right], \\ y_2(x, \rho) &:: e^{-i\rho x} \left[1 + \sum_{v=1}^N \frac{(-1)^v u_v(x)}{\rho^v} + O\left(\frac{1}{\rho^{N+1}}\right) \right], \end{aligned} \quad (2.1)$$

where N positive integer depends on the smoothness of the function $q(x)$ and the functions $u_v(x)$, $v=1, 2, \dots, N$ admit the representations:

$$\begin{aligned}
 u_1(x) &= \frac{1}{2i} \int_0^x q(\xi) d\xi, \\
 u_2(x) &= \frac{1}{4} \left[q(x) - q(0) - \frac{1}{2} \left(\int_0^x q(\xi) d\xi \right)^2 \right], \\
 u_{v+2}(x) &= \frac{1}{2i} \int_0^x \left\{ \dots \frac{d^2}{d\xi^2} + q(\xi) \right\} u_{v+1}(\xi) d\xi; \quad v = 1, 2, 3, \dots, N-2 \quad (2.2)
 \end{aligned}$$

We note that $u_v(0) = 0$, for $v = 1, 2, \dots, N$. By means of the asymptotic formulae (2.1) and Equation (2.2) we can prove that

$$\begin{aligned}
 \delta(x, \rho) &= \delta(0, \rho) + W[y_1, y_2] \Big|_{x=0} \\
 &= -2i\rho + i \left[\frac{q(0)}{2\rho} - \frac{q'''(0) - 4q(0)q'(0)}{4\rho^3} + \dots \right] \quad (2.3)
 \end{aligned}$$

3. ASYMPTOTIC FORMULAE FOR EIGENVALUES OF THE PROBLEM (0.1)-(0.2) IN THE COMMENSURABLE CASE

In Saleh [9] proved that the eigenvalues of the problem (0.1)-(0.2) ($\lambda = \rho^2$) are found from the condition :

$$f(\rho) = \frac{\Delta(\rho)}{\delta(\rho)} = 0, \quad (3.1)$$

where
$$\Delta(\rho) = \det \|U_i(y_k)\|_{i,k=1}^2 \quad (3.2)$$

Upon using Equations (2.1), (2.2) it is easy to see that

$$\Delta(\rho) = \sum_{k=1}^{2n^2-6} A_k(\rho) \left[\gamma_k^{(0)} + \frac{\beta_k^{(1)}}{\rho} + \frac{\beta_k^{(2)}}{\rho^2} + \dots + \frac{\beta_k^{(N)}}{\rho^N} + \dots \right], \quad (3.3)$$

where $A_k(\rho) = e^{i\xi_k \pi \rho}$, $\xi_1 = -1$, $\xi_2 = -(1 - x_3)$, $\xi_{2n^2-6} = -\xi_1 = 1$,

and $\gamma_k^{(0)}$, $\beta_k^{(j)}$, ($j=1,2,3,\dots$) are calculated in terms of the constants α_v, β_v ($v=1,2,\dots,n$) and the function $q(x)$, for example,

$$\begin{aligned}
 \gamma_1^{(0)} &= (\alpha_1 \beta_2 - \beta_1 \alpha_2) = -\gamma_{2n^2-6}^{(0)}, \\
 \beta_1^{(1)} &= -\beta_{2n^2-6}^{(1)} = \frac{1}{2i} (\beta_1 \alpha_2 - \alpha_1 \beta_2) \int_0^\pi q(\xi) d\xi \quad (3.4)
 \end{aligned}$$

Using the results of Saleh [9], we deduce that in commensurable case the problem (0.1)-(0.2) has $2m$ series of eigenvalues which have the following asymptotic formula:

$$\rho_{k,s} \sim -2mk - \frac{m}{i\pi} \ln a_0^{(s)} + \frac{a^{(s)}}{2a_0^{(s)} k} - \frac{a^{(s)} \ln a^{(s)}}{4ia_0^{(s)} \pi k^2} + \dots, \quad (3.5)$$

where $m = \frac{1}{d}$; $d = \min\{\xi_1, \xi_2, \xi_3, \dots, \xi_{2n^2-6}\}$

4. THE ASYMPTOTIC FORMULAE FOR THE GREEN'S FUNCTION OF THE PROBLEM (0.1)-(0.2) IN COMMENSURABLE CASE

It is well-known that the Green's function of differential equation of the second order is given by the formula:

$$G(x, \xi, \rho) = \frac{1}{\Delta(\rho)} \begin{vmatrix} y_1(x, \rho) & y_2(x, \rho) & g(x, \xi, \rho) \\ U_1(y_1) & U_1(y_2) & U_1(g) \\ U_2(y_1) & U_2(y_2) & U_2(g) \end{vmatrix}, \quad (4.1)$$

$$\text{where} \quad g(x, \xi, \rho) = \pm \frac{1}{2\delta(0, \rho)} \left[y_1(x, \rho) y_2(\xi, \rho) - y_2(x, \rho) y_1(\xi, \rho) \right] \quad (4.2)$$

(The positive sign being taken if $x > \xi$, and the negative sign if $x < \xi$).

If we divide the ρ -plane into four regions $\tilde{S}_0, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3$ such that:

$$\begin{aligned} \tilde{S}_0 &= \left\{ \rho : |\rho| > R, \quad 0 < \arg \rho \leq \frac{\pi}{2} - \theta \right\}, \\ \tilde{S}_1 &= \left\{ \rho : |\rho| > R, \quad \frac{\pi}{2} - \theta < \arg \rho \leq \pi \right\}, \\ \tilde{S}_2 &= \left\{ \rho : |\rho| > R, \quad \pi < \arg \rho \leq \frac{3\pi}{2} - \theta \right\}, \\ \tilde{S}_3 &= \left\{ \rho : |\rho| > R, \quad \frac{3\pi}{2} - \theta < \arg \rho \leq 2\pi \right\}, \end{aligned} \quad (4.3)$$

we see that the Green's function of the problem (0.1)-(0.2) in the commensurable case has the following asymptotic formula:

$$\begin{aligned} G(x, \xi, \rho) &\sim e^{i(x-\xi)\rho} \sum_{v=0}^{\infty} \frac{\phi_{v,j}^{(1)}}{\rho^v}; \quad \rho \in \tilde{S}_0^{\delta}, \quad \xi < x < x_j \pi, \\ G(x, \xi, \rho) &\sim e^{-i(x-\xi)\rho} \sum_{v=0}^{\infty} \frac{\phi_{v,j}^{(2)}}{\rho^v}; \quad \rho \in \tilde{S}_0^{\delta}, \quad \xi < x_j \pi < x, \end{aligned} \quad (4.4)$$

$$\text{where} \quad \tilde{S}_0^{\delta} = \tilde{S}_0 \setminus Q_{k,s}^{\delta}$$

and,

$$Q_{k,s}^{\delta} = \left\{ \rho : |\rho - \rho_{k,s}| \leq \delta, \quad \Delta(\rho_{k,s}) = 0 \right\}$$

Since $H_{k,s}(x, \xi)$ the residue of Green's function $G(x, \xi, \rho)$ for problem (0.1)-(0.2) in the points $\rho_{k,s}$ and from the assumption that the eigenvalues $\rho_{k,s}$ are simple, then

$$H_{k,s}(x, \xi, \rho_{k,s}) = \lim_{\rho \rightarrow \rho_{k,s}} (\rho - \rho_{k,s}) G(x, \xi, \rho) \quad (4.5)$$

Upon using the asymptotic formulae (4.4) for $G(x, \xi, \rho)$ in \tilde{S}_0^{δ} and Equation (4.5) we have for

$H_{k,s}(x, \xi)$ the following asymptotic formulae

$$\begin{aligned} H_{k,s}(x, \xi, \rho_{k,s}) &\sim e^{i(x-\xi)\rho_{k,s}} \sum_{v=0}^{\infty} \frac{\phi_{v,j}^{(3)}}{\rho_{k,s}^v}, \quad \rho_{k,s} \in \tilde{S}_0^{\delta}, \quad \xi < x < x_j \pi \\ H_{k,s}(x, \xi, \rho_{k,s}) &\sim e^{-i(x-\xi)\rho_{k,s}} \sum_{v=0}^{\infty} \frac{\phi_{v,j}^{(4)}}{\rho_{k,s}^v}, \quad \rho_{k,s} \in \tilde{S}_0^{\delta}, \quad \xi < x_j \pi < x, \end{aligned} \quad (4.6)$$

where the functions $\phi_{v,j}^{(1)}, \phi_{v,j}^{(2)}, \phi_{v,j}^{(3)}$ and $\phi_{v,j}^{(4)}$ are defined in terms of the constants

α_k, β_k , ($k = 1, 2, \dots, n$) and the function $q(x)$, $j = 3, 4, 5, \dots, n$. We note that for the functions $\Delta(\rho)$, $\Delta'(\rho)$ have asymptotic formulae:

$$\Delta(\rho) \sim e^{-i\rho\pi} \sum_{v=0}^{\infty} \frac{\psi_v^{(1)}}{\rho^v}, \quad \rho \in \tilde{S}_0^{\delta} \quad (4.7)$$

$$\Delta'(\rho) \sim e^{-i\rho\pi} \sum_{v=0}^{\infty} \frac{\psi_v^{(2)}}{\rho^v}, \quad \rho \in \tilde{S}_0^{\delta} \quad (4.8)$$

5. REGULARIZED SUM FOR EIGENFUNCTIONS OF THE PROBLEM (0.1)-(0.2) IN THE COMMENSURABLE CASE

Now we wish to evaluate the functions $R_{k,s}(\sigma, x, \xi)$ which ensure the convergence of the series (0.3). We must first estimate the functions $\rho_{k,s}^{-\sigma} H_{k,s}(x, \xi)$ in \tilde{S}_0^{σ} . From Equation (3.5), we have

$$\rho_{k,s}^{-\sigma} \sim \sum_{v=0}^{\infty} \frac{Q_v^{(s)}(\sigma)}{k^{v+\sigma}} \quad (5.1)$$

In the asymptotic formula (5.1)

$$\begin{aligned} Q_0^{(s)}(\sigma) &= (-2m)^{-\sigma}, \quad Q_1^{(s)}(\sigma) = (-1)^{-\sigma+1} (2m)^{-\sigma-1} \frac{m}{i\pi} \ln a_0^{(s)}, \\ Q_2^{(s)}(\sigma) &= (-2m)^{-\sigma} \left[\binom{-\sigma}{1} \frac{a_1^{(s)}}{2a_0^{(s)}(-2m)} + \binom{-\sigma}{2} \frac{(\ln a_0^{(s)})^2}{4\pi^2} \right], \dots \end{aligned} \quad (5.2)$$

From Equation (3.5) we get

$$\rho_{k,s}^{-\sigma} H_{k,s}(x, \xi) \sim e^{i(x-\xi)(-2mk - \frac{m}{i\pi} \ln a_0^{(s)})} \sum_{n=0}^{\infty} \frac{\psi_n^{(s)}(x, \xi)}{k^n}, \quad (5.3)$$

where the functions $\psi_n^{(s)}(x, \xi)$ are polynomials of (x, ξ) .

Upon using Equations (4.5), (5.1) and (5.2), we have

$$\rho_{k,s}^{-\sigma} H_{k,s}(x, \xi) \sim e^{i(x-\xi)(-2mk - \frac{m}{i\pi} \ln a_0^{(s)})} \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)}(x, \xi) Q_1^{(s)}(\sigma+p-1) \phi_{p-1}^{(s)}}{k^{n+\sigma}} \quad (5.4)$$

For large number τ , we consider the function

$$\Phi_{\tau}(\sigma) = \sum_{(k)} \sum_{(s)} \left[\rho_{k,s}^{-\sigma} H_{k,s}(x, \xi) - e^{i(x-\xi)(-2mk - \frac{m}{i\pi} \ln a_0^{(s)})} \sum_{n=0}^{\tau} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)}(x, \xi) Q_1^{(s)}(\sigma+p-1) \phi_{p-1}^{(s)}}{k^{n+\sigma}} \right]. \quad (5.5)$$

It is clear that the function $\Phi_{\tau}(\sigma)$ may be extended to analytic function in the half plane $\operatorname{Re} \sigma > -\tau$

THEOREM 5.1 If $\operatorname{Re} \sigma > -\tau$ then

$$\begin{aligned} & \sum_{(k)} \sum_{(s)} \left[\rho_{k,s}^{-\sigma} H_{k,s}(x, \xi) - e^{i(x-\xi)(-2mk - \frac{m}{i\pi} \ln a_0^{(s)})} \sum_{n=0}^{\tau} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)}(x, \xi) Q_1^{(s)}(\sigma+p-1) \phi_{p-1}^{(s)}}{k^{n+\sigma}} \right] \\ &= \phi_{-\sigma+1, j}^{(1)} - \sum_{(s)} \sum_{n=0}^{\tau} \sum_{p=0}^n \sum_{\ell=0}^p e^{i(x-\xi)(-\frac{m}{i\pi} \ln a_0^{(s)})} F(e^{i(x-\xi)(-2m)}, n+\sigma) \psi_{n-p}^{(s)}(x, \xi) Q_1^{(s)}(\sigma+p-1) \phi_{p-1}^{(s)}, \end{aligned} \quad (5.6)$$

where

$$F(z, \sigma) = \sum_{k=1}^{\infty} \frac{z^k}{k^{\sigma}}.$$

THEOREM 5.2 If $\sigma = -2$, we have

$$\sum_{(k)(s)} \lambda_{k,s} H_{k,s}(x, \xi) e^{i(x-\xi)(-2mk - \frac{m}{\ln a_0}(s))} \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)}(x, \xi) Q_1^{(S)}(-2+p-l) \phi_{p-l}^{(s)}}{k^{n-2}} \Bigg] \\ = \phi_{3,1}^{(1)} - \sum_{(s)n=0}^3 \sum_{p=1}^n \sum_{\ell=0}^p e^{i(x-\xi)(-\frac{m}{i\pi} \ln a_0(s))} F(e^{i(x-\xi)(-2m)}, n-2) \psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell}^{(s)}. \quad (5.7)$$

REMARK. From the definition of $F(z, \sigma)$ it is clear that this function satisfies the following properties

1. $F(z, \sigma) = z\Phi(z, \sigma, 1)$, where

$$\Phi(z, \sigma, v) = \sum_{n=0}^{\infty} (v+n)^{-\sigma} z^n = \frac{1}{\Gamma(\sigma)} \int_0^{\infty} \frac{t^{\sigma-1} e^{-(v-1)t}}{e^t - z} dt,$$

Re $v > 0$ and either $|z| \leq 1$, $z \neq 1$, Re $\sigma > 0$ or $z = 1$, Re $\sigma > 1$.

2. $F(z, -m) = (-1)^{m+1} F(\frac{1}{z}, -m)$, $m = 1, 2, 3, \dots$

3. $F(z, s) + e^{i\pi s} F(\frac{1}{z}, s) = \frac{(2\pi)^s}{\Gamma(s)} e^{i\pi s/2} \zeta(1-s, \frac{\log z}{2\pi i})$

4. Equations (0.2), (0.3) furnish the analytical continuation of the series $\sum_{n=1}^{\infty} \frac{z^n}{n^\sigma}$ beyond the circle of convergence $|z| = 1$

If $F_0(z)$ denotes the principal branch of $F(z)$ in the cut z -plane $[0 < \arg(z-1) < 2\pi]$, the cut being imposed from 1 to ∞ along the real axis, the difference of the values of $F_0(z)$ between a point on the upper edge of the cut and a point on the lower edge, according to (0.3),

$$F_0(x, s) - F_0(xe^{2i\pi}, s) = 2\pi i (\log x)^{s-1} / \Gamma(s)$$

Hence, if we cross the cut, from the upper half-plane to the lower half-plane, we obtain for the continuation $F_1(z)$ of $F_0(z)$

$$F_1(z) = F_0(z) + 2\pi i (\log z)^{s-1} / \Gamma(s)$$

The analogous formula for the inverse process of continuation is

$$F_2(z) = F_0(z) - 2\pi i (\log z)^{s-1} / \Gamma(s)$$

5. $F(e^{it}, -m) = (i \frac{d}{dt})^m \frac{1}{e^{-it} - 1}$; $m = 1, 2, 3, \dots$

The previous properties of $F(z, \sigma)$ are proved in A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi [13]

6. D. Klusch [14] considered the generalized zeta function in the form

$$L(x, a, s) = \sum_{n \geq 0} e^{(2\pi i n x)(n+a)^{-s}}$$

($a \in \mathbb{R}^+$; x is not integer, Re $s > 0$; and if x is an integer, Re $s > 1$)

and studied some further properties of the function $L(x, a, s)$ resulting from the Taylor expansion of the function $W(\zeta) = L(x, a + \zeta, s)$ in the neighbourhood $\zeta = 0$

Now, we consider the following examples:

6. EXAMPLES

1. Consider the problem

$$-y''(x) = \lambda y(x) \quad 0 \leq x \leq \pi, \quad (6.1)$$

$$y(0) = y(\pi) = 0, \quad (6.2)$$

It's clear that the eigenvalues of problem (6.1), (6.2) are $\lambda_n = n^2$ and the corresponding eigenfunctions are $y_n(x) = \sin nx$, so the regularized trace of problem (6.1), (6.2) is $\sum_{n=1}^{\infty} (\lambda_n - n^2) = 0$, and the regularized sum of eigenfunctions of problem (6.1)-(6.2) is given by the following formula

$$\sum_{(n)} \left[\lambda_n H_n(x, \xi) + \frac{n}{\pi} \sin nx \sin n\xi \right] = 0. \quad (6.3)$$

2. Consider the Sturm Liouville problem

$$-y''(x) + q(x)y(x) = \lambda y, \quad \lambda = p^2, \quad 0 \leq x \leq \pi \quad (6.4)$$

$$y(0) = y(\pi) = 0, \quad (6.5)$$

where $q(x)$ is a sufficiently smooth function defined in the interval $[0, \pi]$.

Let $y_1(x, \rho)$, $y_2(x, \rho)$ are two independent solutions of Equation (6.4) such that

$$y_j^{(k-1)}(0, \rho) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

Then from Equations (2.1), (2.2) and (2.3), we have

$$y_1(x, \rho) = \sum_{v=0}^N \frac{A_v(x, \rho)}{\rho^v} + O\left(\frac{1}{\rho^{N+1}}\right) \quad (6.6)$$

$$y_2(x, \rho) = \sum_{v=1}^N \frac{B_v(x, \rho)}{\rho^v} + O\left(\frac{1}{\rho^{N+1}}\right) \quad (6.7)$$

where

$$\begin{aligned} A_0(x, \rho) &= \cos \rho x \\ A_1(x, \rho) &= i u_1(x) \sin \rho x, \\ A_2(x, \rho) &= u_2(x) \cos \rho x, \\ A_3(x, \rho) &= i \left[u_3(x) + i u_2'(0) \right] \sin \rho x, \dots \\ B_1(x, \rho) &= \sin \rho x \\ B_2(x, \rho) &= -i u_1(x) \cos \rho x, \\ B_3(x, \rho) &= -i (i u_2(x) - u_1'(0)) \sin \rho x, \dots \\ B_4(x, \rho) &= \left[2 u_1(x) u_1'(0) - i u_3(x) \right] \cos \rho x, \dots \end{aligned} \quad (6.8)$$

and N is a positive integer depending on the smoothness of the function $q(x)$.

Since $\Delta(\rho) = \det \|U_j(y_k)\|_{j,k}^2$, then

$$\Delta(\rho) = y_2(\pi, \rho) = \sum_{v=1}^N \frac{B_v(\pi, \rho)}{\rho^v} + O\left(\frac{1}{\rho^{N+1}}\right)$$

From the last formula, we can obtain the roots of the function $\Delta(\rho)$ which are eigenvalues of the problem (6.4)-(6.5). Upon using the successive approximation, we get the following asymptotic formula for the zeros of the function $\Delta(\rho)$:

$$\rho_n \sim n + \frac{c_{-1}}{n} + \frac{c_{-3}}{n^3} + \dots, \quad (6.9)$$

where

$$c_{-1} = \frac{i}{\pi} u_1(\pi), \quad c_{-3} = \frac{1}{2\pi i} \left[3 \left\{ u_1(\pi) u_2(\pi) - i u_1(\pi) u_1'(0) - u_3(\pi) \right\} - u_1^3(\pi) \right], \dots \quad (6.10)$$

Then

$$\lambda_n \sim n^2 + c_0 + \frac{c_2}{n^2} + \frac{c_4}{n^4} + \dots, \quad (6.11)$$

where
$$c_0 = \frac{1}{\pi} \int_0^\pi q(t) dt; \quad c_2 = (c_{-1})^2 + 2c_{-3}, \dots$$

In the paper [2] I. M. Gelfand, B. M. Levitan have proved that

$$\sum_{n=1}^{\infty} (\lambda_n - n^2 - c_0) = \frac{1}{2} c_0 - \frac{1}{4} [q(0) + q(\pi)] \quad (6.12)$$

Upon using the results in H.F. Weinberger[14], we deduce that the Green's function of the problem (6.4)-(6.5) is given by the following formula:

$$G(x, \xi, \rho) = \begin{cases} \frac{y_2(\xi, \rho)}{y_2(\pi, \rho)} [y_1(x, \rho) y_2(\pi, \rho) - y_2(x, \rho) y_1(\pi, \rho)] & x \geq \xi \\ \frac{y_2(x, \rho)}{y_2(\pi, \rho)} [y_1(\xi, \rho) y_2(\pi, \rho) - y_2(\xi, \rho) y_1(\pi, \rho)] & x \leq \xi \end{cases} \quad (6.13)$$

From the definitions of $H_k(x, \xi, \rho_k)$, we deduce that

$$H_k(x, \xi, \rho_k) = \begin{cases} \frac{y_2(\xi, \rho_k) [y_1(x, \rho_k) y_2(\pi, \rho_k) - y_2(x, \rho_k) y_1(\pi, \rho_k)]}{y_2'(\pi, \rho_k)} & x \geq \xi \\ \frac{y_2(x, \rho_k) [y_1(\xi, \rho_k) y_2(\pi, \rho_k) - y_2(\xi, \rho_k) y_1(\pi, \rho_k)]}{y_2'(\pi, \rho_k)} & x \leq \xi \end{cases} \quad (6.14)$$

where $y_2'(\pi, \rho_k) = \frac{d}{d\rho} [y_2(\pi, \rho)] \big|_{\rho=\rho_k}$

Substituting Equations (6.6), (6.7) and (6.8) in (6.13), (6.14), we get the following asymptotic formulae:

$$G(x, \xi, \rho) \sim \begin{cases} e^{i\rho(x-\xi)} \sum_{v=0}^{\infty} \frac{\varphi_v^{(1)}(x, \xi)}{\rho^v}; & x > \xi \\ e^{-i\rho(x-\xi)} \sum_{v=0}^{\infty} \frac{\varphi_v^{(2)}(x, \xi)}{\rho^v}; & x \leq \xi \end{cases} \quad (6.15)$$

and

$$H_k(x, \xi, \rho_k) \sim \begin{cases} e^{i\rho_k(x-\xi)} \sum_{v=0}^{\infty} \frac{\varphi_v^{(3)}(x, \xi)}{\rho_k^v}, & x > \xi \\ e^{-i\rho_k(x-\xi)} \sum_{v=0}^{\infty} \frac{\varphi_v^{(4)}(x, \xi)}{\rho_k^v}; & x \leq \xi \end{cases} \quad (6.16)$$

In formulae (6.15) and (6.16), the functions $\varphi_j^{(k)}(x, \xi)$ ($k=1,2,3,4$, $j=0,1,2,3,\dots$) can be expressed in terms of the potential $q(x)$ and its derivatives. For example:

$$\begin{aligned} \varphi_0^{(1)} = \varphi_0^{(2)} = 0, \quad \varphi_1^{(1)} = \varphi_1^{(2)} = \frac{1}{2i}, \quad \varphi_2^{(1)} = \frac{i}{2}(u_1(x) - u_1(\xi)), \quad \varphi_2^{(2)} = \frac{i}{2}(u_1(\xi) - u_1(x)), \\ \varphi_3^{(2)} = \varphi_3^{(1)} = \frac{i}{2}[u_2(x) + u_1(\xi)u_1(x) + u_2(\xi) + iu_1'(0)], \dots \end{aligned}$$

$$\begin{aligned}\varphi_0^{(3)} = \varphi_0^{(4)} = 0, \quad \varphi_1^{(3)} = \varphi_1^{(4)} = \frac{-1}{2\pi}, \quad \varphi_2^{(3)} = \frac{1}{2\pi} [u_1(x) - u_1(\xi)], \quad \varphi_2^{(4)} = \frac{1}{2\pi} [u_1(\xi) - u_1(x)] \dots \\ \phi_3^{(3)} = \frac{1}{2\pi} \left[u_2(x) + u_2(\xi) - u_1(x)u_1(\xi) + iu_1'(0) + \frac{i}{\pi} (u_1(x) + u_1(\pi) - u_1(\xi) - 2u_1'(0)) - \frac{1}{\pi^2} \right] \quad (6.17)\end{aligned}$$

Using the asymptotic formula (6.9) we have

$$\rho_k^2 = \sum_{v=0}^{\infty} \frac{Q_v}{k^{v/2}}, \quad (6.18)$$

where $Q_0 = 1 \quad Q_1 = 0 \quad Q_2 = 2C_{-1} \quad Q_3 = 0, \quad Q_4 = 2C_{-3}$

When we deal with the problem (6.1)-(6.2), the formula (5.3) takes the form :

$$e^{i\rho_k(x-\xi)} \sim e^{ik(x-\xi)} \sum_{n=0}^{\infty} \frac{\psi_n(x, \xi)}{k^n}, \quad (6.19)$$

where $\psi_0(x, \xi) = 1 \quad \psi_1(x, \xi) = iC_{-1}(x - \xi)$

$$\psi_2(x, \xi) = -\frac{1}{2}C_{-1}^2(x - \xi)^2 \quad \psi_3 = iC_{-3}(x - \xi) + \frac{(iC_{-1}(x - \xi))^3}{6}, \dots \quad (6.20)$$

From the formulae (6.18), (6.19), we have

$$\lambda_k H_k(x, \xi) \sim \begin{cases} e^{ik(x-\xi)} \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p} Q_{\ell} \varphi_{p-\ell}^{(3)}}{k^{n-2}} & x > \xi \\ e^{-ik(x-\xi)} \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p} Q_{\ell} \varphi_{p-\ell}^{(4)}}{k^{n-2}}, & x \leq \xi \end{cases} \quad (6.21)$$

Using formulae (6.17), (6.18), (6.20) and (6.21), we have the following theorem.

THEOREM 6.1 For the problem (6.1)-(6.2), the regularized sum for the eigenfunctions is given by the following formulae :

(1) If $x > \xi$, then :

$$\begin{aligned}\sum_{k=1}^{\infty} \left[\lambda_k H_k(x, \xi) + \frac{1}{2\pi} e^{ik(x-\xi)} \left\{ k + u_1(\xi) - u_1(x) - 2\pi(x - \xi)u_1(\pi) + \frac{1}{k} [u_1(x)u_1(\xi) - u_2(x) - \right. \right. \\ \left. - u_2(\xi) - iu_1'(0) + \frac{i}{\pi} (u_1(\xi) - u_1(x) - u_1(\pi) + 2u_1'(0)) + \frac{1}{\pi^2} + 2\pi(x - \xi)u_1(x) - \right. \\ \left. - u_1(\xi)u_1(\pi) + 2\pi^2(x - \xi)^2 u_1^2(\pi) \right\} \right] = \frac{i}{2} [u_2(x) + u_1(\xi)u_1(x) + u_2(\xi) + u_1'(0)] + \\ + \frac{1}{2\pi} \left\{ F(e^{i(x-\xi)}, -1) + (u_1(\xi) - u_1(x) - 2\pi(x - \xi)u_1(\pi)) F(e^{i(x-\xi)}, 0) + \right. \\ \left. + [u_1(x)u_1(\xi) - u_2(x) - u_2(\xi) - iu_1'(0) + \frac{i}{\pi} (u_1(\xi) - u_1(x) - u_1(\pi) + 2u_1'(0)) + \right. \\ \left. + \frac{1}{\pi^2} + 2\pi(x - \xi)(u_1(x) - u_1(\xi))u_1(\pi) + 2\pi^2(x - \xi)^2 u_1^2(\pi)] F(e^{i(x-\xi)}, 0) \right\} \quad (6.22)\end{aligned}$$

(2) If $x < \xi$, then :

$$\begin{aligned}\sum_{k=1}^{\infty} \left[\lambda_k H_k(x, \xi) + \frac{1}{2\pi} e^{-ik(x-\xi)} \left\{ k + u_1(x) - u_1(\xi) - 2\pi(x - \xi)u_1(\pi) + \right. \right. \\ \left. + \frac{1}{k} [u_1(x)u_1(\xi) - u_2(x) - u_2(\xi) - iu_1'(0) + \frac{i}{\pi} (u_1(x) - u_1(\xi) - u_1(\pi) + \right. \\ \left. + 2u_1'(0)) + \frac{1}{\pi^2} + 2\pi(x - \xi)(u_1(\xi) - u_1(x))u_1(\pi) + 2\pi^2(x - \xi)^2 u_1^2(\pi) \right\} \right] \\ = \frac{i}{2} [u_2(x) + u_1(\xi)u_1(x) + u_2(\xi) + u_1'(0)] + \frac{1}{2\pi} \left\{ F(e^{-i(x-\xi)}, -1) + (u_1(x) - \right. \\ \left. - u_1(\xi) - 2\pi(x - \xi)u_1(\pi)) F(e^{-i(x-\xi)}, 0) + [u_1(x)u_1(\xi) - u_2(x) - u_2(\xi) - iu_1'(0) + \right. \\ \left. + \frac{1}{\pi} (u_1(x) - u_1(\xi) - u_1(\pi) + 2u_1'(0)) + \frac{1}{\pi^2} + 2\pi(x - \xi)(u_1(\xi) - u_1(x))u_1(\pi) + 2\pi^2(x - \xi)^2 u_1^2(\pi) \right\} \quad (6.23)\end{aligned}$$

3. Consider the Sturm-Liouville problem

$$-y''(x) + q(x)y(x) = \lambda y, \quad \lambda = \rho^2, \quad 0 \leq x \leq \pi \quad (6.24)$$

$$y(0) = y\left(\frac{\pi}{2}\right) + y(\pi) = 0, \quad (6.25)$$

where $q(x)$ is a sufficiently smooth function defined on the interval $[0, \pi]$.

Upon using the definition of $\Delta(\rho)$ and the formulae (6.6), (6.7), (6.8), we have:

$$\begin{aligned} \Delta(\rho) = & \frac{\sin \rho \pi + \sin \rho \frac{\pi}{2}}{\rho} + \frac{-i \left[u_1(\pi) \cos \rho \pi + u_1\left(\frac{\pi}{2}\right) \cos \rho \frac{\pi}{2} \right]}{\rho^2} + \\ & + \frac{-i \left[(iu_2(\pi) - u_1'(0)) \sin \rho \pi + (iu_2\left(\frac{\pi}{2}\right) - u_1'(0)) \sin \rho \frac{\pi}{2} \right]}{\rho^3} + \\ & + \frac{\left[2u_1(\pi)u_1'(0) - iu_3(\pi) \right] \cos \rho \pi + \left[2u_1\left(\frac{\pi}{2}\right)u_1'(0) - iu_3\left(\frac{\pi}{2}\right) \right] \cos \rho \frac{\pi}{2}}{\rho^4} + \dots \end{aligned} \quad (6.26)$$

To find the eigenvalues of problem (6.24)-(6.25), we put

$$V = e^{-\frac{x\rho}{2}} = \sum_{v=0}^{\infty} \frac{a_v}{\rho^v}; \quad \Delta(\rho) = 0 \quad \text{then}$$

$$\begin{aligned} V^4 \left[-\frac{1}{2i\rho} - \frac{iu_1(\pi)}{2\rho^2} + \frac{i[iu_2(\pi) - u_1'(0)]}{3i\rho^3} + \dots \right] + V^3 \left[-\frac{1}{2i\rho} - \frac{iu_1\left(\frac{\pi}{2}\right)}{2\rho^2} + \frac{i[iu_2\left(\frac{\pi}{2}\right) - u_1'(0)]}{3i\rho^3} + \dots \right] + \\ + V \left[\frac{1}{2i\rho} - \frac{iu_1\left(\frac{\pi}{2}\right)}{2\rho^2} + \frac{i[iu_2\left(\frac{\pi}{2}\right) - u_1'(0)]}{3i\rho^3} + \dots \right] + \left[\frac{1}{2i\rho} - \frac{iu_1(\pi)}{2\rho^2} + \frac{i[iu_2(\pi) - u_1'(0)]}{3i\rho^3} + \dots \right] = 0, \end{aligned} \quad (6.27)$$

and

$$\begin{aligned} \left\{ \sum_{v=0}^{\infty} \frac{a_v}{\rho^v} \right\}^4 \left[-\frac{1}{2i\rho} - \frac{iu_1(\pi)}{2\rho^2} + \frac{i[iu_2(\pi) - u_1'(0)]}{3i\rho^3} + \dots \right] + \left\{ \sum_{v=0}^{\infty} \frac{a_v}{\rho^v} \right\}^3 \left[-\frac{1}{2i\rho} - \frac{iu_1\left(\frac{\pi}{2}\right)}{2\rho^2} + \frac{i[iu_2\left(\frac{\pi}{2}\right) - u_1'(0)]}{3i\rho^3} + \dots \right] + \\ + \left\{ \sum_{v=0}^{\infty} \frac{a_v}{\rho^v} \right\} \left[\frac{1}{2i\rho} - \frac{iu_1\left(\frac{\pi}{2}\right)}{2\rho^2} + \frac{i[iu_2\left(\frac{\pi}{2}\right) - u_1'(0)]}{3i\rho^3} + \dots \right] + \left[\frac{1}{2i\rho} - \frac{iu_1(\pi)}{2\rho^2} + \frac{i[iu_2(\pi) - u_1'(0)]}{3i\rho^3} + \dots \right] = 0 \end{aligned} \quad (6.28)$$

Equating the coefficients of ρ^{-k} ($k=1,2,3,\dots$) to zero, we have

$$-a_0^4 - a_0^3 + a_0 + 1 = 0 \quad (6.29)$$

Solving the Equation (6.29), we have

$$a_0^{(1)} = 1, \quad a_0^{(2)} = -1, \quad a_0^{(3)} = \frac{-1+i\sqrt{3}}{2}, \quad a_0^{(4)} = \frac{-1-i\sqrt{3}}{2} \quad (6.30)$$

Upon using the result in [9], we have for the eigenvalues of the problem (6.24)-(6.25), the following asymptotic formula

$$\rho_{k,s} - 4k - \frac{2}{i\pi} \ln a_0^{(s)} + \frac{a_1^{(s)}}{2a_0^{(s)}k\pi} - \frac{a_1^{(s)} \ln a_0^{(s)}}{4ia_0^{(s)}\pi k^2} + \dots \quad (6.31)$$

where $s=1,2,3,4$, and

$$a_1^{(1)} = \frac{1}{6}u_1(\pi), \quad a_1^{(2)} = -\frac{1}{2}u_1(\pi), \quad a_1^{(3)} = \frac{u_1(\pi)}{4(a_0^{(3)})^3 + 3(a_0^{(3)})^2 - 1}, \quad a_1^{(4)} = \frac{u_1(\pi)}{4(a_0^{(4)})^3 + 3(a_0^{(4)})^2 - 1}, \quad (6.32)$$

Using the formula (6.31), we have

$$\rho_{k,s}^2 = \sum_{v=0}^{\infty} \frac{Q_v^{(s)}}{k^{v-1}}, \quad (6.33)$$

where

$$Q_0^{(s)} = 16, \quad Q_1^{(s)} = -16i \ln a_0^{(s)}, \quad Q_2^{(s)} = -\frac{4}{\pi^2} (\ln a_0^{(s)})^2 - \frac{4a_1^{(s)}}{a_0^{(s)}\pi}, \quad Q_3^{(s)} = \frac{-2i}{\pi^2} (\pi-1)a_1^{(s)} \ln a_0^{(s)}, \dots \quad (6.34)$$

According to (4.1), we see that the Green's function $G(x, \xi, \rho)$ of the problem (6.24)-(6.25) has the following asymptotic formulae:

$$G(x, \xi, \rho) = \begin{cases} e^{i\rho(x-\xi)} \sum_{v=0}^{\infty} \frac{\varphi_{v,1}^{(1)}}{\rho^v}; & \xi < x < \frac{\pi}{2}; \quad \rho \in \tilde{S}_0^\sigma \\ e^{-i\rho(x-\xi)} \sum_{v=0}^{\infty} \frac{\varphi_{v,1}^{(2)}}{\rho^v}; & x < \xi < \frac{\pi}{2}; \quad \rho \in \tilde{S}_0^\sigma \\ e^{i\rho(x-\xi)} \sum_{v=0}^{\infty} \frac{\varphi_{v,2}^{(1)}}{\rho^v}; & \frac{\pi}{2} < \xi < x; \quad \rho \in \tilde{S}_0^\sigma \\ e^{-i\rho(x-\xi)} \sum_{v=0}^{\infty} \frac{\varphi_{v,2}^{(2)}}{\rho^v}; & \frac{\pi}{2} < x < \xi; \quad \rho \in \tilde{S}_0^\sigma, \end{cases} \quad (6.35)$$

where

$$\begin{aligned} \varphi_{0,1}^{(1)} &= \varphi_{0,1}^{(2)} = \varphi_{0,2}^{(1)} = \varphi_{0,2}^{(2)} = 0 & \varphi_{1,1}^{(1)} &= \varphi_{1,1}^{(2)} = \varphi_{1,2}^{(1)} = \varphi_{1,2}^{(2)} = \frac{1}{2i} \\ \varphi_{2,1}^{(1)} &= \varphi_{2,2}^{(1)} = -\varphi_{2,1}^{(2)} = -\varphi_{2,2}^{(2)} = \frac{i}{2} (u_1(x) - u_1(\xi)) \\ \varphi_{3,1}^{(1)} &= \varphi_{3,1}^{(2)} = \varphi_{3,2}^{(1)} = \varphi_{3,2}^{(2)} = \frac{i}{2} [u_2(x) + u_1(\xi)u_1(x) + u_2(\xi) + iu_1'(0)] \end{aligned} \quad (6.36)$$

Upon using the asymptotic formulae (6.35) for $G(x, \xi, \rho)$ in \tilde{S}_0^σ and the definition of the function $H_{k,s}(x, \xi)$, we have

$$H_{k,s}(x, \xi) = \begin{cases} e^{i\rho_{k,s}(x-\xi)} \sum_{v=0}^{\infty} \frac{\phi_{v,1}^{(3)}}{\rho_{k,s}^v}; & \xi < x < \frac{\pi}{2}; \quad \rho_{k,s} \in \tilde{S}_0^\sigma, \\ e^{-i\rho_{k,s}(x-\xi)} \sum_{v=0}^{\infty} \frac{\phi_{v,1}^{(4)}}{\rho_{k,s}^v}; & \xi < x < \frac{\pi}{2}; \quad \rho_{k,s} \in \tilde{S}_0^\sigma, \\ e^{i\rho_{k,s}(x-\xi)} \sum_{v=0}^{\infty} \frac{\phi_{v,2}^{(3)}}{\rho_{k,s}^v}; & \frac{\pi}{2} < \xi < x; \quad \rho_{k,s} \in \tilde{S}_0^\sigma, \\ e^{-i\rho_{k,s}(x-\xi)} \sum_{v=0}^{\infty} \frac{\phi_{v,2}^{(4)}}{\rho_{k,s}^v}; & \frac{\pi}{2} < x < \xi; \quad \rho_{k,s} \in \tilde{S}_0^\sigma, \end{cases} \quad (6.37)$$

where

$$\phi_{0,1}^{(3)} = \phi_{0,1}^{(4)} = \phi_{0,2}^{(3)} = \phi_{0,2}^{(4)} = 0; \quad \phi_{1,1}^{(3)} = \phi_{1,1}^{(4)} = \phi_{1,2}^{(3)} = \phi_{1,2}^{(4)} = -\frac{1}{2\pi};$$

$$\begin{aligned}
\phi_{2,1}^{(3)} &= \phi_{2,2}^{(3)} = -\phi_{2,1}^{(4)} = -\phi_{2,2}^{(4)} = \frac{1}{2\pi} (u_1(\xi) - u_1(x)) \\
\phi_{3,1}^{(3)} &= \phi_{3,2}^{(3)} = -\phi_{3,1}^{(4)} = -\phi_{3,2}^{(4)} = \frac{1}{2\pi} [u_2(x) + u_2(\xi) - u_1(x)u_1(\xi) + \\
&\quad + iu'_1(0) + \frac{i}{\pi} (u_1(x) + u_1(\pi) - u_1(\xi) - 2u'(0) - \frac{1}{\pi^2})], \dots
\end{aligned} \tag{6.38}$$

From formulae (6.33) and (6.37), we get

$$\rho_{k,s}^2 H_{k,s}(x, \xi) = \begin{cases} e^{-2i(x-\xi)(2k-\frac{1}{\pi} \ln a_0^{(s)})} \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,1}^{(3)}}{k^{n-2}} & \xi < x < \frac{\pi}{2} \\ e^{2i(x-\xi)(2k-\frac{1}{\pi} \ln a_0^{(s)})} \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,1}^{(4)}}{k^{n-2}} & x < \xi < \frac{\pi}{2} \\ e^{-2i(x-\xi)(2k-\frac{1}{\pi} \ln a_0^{(s)})} \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,2}^{(3)}}{k^{n-2}} & \frac{\pi}{2} < \xi < x \\ e^{2i(x-\xi)(2k-\frac{1}{\pi} \ln a_0^{(s)})} \sum_{n=0}^{\infty} \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,2}^{(4)}}{k^{n-2}} & \frac{\pi}{2} < x < \xi \end{cases} \tag{6.39}$$

where,...

$$\begin{aligned}
\psi_0^{(s)}(x, \xi) &= 1, \quad \psi_1^{(s)}(x, \xi) = \frac{ia_1^{(s)}(x - \xi)}{2a_0^{(s)}\pi}; \quad \psi_2^{(s)} = -\frac{1}{8}(x - \xi)^2 \left(\frac{a_1^{(s)}}{a_0^{(s)}\pi}\right)^2, \\
\psi_3^{(s)}(x, \xi) &= \frac{-(a_1^{(s)} \ln a_0^{(s)})(x - \xi)}{\pi a_0^{(s)}} + \frac{i\left(\frac{a_1^{(s)}(x - \xi)}{2a_0^{(s)}\pi}\right)^3}{6}, \dots
\end{aligned} \tag{6.40}$$

Upon using formulae (6.36), (6.38), (6.39) and (6.40), we have the following theorem :

THEOREM 6.2

1) If $\xi < x < \frac{\pi}{2}$, then the regularized sum of the first order for eigenfunctions of the problem

(6.24)-(6.25) are given by the followig formulae

$$\begin{aligned}
& \sum_{k=1}^{\infty} \sum_{s=1}^4 \left[\lambda_{k,s} H_{k,s}(x, \xi) - e^{i(x-\xi)(-4k-\frac{2}{i\pi} \ln a_0^{(s)})} \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,1}^{(3)}}{k^{n-2}} \right] \\
&= \frac{i}{2} [u_2(x) + u_1(\xi)u_1(x) + u_2(\xi) + iu'_1(0)] - \sum_{s=1}^4 \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p e^{i(x-\xi)(-\frac{2}{i\pi} \ln a_0^{(s)})} \\
&\quad \cdot F(e^{-4i(x-\xi)}, n-2) \psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,1}^{(3)}
\end{aligned} \tag{6.41}$$

(2) If $x < \xi < \frac{\pi}{2}$, then the regularized sum of the first order for eigenfunctions of the problem

(6.24)-(6.25) are given by the followig formulae

$$\begin{aligned}
& \sum_{k=1}^{\infty} \sum_{s=1}^4 \left[\lambda_{k,s} H_{k,s}(x, \xi) - e^{-i(x-\xi)(-4k-\frac{2}{i\pi} \ln a_0^{(s)})} \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,1}^{(4)}}{k^{n-2}} \right] \\
&= \frac{i}{2} [u_2(x) + u_1(\xi)u_1(x) + u_2(\xi) + iu'_1(0)] - \sum_{s=1}^4 \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p e^{-i(x-\xi)(-\frac{2}{i\pi} \ln a_0^{(s)})}
\end{aligned}$$

$$F(e^{4(x-\xi)}, n-2) \psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,1}^{(4)} \quad (6.42)$$

(3) If $\frac{\pi}{2} < \xi < x$, then the regularized sum of the first order for eigenfunctions of the problem

(6.24)-(6.25) are given by the followig formulae.

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum_{s=1}^4 \left[\lambda_{k,s} H_{k,s}(x, \xi) - e^{l(x-\xi)(-4k - \frac{2}{i\pi} \ln a_0^{(s)})} \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,2}^{(3)}}{k^{n-2}} \right] \\ &= \frac{i}{2} [u_2(x) + u_2(\xi) + u_1(x)u_1(\xi) + iu_1'(0)] - \sum_{s=1}^4 \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p e^{l(x-\xi)(-\frac{2}{i\pi} \ln a_0^{(s)})} \\ & \cdot F(e^{4(x-\xi)}, n-2) \psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,2}^{(3)} \end{aligned} \quad (6.43)$$

(4) If $\frac{\pi}{2} < x < \xi$, then the regularized sum of the first order for eigenfunctions of the problem

(6.24)-(6.25) are given by the followig formulae

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum_{s=1}^4 \left[\lambda_{k,s} H_{k,s}(x, \xi) - e^{-l(x-\xi)(-4k - \frac{2}{i\pi} \ln a_0^{(s)})} \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p \frac{\psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,2}^{(4)}}{k^{n-2}} \right] \\ &= \frac{i}{2} [u_2(x) + u_2(\xi) + u_1(x)u_1(\xi) + iu_1'(0)] - \sum_{s=1}^4 \sum_{n=0}^3 \sum_{p=0}^n \sum_{\ell=0}^p e^{l(x-\xi)(-\frac{2}{i\pi} \ln a_0^{(s)})} \\ & F(e^{4(x-\xi)}, n-2) \psi_{n-p}^{(s)} Q_{\ell}^{(s)} \phi_{p-\ell,2}^{(4)} \end{aligned} \quad (6.44)$$

REFERENCES

- [1] **SADOVNICHII V. A.**, Theory of Operators (Moscow University 1979-Moscow SSSR), 192-226. [Russian]. Translated from the second Russian edition by Roger cooke. Contemporary soviet Mathematics, consultants Bureau, New York, 1991.
- [2] **GELFAND I. M., LEVITAN M. B.**, " On a simple identity for eigenvalues of a differential operator of the second order" *Dokl Akad Nauk SSSR* **88** (1953) 593-596 [Russian]
- [3] **CHARLES J. A., HALBERG JR. and KRAMER V. A.** "Ageneralization of the one trace concept " *Duke Math. J.* **27** (1960) 607-617.
- [4] **LIDSKY V. B., SADOVNICHII V.A.** " Regularized sums of the roots of the class of integral functions " *Dokl. Akad. Nauk SSSR* **176** No 2. (1967) 259-262. [Russian]
- [5] **LIDSKY V. B., SADOVNICHII V. A.** " Regularized sums of the roots of the one class of integral functions." *Functional analysis* V. 1 No 2 (1967) 52-59 [Russian]
- [6] **LIDSKY V. B., SADOVNICHII V. A.** " Förmulae of traces in case of Orr-Sommerfield equation. *Esv. Akad. Nauk SSSR. ser. Math.* V. **32** No 3(1968) 633-648. [Russian].
- [7] **SADOVNICHII V. A., LYUBISHKIN V. A. and BELABBASY U.** " On regularized sums of the roots of integral function of some class ". *Dokel Akad. Nauk SSSR.* V. **254**, No 6 (1980). 1346-1348 [Russian]

- [8] **SADOVNICHII V. A., LYUBISHKIN V. A.** "Regularized sums of roots of class of integral functions of an exponential type". *Dokl Akad. Nauk SSSR*. V. **256**, No 4 (1981) 794-798 [Russian]
- [9] **SALEH S. A.** " Spectral properties of some class of non-self adjoint problem with multi-point conditions" PH. D. Moscow university (1984). [Russian].
- [10] **SADOVNICHII V. A.** " On the trace with weight and asymptotic spectral functions " *Differential equations* V. **10** No10 (1974) 1808-1818 [Russian]
- [11] **MARCHENKO V. A.** " Sturm-Liouville operators and Applications"
Translated from the Russian by A. Lach. Birkhauser Verlag. Basel. Boston-Stuttgart(1986)
51
- [12] **NAIMARK M. A.** " Linear differential operators " Nauk. Moscow (1969) [Russian].
Translated by E. R. Dawson, Queen's college, Dundee. English translation edited by
W. N. Everitt Professor of Mathematics, Queen's College, Dundee.
- [13] **ERDELYI A., MAGNUS W., OBERHETTINGER F., and TRICOMI F. G.** "Higher transcendental functions "V. 1 ..New york-Toronto-London McGRAW-HILL BOOK COMPANY, INC 1953 26-27.
- [14] **KLUSCH D.** " On the Taylor expansion of the Lerch-Zeta- function " *Journal of Mathematical analysis and applications* **170** , (1992) 513-523

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru