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NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION OF DELAY DIFFERENTIAL EQUATION WITH A PIECEWISE CONSTANT ARGUMENT

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ABSTRACT: The characteristic equation for an equation with continuous and piecewise constant argument in the form

$$\dot{x}(t) + p\,x(t-\tau) + q\,x([t-k]) = 0 \text{ where } p,q \in \Re \text{ },\tau \in \Re^{+} \text{ and } k \in \aleph.$$

is presented , which when $q=0$ reduces to

$$f(\lambda) = \lambda + e^{-\lambda\tau} = 0$$

and when $p=0$ reduces to

$$\lambda - 1 + q\,\lambda^{-k} = 0.$$

Also, the necessary and sufficient conditions for oscillation are obtained.

KEY WORDS: Oscillations , Delay differential equations.

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1. INTRODUCTION

The study of equations with piecewise constant argument was originated by the work of Wiener and his collaborators. See [1,2,3,4,5 and 6] and the references cited therein . In addition to its own interest this area has stimulated much activity in the study of delay difference equations.

As usual , a solution x(t) is called oscillatory if it has arbitrarily large zeros .Otherwise , the solution is called nonoscillatory.An equation is called oscillatory if all its solutions are oscillatory.

Let [.] denote the greatest-integer function,$\aleph$ the set of non-negative integers and $\Re$ the set of real numbers.

Consider

$$\dot{x}(t) + p\,x(t-\tau) + q\,x([t-k]) = 0 \tag{1.1}$$

where p,q $\in \Re$,$\tau \in \Re^{+}$ and $k \in \aleph$.

By a solution of Eqn.(1.1), we mean a function x which is defined on the set $\{-k, ..., -1, 0\} \cup [-\tau,\infty)$ and satisfies the following properties:

(a) x is continuous on $[-\tau,\infty)$.

(b) the derivative $\dot{x}$ exists at each point $t \in (0,\infty)$ with the possible exception of the points $t \in \aleph$, where one side derivatives exist.

(c) Eqn.(1.1) is satisfied on each interval $[n,n+1)$ for $n \in \aleph$.

Let $\phi \in C([-\tau,0],\Re)$ and $a_{-k},...,a_{-1},a_0$ be given real numbers such that

$$a_{-j} = \phi(-j) \text{ for } j \le \tau,\ j=0,1,2,...,k, \tag{1.2}$$

then one can show that Eqn. (1.1) has a unique solution satisfying the initial conditions

$$x(t) = \phi(t) \qquad -\tau \le t \le 0 \tag{1.3a}$$

$$x(-j) = a_{-j} \qquad j=0,1,...,k. \tag{1.3b}$$

When q =0, Eqn. (1.1) reduces to

$$\dot{u}(t) + p\,u(t-\tau) = 0 \tag{1.4}$$

which is oscillatory if and only if its characteristic equation

$$f(\lambda) = \lambda + e^{-\lambda\tau} = 0 \tag{1.5}$$

has no real roots, or equivalently, to

$$p\tau > \frac{1}{e}. \tag{1.6}$$

On the other hand, when p=0, Eqn.(1.1) reduces to

$$\dot{v}(t) + q\,v([t-k]) = 0 \tag{1.7}$$

which is oscillatory if and only if the following equation

$$\lambda - 1 + q\,\lambda^{-k} = 0 \tag{1.8}$$

has no positive real roots, or equivalently,

$$q > \frac{k^k}{(k+1)^{k+1}} \qquad ,\ k \ge 1 \tag{1.9a}$$

$$q \ge 1 \qquad ,\ k = 0 \tag{1.9b}$$

An open question arises (see [4] ,p. 223) for obtaining a characteristic equation for equation (1.1) which reduces to Eqn.(1.5) when q=0 and reduces to Eqn. (1.8) when p=0 and also obtaining the necessary and sufficient conditions for oscillation of all solutions of

$$\dot{x}(t) + p\,x(t-1) + q\,x([t-1]) = 0 \tag{1.10}$$

2. THE MAIN RESULTS

In the following , a characteristic equation associated with equation (1.1) will be presented in Theorem **2.1** . Also the necessary and sufficient conditions for oscillation are obtained through Theorems **2.2 and 2.3** .

THEOREM 2.1. The characteristic equation associated with equation (1.1) is

$$f(\lambda) = \lambda - 1 + \frac{p\lambda^{-\tau}}{\ln\lambda}(\lambda - 1) + q\lambda^{-k} = 0 \tag{2.1}$$

which reduces to Eqn. (1.5) when q=0 and reduces to Eqn. (1.8) when p=0.

PROOF: Consider Eqn.(1.1) and assume that the initial conditions (1.3a) and (!.3b) are satisfied .For $t \in [n,n+1)$, we have $[t-k] = n-k$ and one can write

$$\dot{x}(t) + p\,x(t-\tau) + q\,a_{n-k} = 0 \qquad , \ t \in [n,n+1) \tag{2.2a}$$

$$x(n) = a_n \qquad , \ n \in \aleph \tag{2.2b}$$

Integrating (2.2a) from n to t , we get

$$x(t) - a_n + p\int_n^t x(s-\tau)\,ds + q\,a_{n-k}(t-n) = 0\,. \tag{2.3}$$

By using the continuity of x(t) as $t \to n+1$, we find

$$a_{n+1} - a_n + p\int_n^{n+1} x(s-\tau)\,ds + q\,a_{n-k} = 0\,. \tag{2.4}$$

Assume that $x(t) = e^{\lambda t}$, $t \in [n,n+1)$, then from (2.4) ,we get

$$f(\lambda) = e^{\lambda} - 1 + \frac{p e^{-\lambda\tau}}{\lambda}(\lambda - 1) + q e^{-\lambda k} = 0 \tag{2.5}$$

Putting $e^{\lambda} = \gamma$ in Eqn.(2.5),then

$$F(\gamma) = \gamma - 1 + \frac{p\gamma^{-\tau}}{\ln\gamma}(\gamma - 1) + q\gamma^{-k} = 0\,. \tag{2.6}$$

and consequently Eqn.(2.6) has no positive real roots if and only if Eqn.(2.5) has no real roots.Assume that Eqn.(2.5) has no real roots , then $\lambda \neq 0$, and consequently $\gamma \neq 1$. If p= 0, then Eqn. (2.6) reduces to Eqn.(1.8),also if q=0 ,then Eqn.(2.6) reduces to Eqn.(1.5) .

THEOREM 2.2. Equation (1.1) is oscillatory if and only if its characteristic equation (2.6) has no positive real roots.

PROOF: Assume that the characteristic equation (2.6) has a positive real root γ_0, then γ_0^t is a solution of Eqn.(2.4) which is a nonoscillatory solution and consequently Eqn.(1.1) is not oscillatory .On the other hand , assume that $x(t) > 0\ \forall\ t \in [n,n+1)$ for sufficiently large n and $F(\gamma)$ has no positive real roots .As $F(\infty) = \infty$, it follows that $F(\gamma) > 0\ \ \forall \gamma \in (0,\infty)$.For seeking the contradiction , choose :

(i) $p \le 0$ and $q \le 0$ then $F(\gamma) < 0\ \ \forall\ \gamma \in (0,1)$,

(ii) $p \ge 0$, $q < 0$ with $p < |q|$ and $\tau \le k$ then $F(\gamma) < 0\ \ \forall\ \gamma \in (0,1)$,

(iii) $p < 0$, $q \ge 0$ with $q < |p|\,(1 - 1/e)$ and $\tau = k$ then $F(1/e) < 0$,

(iv) $p \ge 0$, $q \ge 0$ with $p + q \le 1/8e^k$ and $\tau \le k$ then $F(1/e) < 0$,

which is a contradiction.

THEOREM 2.3. If $p,q \in \Re^+$, then all solutions of equation (1.1) are oscillatory if and only if

$$pe\tau + q\frac{(k+1)^{k+1}}{k^k} > 1 \quad , \ k \ge 1 \tag{2.7}$$

PROOF: Assume that Eqn.(1.1) has a nonoscillatory solution ,then the characteristic Eqn.(2.6) has a positive real root $\gamma_0 \in (0,1)$. Otherwise $F(\gamma_0) > 0\ \forall \gamma_0 \in [1,\infty)$ and therefore , we have

$$F(\gamma_0)=\gamma_0-1+\frac{p\gamma_0^{\ -\tau}}{\ln\gamma_0}(\gamma_0-1)+q\gamma_0^{\ -k}=0\,,\ \gamma_0\in(0,1)$$

and then

$$0=(\gamma_0-1)\{1+\frac{p\gamma_0^{\ -\tau}}{\ln\gamma_0}+q\gamma_0^{\ -k}/(\gamma_0-1)\}$$

$$0=1+\frac{p\gamma_0^{\ -\tau}}{\ln\gamma_0}+q\gamma_0^{\ -k}/(\gamma_0-1)$$

$$\le 1-pe\tau\ -q\frac{(k+1)^{k+1}}{k^k}$$

which is a contradiction .On the other hand ,assume that

$$pe\tau\ +q\frac{(k+1)^{k+1}}{k^k}\le 1\ ,k\ge 1\ .$$

Now , we study the following cases:

(1) q=0 ,p>0.

Since $F(\gamma)>0,\forall\ \gamma\in(1,\infty)$ and $F(e^{-\frac{1}{\tau}})\le 0$,then there exists $\gamma_1\in\Re^+$ such that $F(\gamma_1)=0$. i.e. the characteristic equation has a positive real root and consequently equation (1.1) is not oscillatory.

(2) p=0, q>0.

In this case , $F(\gamma)>0,\forall\ \gamma\in(1,\infty)$and $F(\frac{k}{k+1})\le 0$.Therefore , the characteristic equation has a positive real root and then equation(1.1) has a nonoscillatory solution.

(3) p>0 ,q>0.

Since $pe\tau\ +q\frac{(k+1)^{k+1}}{k^k}\le 1$,

then,

$$q\frac{(k+1)^{k+1}}{k^k}+\frac{p\frac{(k+1)^{\tau+1}}{k^\tau}}{\ln(\frac{k}{k+1})}<pe\tau\ +q\frac{(k+1)^{k+1}}{k^k}\le 1,\ k\ge 1\ . \tag{2.8}$$

It is clear that the characteristic equation has no real roots in $(1,\infty)$ and $F(\gamma)>0$,but

$$F(\frac{k}{k+1})=\frac{k}{k+1}-1+\frac{p\frac{(k+1)^{\tau}}{k^\tau}}{\ln(\frac{k}{k+1})}+q\frac{(k+1)^k}{k^k}$$

$$=-\frac{1}{k+1}+\frac{q}{(k+1)}\frac{(k+1)^{k+1}}{k^k}+\frac{p}{k+1}\frac{\frac{(k+1)^{\tau+1}}{k^\tau}}{\ln(\frac{k}{k+1})}$$ From (2.8),it follows that $F(\frac{k}{k+1})\le 0$ and consequently equation (1.1) has a nonoscillatory solution.

REMARK. If $\tau=k=1$ and $p,q\in\Re^+$,then $pe+4q>1$ is a necessary and sufficient condition for oscillation of

$$\dot{x}(t)+p\,x(t-1)+q\,x([t-1])=0\,.$$

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