

ASYMPTOTIC THEORY FOR A CRITICAL CASE FOR A GENERAL FOURTH-ORDER DIFFERENTIAL EQUATION

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(Received November 27, 1996 and in revised form November 11, 1997)

ABSTRACT. In this paper we identify a relation between the coefficients that represents a critical case for general fourth-order equations. We obtained the forms of solutions under this critical case

KEY WORDS AND PHRASES: Asymptotic, eigenvalues.

1991 AMS SUBJECT CLASSIFICATION CODES: 34E05.

1. INTRODUCTION

We consider the general fourth-order differential equation

$$(p_0 y'')'' + (p_1 y')' + \frac{1}{2} \sum_{j=0}^1 \left[\{q_{2-j} y^{(j+1)}\} + \{q_{2-j} y^{(j+1)}\}^{(j)} \right] + p_2 y = 0 \quad (1.1)$$

where x is the independent variable and the prime denotes d/dx . The functions $p_i(x)$ ($0 \leq i \leq 2$) and $q_i(x)$ ($i = 1, 2$) are defined on an interval $[a, \infty)$ and are not necessarily real-valued and are all nowhere zero in this interval. Our aim is to identify relations between the coefficients that represent a critical case for (1.1) and to obtain the asymptotic forms of our linearly independent solutions under this case. Al-Hammadi [1] considered (1.1) with the case where p_0 and p_2 are the dominate coefficients and we give a complete analysis for this case. Similar fourth-order equations to (1.1) have been considered previously by Walker [2, 3] and Al-Hammadi [4]. Eastham [5] considered a critical case for (1.1) with $p_1 = q_2 = 0$ and showed that this case represents a borderline between situations where all solutions have a certain exponential character as $x \rightarrow \infty$ and where only two solutions have this character.

The critical case for (1.1) that has been referred, is given by:

$$\frac{q'_i}{q_i} \sim \text{const. } \frac{p_2}{q_2} \quad (i = 1, 2), \quad \frac{\left(p_1 q_1^{-1/2} \right)'}{p_1 q_1^{-1/2}} \sim \text{const. } \frac{p_2}{q_2}. \quad (1.2)$$

We shall use the recent asymptotic theorem of Eastham [6, section 2] to obtain the solutions of (1.1) under the above case. The main theorem for (1.1) is given in section 4 with discussion in section 5.

2. A TRANSFORMATION OF THE DIFFERENTIAL EQUATION

We write (1.1) in the standard way [7] as a first order system

$$Y' = AY, \quad (2.1)$$

where the first component of Y is y and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2}q_1p_0^{-1} & p_0^{-1} & 0 \\ -\frac{1}{2}q_2 & -p_1 + \frac{1}{4}q_1^2p_0^{-1} & -\frac{1}{2}p_0^{-1}q_1 & 1 \\ -p_2 & -\frac{1}{2}q_2 & 0 & 0 \end{bmatrix}. \quad (2.2)$$

As in [4], we express A in its diagonal form

$$T^{-1}AT = \Lambda, \quad (2.3)$$

and we therefore require the eigenvalues λ_j and eigenvectors v_j ($1 \leq j \leq 4$) of A .

The characteristic equation of A is given by

$$p_0\lambda^4 + q_1\lambda^3 + p_1\lambda^2 + q_2\lambda + p_2 = 0. \quad (2.4)$$

An eigenvector v_j of A corresponding to λ_j is

$$v_j = \left(1, \lambda_j, p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j, -\frac{1}{2}q_2 - p_2\lambda_j^{-1} \right)^t \quad (2.5)$$

where the superscript t denotes the transpose. We assume at this stage that the λ_j are distinct, and we define the matrix T in (2.3) by

$$T = (v_1 \ v_2 \ v_3 \ v_4). \quad (2.6)$$

Now from (2.2) we note that EA coincides with its own transpose, where

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.7)$$

Hence, by [8, section 2(i)], the v_j have the orthogonality property

$$(Ev_k)^t v_j = 0 \quad (k \neq j). \quad (2.8)$$

We define the scalars m_j ($1 \leq j \leq 4$) by

$$m_j = (Ev_j)^t v_j, \quad (2.9)$$

and the row vectors

$$r_j = (Ev_j)^t. \quad (2.10)$$

Hence, by [8, section 2]

$$T^{-1} = \begin{bmatrix} m_1^{-1}r_1 \\ m_2^{-1}r_2 \\ m_3^{-1}r_3 \\ m_4^{-1}r_4 \end{bmatrix}, \quad (2.11)$$

and

$$m_j = 4p_0\lambda_j^3 + 3q_1\lambda_j^2 + 2p_2\lambda_j + q_2. \quad (2.12)$$

Now we define the matrix U by

$$U = (v_1 \ v_2 \ v_3 \ \epsilon_1 \ v_4) = TK, \quad (2.13)$$

where

$$\epsilon_1 = \frac{p_0 p_1}{q_1^2}, \quad (2.14)$$

the matrix K is given by

$$K = dg(1, 1, 1, \epsilon_1). \quad (2.15)$$

By (2.3) and (2.13), the transformation

$$Y = UZ \quad (2.16)$$

takes (2.1) into

$$Z' = (\Lambda - U^{-1}U')Z. \quad (2.17)$$

Now by (2.13),

$$U^{-1}U' = K^{-1}T^{-1}T'K + K^{-1}K', \quad (2.18)$$

where

$$K^{-1}K' = dg(0, 0, 0, \epsilon_1^{-1}\epsilon_1'), \quad (2.19)$$

and we use (2.15).

Now we write

$$U^{-1}U' = \phi_{ij} \quad (1 \leq i, j \leq 4), \quad (2.20)$$

and

$$T^{-1}T' = \psi_{ij} \quad (1 \leq i, j \leq 4), \quad (2.21)$$

then by (2.18) to (2.21), we have

$$\phi_{ij} = \psi_{ij}, \quad (1 \leq i, j \leq 3), \quad (2.22)$$

$$\phi_{44} = \psi_{44} + \epsilon_1^{-1}\epsilon_1', \quad (2.23)$$

$$\phi_{i4} = \psi_{i4}\epsilon_1 \quad (1 \leq i \leq 3), \quad (2.24)$$

$$\phi_{j4} = \epsilon_1^{-1}\psi_{j4} \quad (1 \leq j \leq 3). \quad (2.25)$$

Now to work out ϕ_{ij} ($1 \leq i, j \leq 4$), it suffices to deal with ψ_{ij} of the matrix $T^{-1}T'$. Thus by (2.6), (2.10), (2.11) and (2.12) we obtain

$$\psi_{ii} = \frac{1}{2} \frac{m'_i}{m_i} \quad (1 \leq i \leq 4) \quad (2.26)$$

and, for $i \neq j$, $1 \leq i, j \leq 4$

$$\psi_{ij} = m_i^{-1} \left\{ \lambda'_j \left(p_0 \lambda_i^2 + \frac{1}{2} q_1 \lambda_i \right) + \lambda_i \left(p_0 \lambda_j^2 + \frac{1}{2} q_1 \lambda_j \right)' - \frac{1}{2} q'_2 - (p_2 \lambda_j^{-1})' \right\}. \quad (2.27)$$

Now we need to work out (2.26) and (2.27) in some detail in terms of p_0, p_1, p_2, q_1 and q_2 and then (2.22)-(2.25) in order to determine the form of (2.17).

3. THE MATRICES L , $T^{-1}T$ AND $U^{-1}U$

In our analysis, we impose a basic condition on the coefficients, as follows:

(I) p_i ($0 \leq i \leq 2$) and q_i ($i = 1, 2$) are nowhere zero in some interval $[a, \infty)$, and

$$\frac{p_i}{q_{i+1}} = o\left(\frac{q_{i+1}}{p_{i+1}}\right) \quad (i = 0, 1) \quad (x \rightarrow \infty) \quad (3.1)$$

and

$$\frac{q_1}{p_1} = o\left(\frac{p_1}{q_2}\right). \quad (3.2)$$

If we write

$$\epsilon_1 = \frac{p_0 p_1}{q_1^2}, \quad \epsilon_2 = \frac{q_1 q_2}{p_1^2}, \quad \epsilon_3 = \frac{p_2 p_1}{q_2^2}, \quad (3.3)$$

then by (3.1) and (3.2) for $(1 \leq i \leq 3)$

$$\epsilon_i = o(1) \quad (x \rightarrow \infty). \quad (3.4)$$

Now as in [4], we can solve the characteristic equation (2.4) asymptotically as $x \rightarrow \infty$. Using (3.1), (3.2) and (3.3) we obtain the distinct eigenvalues λ_j as

$$\lambda_1 = -\frac{p_2}{q_2}(1 + \delta_1), \quad (3.5)$$

$$\lambda_2 = -\frac{q_2}{p_1}(1 + \delta_2), \quad (3.6)$$

$$\lambda_3 = -\frac{p_1}{q_1}(1 + \delta_3), \quad (3.7)$$

and

$$\lambda_4 = -\frac{q_1}{p_0}(1 + \delta_4), \quad (3.8)$$

where

$$\delta_1 = 0(\epsilon_3), \quad \delta_2 = 0(\epsilon_2) + 0(\epsilon_3), \quad \delta_3 = 0(\epsilon_1) + 0(\epsilon_2), \quad \delta_4 = (\epsilon_1). \quad (3.9)$$

Now by (3.1) and (3.2), the ordering of λ_j is such that

$$\lambda_j = o(\lambda_{j+1}) \quad (x \rightarrow \infty, 1 \leq j \leq 3). \quad (3.10)$$

Now we work out $m_j (1 \leq j \leq 4)$ asymptotically as $x \rightarrow \infty$, hence by (3.3)-(3.9), (2.12) gives for $(1 \leq j \leq 4)$

$$m_1 = q_2\{1 + 0(\epsilon_3)\}, \quad (3.11)$$

$$m_2 = -q_2\{1 + 0(\epsilon_2) + 0(\epsilon_3)\}, \quad (3.12)$$

$$m_3 = \frac{p_1^2}{q_1}\{1 + 0(\epsilon_1) + 0(\epsilon_2)\}, \quad (3.13)$$

and

$$m_4 = -\frac{q_1^3}{p_0^2}\{1 + 0(\epsilon_1)\}. \quad (3.14)$$

Also on substituting $\lambda_j (j = 1, 2, 3, 4)$ into (2.12) and using (3.5)-(3.8) respectively and differentiating, we obtain

$$m'_1 = q_2'\{1 + 0(\epsilon_3)\} + q_2\{0(\epsilon'_3) + 0(\epsilon_3\delta'_1) + 0(\epsilon'_2\epsilon_3^2) + 0(\epsilon'_1\epsilon_2^2\epsilon_3^3)\}, \quad (3.15)$$

$$m'_2 = -q'_2\{1 + 0(\epsilon_2) + 0(\epsilon_3)\} + q_2\{0(\delta'_2) + 0(\epsilon'_2) + 0(\epsilon'_1\epsilon_2^2)\}, \quad (3.16)$$

$$m'_3 = \left(\frac{p'_1}{q_1}\right)' \{1 + 0(\epsilon_1) + 0(\epsilon_2)\} + \frac{p'_1}{q_1}\{0(\delta'_3) + 0(\epsilon'_2) + 0(\epsilon'_1)\}, \quad (3.17)$$

and

$$m'_4 = -\left(\frac{q'_1}{p'_0}\right)' \{1 + 0(\epsilon_2)\} + \frac{q'_1}{p'_0}\{0(\epsilon'_2\epsilon_1^2) + 0(\epsilon'_1)\}. \quad (3.18)$$

At this stage we also require the following conditions

$$(II) \quad \frac{p'_0}{p_0} \epsilon_i, \quad \frac{p'_1}{p_1} \epsilon_i, \quad \frac{q'_1}{q_1} \epsilon_i, \quad \frac{q'_2}{q_2} \epsilon_i, \quad \frac{p'_2}{p_2} \epsilon_2, \quad \frac{p'_2}{p_2} \epsilon_3 \quad \text{are all} \\ L(a, \infty) \quad (1 \leq i \leq 3). \quad (3.19)$$

Further, differentiating (3.3) for ϵ_i ($1 \leq i \leq 3$), we obtain

$$\epsilon'_1 = 0\left(\frac{p'_0}{p_0} \epsilon_1\right) + 0\left(\frac{p'_1}{p_1} \epsilon_1\right) + 0\left(\frac{q'_1}{q_1} \epsilon_1\right), \quad (3.20)$$

$$\epsilon'_2 = 0\left(\frac{q'_1}{q_1} \epsilon_2\right) + 0\left(\frac{q'_2}{q_2} \epsilon_2\right) + 0\left(\frac{p'_1}{p_1} \epsilon_2\right), \quad (3.21)$$

and

$$\epsilon'_3 = 0\left(\frac{p'_2}{p_2} \epsilon_3\right) + 0\left(\frac{p'_1}{p_1} \epsilon_3\right) + 0\left(\frac{q'_2}{q_2} \epsilon_3\right). \quad (3.22)$$

For reference shortly, we note on substituting (3.5)-(3.8) into (2.4) and differentiating, we obtain

$$\delta'_1 = 0(\epsilon'_3) + 0(\epsilon'_2\epsilon_3^2) + 0(\epsilon'_1\epsilon_3^3\epsilon_2^2), \quad (3.23)$$

$$\delta'_2 = 0(\epsilon'_2) + 0(\epsilon'_3) + 0(\epsilon'_1\epsilon_3^2), \quad (3.24)$$

$$\delta'_3 = 0(\epsilon'_1) + 0(\epsilon'_2) + 0(\epsilon'_3\epsilon_2^2), \quad (3.25)$$

and

$$\delta'_4 = 0(\epsilon'_1) + 0(\epsilon'_2\epsilon_1^2) + 0(\epsilon'_3\epsilon_1^3\epsilon_2^2). \quad (3.26)$$

Hence by (3.19) and (3.20)-(3.26)

$$\epsilon'_j \quad \text{and} \quad \delta'_j \quad \text{are} \quad L(a, \infty). \quad (3.27)$$

For the diagonal elements ψ_{ii} ($1 \leq i \leq 4$) in (2.26) we can now substitute the estimates (3.11)-(3.18) into (2.26). We obtain

$$\psi_{11} = \frac{1}{2} \frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2} \epsilon_3\right) + 0(\epsilon'_3) + 0(\epsilon_3\delta'_1) + 0(\epsilon'_2\epsilon_3^2) + 0(\epsilon'_1\epsilon_2^2\epsilon_3^3), \quad (3.28)$$

$$\psi_{22} = \frac{1}{2} \frac{q'_2}{q_2} + 0\left(\frac{q'_2}{q_2} \epsilon_2\right) + 0\left(\frac{q'_2}{q_2} \epsilon_3\right) + 0(\delta'_2) + 0(\epsilon'_2) + 0(\epsilon'_1\epsilon_2^2), \quad (3.29)$$

$$\begin{aligned}\psi_{33} = & \frac{1}{2} \left[2 \frac{p'_1}{p_1} - \frac{q'_1}{q_1} \right] + 0 \left(\frac{p'_1}{p_1} \epsilon_1 \right) + 0 \left(\frac{p'_1}{p_1} \epsilon_2 \right) \\ & + 0 \left(\frac{q'_1}{q_1} \epsilon_1 \right) + 0 \left(\frac{q'_1}{q_1} \epsilon_2 \right) + 0(\delta'_3) + 0(\epsilon'_2) + 0(\epsilon'_1),\end{aligned}\quad (3.30)$$

$$\psi_{44} = \frac{1}{2} \left[3 \frac{q'_1}{q_1} - 2 \frac{p'_0}{p_0} \right] + 0 \left(\frac{q'_1}{q_1} \epsilon_1 \right) + 0 \left(\frac{p'_0}{p_0} \epsilon_1 \right) + 0(\delta'_4) + 0(\epsilon'_2 \epsilon_1^2) + 0(\epsilon'_1). \quad (3.31)$$

Now for the non-diagonal elements ψ_{ij} ($i \neq j, 1 \leq i, j \leq 4$), we consider (2.27). Hence (2.27) gives for $i = 1$ and $j = 2$

$$\psi_{12} = m_1^{-1} \left\{ \lambda'_2 \left(p_0 \lambda_1^2 + \frac{1}{2} q_1 \lambda_1 \right) + \lambda_1 \left(p_0 \lambda_2^2 + \frac{1}{2} q_1 \lambda_2 \right)' - \frac{1}{2} q'_2 - (p_2 \lambda_2^{-1})' \right\}. \quad (3.32)$$

Now by (3.5), (3.6), (3.3) and (3.11) we have

$$m_1^{-1} \lambda'_2 \left(p_0 \lambda_1^2 + \frac{1}{2} q_1 \lambda_1 \right) = \frac{1}{2} \left[\frac{q'_2}{q_2} - \frac{p'_1}{p_1} \right] \epsilon_2 \epsilon_3 \{1 + 0(\epsilon_3)\} + 0(\epsilon_2 \epsilon_3 \delta'_2), \quad (3.33)$$

$$\begin{aligned}m_1^{-1} \lambda_1 \left(p_0 \lambda_2^2 + \frac{1}{2} q_1 \lambda_2 \right)' = & 0(\epsilon_2 \epsilon_3 \delta'_2) + 0(\epsilon_2^2 \epsilon_1 \epsilon_3) \left[\frac{p'_0}{p_0} + 2 \frac{q'_2}{q_2} - 2 \frac{p'_1}{p_1} \right] \\ & + 0(\epsilon_2 \epsilon_3) \left[\frac{q'_1}{q_1} + \frac{q'_2}{q_2} - \frac{p'_1}{p_1} \right],\end{aligned}\quad (3.34)$$

$$-\frac{1}{2} q'_2 m_1^{-1} = -\frac{1}{2} \frac{q'_2}{q_2} + 0 \left(\frac{q'_2}{q_2} \epsilon_3 \right), \quad (3.35)$$

and

$$m_1^{-1} (p_2 \lambda_2^{-1})' = 0 \left(\frac{p'_2}{p_2} \epsilon_3 \right) + 0 \left(\frac{p'_1}{p_1} \epsilon_3 \right) + 0 \left(\frac{q'_2}{q_2} \epsilon_3 \right) + 0(\epsilon_3 \delta'_2). \quad (3.36)$$

Hence by (3.33)-(3.36), (3.32) gives

$$\begin{aligned}\psi_{12} = & -\frac{1}{2} \frac{q'_2}{q_2} + 0 \left(\frac{q'_2}{q_2} \epsilon_3 \right) + 0 \left(\frac{p'_1}{p_1} \epsilon_3 \right) + 0 \left(\frac{p'_2}{p_2} \epsilon_3 \right) + 0 \left(\frac{p'_0}{p_0} \epsilon_1 \epsilon_2^2 \epsilon_3 \right) \\ & + 0(\epsilon_3 \delta'_2) + 0 \left(\frac{q'_1}{q_1} \epsilon_2 \epsilon_3 \right).\end{aligned}\quad (3.37)$$

Similar work can be done for the other elements ψ_{ij} , so we obtain

$$\begin{aligned}\psi_{13} = & -\frac{1}{2} \frac{q'_2}{q_2} + 0 \left(\frac{q'_2}{q_2} \epsilon_3 \right) + 0 \left(\frac{p'_1}{p_1} \epsilon_3 \right) + 0 \left(\frac{q'_1}{q_1} \epsilon_3 \right) + 0(\epsilon_3 \delta'_3) \\ & + 0 \left(\frac{p'_0}{p_0} \epsilon_1 \epsilon_3 \right) + 0 \left(\frac{p'_2}{p_2} \epsilon_2 \epsilon_3 \right).\end{aligned}\quad (3.38)$$

$$\begin{aligned}\psi_{14} = & -\frac{1}{2} \frac{q'_2}{q_2} + 0 \left(\frac{q'_2}{q_2} \epsilon_3 \right) + 0 \left(\frac{q'_1}{q_1} \epsilon_1^{-1} \epsilon_3 \right) + 0 \left(\frac{p'_0}{p_0} \epsilon_1^{-1} \epsilon_3 \right) \\ & + 0(\epsilon_1^{-1} \epsilon_3 \delta'_4) + 0 \left(\frac{p'_2}{p_2} \epsilon_1 \epsilon_2 \epsilon_3 \right).\end{aligned}\quad (3.39)$$

$$\begin{aligned}\psi_{21} = & -\frac{1}{2} \frac{q'_2}{q_2} + 0 \left(\frac{q'_2}{q_2} \epsilon_2 \right) + 0 \left(\frac{q'_2}{q_2} \epsilon_3 \right) + 0(\delta'_1) + 0 \left(\epsilon_2 \frac{p'_2}{p_2} \right) \\ & + 0 \left(\epsilon_3 \frac{p'_2}{p_2} \right) + 0 \left(\frac{q'_1}{q_1} \epsilon_2 \epsilon_3 \right) + 0 \left(\frac{p'_0}{p_0} \epsilon_1 \epsilon_2^2 \epsilon_3^2 \right)\end{aligned}\quad (3.40)$$

$$\begin{aligned}\psi_{23} = & \left[\frac{1}{2} \frac{q'_1}{q_1} - \frac{p'_1}{p_1} + \frac{1}{2} \frac{q'_2}{q_2} \right] + 0\left(\frac{q'_1}{q_1} \epsilon_1\right) + 0\left(\frac{q'_1}{q_1} \epsilon_2\right) + 0\left(\frac{q'_1}{q_1} \epsilon_3\right) \\ & + 0\left(\frac{p'_1}{p_1} \epsilon_1\right) + 0\left(\frac{p'_1}{p_1} \epsilon_2\right) + 0\left(\frac{p'_1}{p_1} \epsilon_3\right) + 0\left(\frac{q'_2}{q_2} \epsilon_2\right) + 0\left(\frac{q'_2}{q_2} \epsilon_3\right) \\ & + 0(\delta'_3) + 0\left(\frac{p'_0}{p_0} \epsilon_1\right) + 0\left(\epsilon_2 \epsilon_3 \frac{p'_2}{p_2}\right),\end{aligned}\quad (3.41)$$

$$\begin{aligned}\psi_{24} = & \epsilon_1^{-1} \left[\frac{1}{2} \frac{q'_1}{q_1} + 0\left(\frac{q'_1}{q_1} \epsilon_1\right) + 0\left(\frac{q'_1}{q_1} \epsilon_2\right) + 0\left(\frac{q'_1}{q_1} \epsilon_3\right) + 0\left(\frac{p'_0}{p_0} \epsilon_1\right) \right. \\ & \left. + 0\left(\frac{p'_0}{p_0} \epsilon_2\right) + 0\left(\frac{p'_0}{p_0} \epsilon_3\right) + 0(\delta'_4) + 0\left(\frac{q'_2}{q_2} \epsilon_1\right) + 0\left(\frac{p'_2}{p_2} \epsilon_1^2 \epsilon_2 \epsilon_3\right) \right]\end{aligned}\quad (3.42)$$

$$\psi_{31} = 0\left(\frac{p'_2}{p_2} \epsilon_2\right) + 0\left(\frac{q'_2}{q_2} \epsilon_2\right) + 0(\delta'_1 \epsilon_2) + 0\left(\frac{q'_1}{q_1} \epsilon_2 \epsilon_3\right) + 0\left(\frac{p'_0}{p_0} \epsilon_1 \epsilon_2^2 \epsilon_3^2\right)\quad (3.43)$$

$$\psi_{32} = 0\left(\frac{q'_2}{q_2} \epsilon_2\right) + 0\left(\frac{p'_1}{p_1} \epsilon_2\right) + 0(\epsilon_2 \delta'_2) + 0\left(\epsilon_1 \epsilon_2^2 \frac{p'_0}{p_0}\right) + 0\left(\frac{q'_1}{q_1} \epsilon_2\right) + 0\left(\epsilon_2 \epsilon_3 \frac{p'_2}{p_2}\right),\quad (3.44)$$

$$\begin{aligned}\psi_{34} = & \epsilon_1^{-1} \left[-\frac{1}{2} \frac{q'_1}{q_1} + 0\left(\frac{q'_1}{q_1} \epsilon_1\right) + 0\left(\frac{q'_1}{q_1} \epsilon_2\right) + 0\left(\frac{p'_0}{p_0} \epsilon_1\right) + 0\left(\frac{p'_0}{p_0} \epsilon_2\right) \right. \\ & \left. + 0(\delta'_4) + 0\left(\frac{q'_1}{q_1} \epsilon_1 \epsilon_2\right) + 0\left(\frac{p'_2}{p_2} \epsilon_1^2 \epsilon_2^2 \epsilon_3\right) \right]\end{aligned}\quad (3.45)$$

$$\psi_{41} = \epsilon_1 \left[0\left(\frac{q'_1}{q_1} \epsilon_2 \epsilon_3\right) + 0\left(\frac{q'_2}{q_2} \epsilon_1 \epsilon_2\right) + 0\left(\frac{p'_2}{p_2} \epsilon_1 \epsilon_2\right) + 0(\delta'_1 \epsilon_1 \epsilon_2) + 0\left(\frac{p'_0}{p_0} \epsilon_1 \epsilon_2^2 \epsilon_3^2\right) \right]\quad (3.46)$$

$$\begin{aligned}\psi_{42} = & 0\left(\frac{q'_2}{q_2} \epsilon_1 \epsilon_2\right) + 0\left(\frac{p'_1}{p_1} \epsilon_1 \epsilon_2\right) + 0(\delta'_2 \epsilon_1 \epsilon_2) + 0\left(\frac{q'_1}{q_1} \epsilon_1 \epsilon_2\right) \\ & + 0\left(\frac{p'_0}{p_0} \epsilon_1^2 \epsilon_2^2\right) + 0\left(\frac{p'_2}{p_2} \epsilon_1^2 \epsilon_2 \epsilon_3\right),\end{aligned}\quad (3.47)$$

$$\begin{aligned}\psi_{43} = & \epsilon_1 \left[-\frac{1}{2} \frac{q'_1}{q_1} + 0\left(\frac{p'_1}{p_1} \epsilon_1\right) + 0\left(\frac{q'_1}{q_1} \epsilon_1\right) + 0\left(\frac{q'_1}{q_1} \epsilon_2\right) + 0(\delta'_3 \epsilon_1) \right. \\ & \left. + 0\left(\frac{p'_0}{p_0} \epsilon_1\right) + 0\left(\frac{p'_2}{p_2} \epsilon_1 \epsilon_2^2 \epsilon_3\right) + 0\left(\frac{q'_2}{q_2} \epsilon_1 \epsilon_2\right) \right].\end{aligned}\quad (3.48)$$

Now we need to work out (2.22)–(2.25) in order to determine the form (2.17). Now by (3.28)–(3.31) and (3.37)–(3.48), (2.22)–(2.25) will give:

$$\begin{aligned}\phi_{11} &= \frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_1), & \phi_{22} &= \frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_2) \\ \phi_{33} &= \frac{p'_1}{p_1} - \frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_3), & \phi_{44} &= \frac{p'_1}{p_1} - \frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_4)\end{aligned}\quad (3.49)$$

$$\begin{aligned}
\phi_{12} &= -\frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_5), & \phi_{13} &= -\frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_6) \\
\phi_{14} &= 0(\Delta_7), & \phi_{21} &= -\frac{1}{2} \frac{q'_2}{q_2} + 0(\Delta_8) \\
\phi_{23} &= \frac{1}{2} \left(\frac{q'_1}{q_1} + \frac{q'_2}{q_2} \right) - \frac{p'_1}{p_1} + 0(\Delta_9), & \phi_{24} &= \frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_{10}) \\
\phi_{31} &= 0(\Delta_{11}), & \phi_{32} &= 0(\Delta_{12}), & \phi_{34} &= -\frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_{13}) \\
\phi_{41} &= 0(\Delta_{14}), & \phi_{42} &= 0(\Delta_{15}), & \phi_{43} &= -\frac{1}{2} \frac{q'_1}{q_1} + 0(\Delta_{16}).
\end{aligned} \tag{3.50}$$

where

$$\Delta_i \text{ is } L(a, \infty) \quad (1 \leq i \leq 16) \tag{3.51}$$

by (3.19) and (3.27).

Now by (3.49)–(3.51), we write the system (2.17) as

$$Z' = (\Lambda + R + S)Z \tag{3.52}$$

where

$$R = \begin{bmatrix} -\eta_1 & \eta_1 & \eta_1 & 0 \\ \eta_1 & -\eta_1 & \eta_2 - \eta_1 & -\eta_3 \\ 0 & 0 & -\eta_2 & \eta_3 \\ 0 & 0 & \eta_3 & -\eta_2 \end{bmatrix} \tag{3.53}$$

with

$$\eta_1 = \frac{1}{2} \frac{q'_2}{q_2}, \quad \eta_2 = \frac{(p_1 q_1^{-1/2})'}{p_1 q_1^{-1/2}}, \quad \eta_3 = \frac{1}{2} \frac{q'_1}{q_1}, \tag{3.54}$$

and S is $L(a, \infty)$ by (3.51).

4. THE ASYMPTOTIC FORM OF SOLUTIONS

THEOREM 4.1. Let the coefficients q_1 , q_2 and p_1 in (1.1) be $C^{(2)}[a, \infty)$ and let p_0 and p_2 to be $C^{(1)}[a, \infty)$. Let (3.1), (3.2) and (3.19) hold. Let

$$\eta_k = \omega_k \frac{p_2}{q_2} (1 + \psi_k) \tag{4.1}$$

where ω_k ($1 \leq k \leq 3$) are “non-zero” constants and $\psi_k(x) \rightarrow 0$ ($1 \leq k \leq 3, x \rightarrow \infty$). Also let

$$\psi'_k(x) \text{ is } L(a, \infty) \quad (1 \leq k \leq 3). \tag{4.2}$$

Let

$$\begin{aligned}
\operatorname{Re} I_j(x) \quad (j = 1, 2) \quad \text{and} \quad \operatorname{Re} \left[\frac{1}{2} (\lambda_3 + \lambda_4 + \eta_2 + \eta_4 - \lambda_1 - \lambda_2) \pm I_1 \pm I_2 \right] \\
\text{be of one sign in } [a, \infty)
\end{aligned} \tag{4.3}$$

where

$$I_1 = [4\eta_1^2 + (\lambda_1 - \lambda_2)^2]^{1/2}, \tag{4.4}$$

$$I_2 = [4\eta_3^2 + (\lambda_3 - \lambda_4)^2]^{1/2}. \tag{4.5}$$

Then (1.1) has solutions

$$y_k \sim q_2^{-1/2} \exp\left(\frac{1}{2} \int_a^x [\lambda_1 + \lambda_2 + (-1)^{k+1} I_1] dt\right), \quad (k = 1, 2) \quad (4.6)$$

$$y_3 \sim q_1^{1/2} p_1^{-1} \exp\left(\frac{1}{2} \int_a^x [\lambda_3 + \lambda_4 + I_2] dt\right), \quad (4.7)$$

$$y_4 = o\left\{ q_1^{1/2} p_1^{-1} \exp\left(\frac{1}{2} \int_a^x [\lambda_3 + \lambda_4 - I_2] dt\right) \right\}. \quad (4.8)$$

PROOF. As in [4] we apply Eastham Theorem [6, section 2] to the system (3.52) provided only that Λ and R satisfy the conditions and we shall use (3.53), (3.54), (4.1) and (4.2). We first require that

$$\eta_k = o\{(\lambda_i - \lambda_j)\} \quad (i \neq j, 1 \leq i, k, j, \leq 4, k \neq 3), \quad (4.9)$$

this being [6, (2.1)] for our system. By (4.1), (3.54), (3.5)–(3.8), this requirement is implied by (3.1) and (3.2).

We also require that

$$\{\eta_k(\lambda_i - \lambda_j)^{-1}\}' \in L(a, \infty) \quad (1 \leq k \leq 3), \quad (4.10)$$

for ($i \neq j$) this being [9, (2.2)] for our system. By (4.1), (3.54), (3.5)–(3.8), this requirement is implied by (3.19) and (4.2). Finally we require the eigenvalues μ_k ($1 \leq k \leq 4$) of $\Lambda + R$ satisfy the dichotomy condition [10], as in [4], the dichotomy condition holds if

$$\mu_j - \mu_k = f + g \quad (j \neq k, 1 \leq j, k \leq 4) \quad (4.11)$$

where f has one sign in $[a, \infty)$ and $g \in L(a, \infty)$ [6, (1.5)]. Now by (2.3) and (3.53)

$$\mu_k = \frac{1}{2}(\lambda_1 + \lambda_2 - 2\eta_1) + \frac{1}{2}(-1)^{k+1} I_1, \quad (k = 1, 2) \quad (4.12)$$

$$\mu_k = \frac{1}{2}(\lambda_3 + \lambda_4 - 2\eta_2) + \frac{1}{2}(-1)^{k+1} I_2, \quad (k = 3, 4). \quad (4.13)$$

Thus by (4.3), (4.11) holds since (3.52) satisfies all the conditions for the asymptotic result [6, section 2], it follows that as $x \rightarrow \infty$, (2.17) has four linearly independent solutions,

$$Z_k(x) = \{e_k + o(1)\} \exp\left(\int_a^x \mu_k(t) dt\right), \quad (4.14)$$

where e_k is the coordinate vector with k -th component unity and other components zero. We now transform back to Y by means of (2.13) and (2.16). By taking the first component on each side of (2.16) and making use of (4.12) and (4.13) and carrying out the integration of $-\frac{1}{2} \frac{q_2}{q_1}$ and $\frac{(q_1^{1/2} p_1^{-1})}{q_1^{1/2} p_1^{-1}}$ for ($1 \leq k \leq 4$) respectively we obtain (4.6), (4.7) and (4.8) after an adjustment of a constant multiple in y_k ($1 \leq k \leq 3$).

5. DISCUSSION

(i) In the familiar case the coefficients which are covered by Theorem 4.1 are

$$p_i(x) = c_i x^{\alpha_i} (i = 0, 1, 2,), \quad q_i(x) = c_{i+2} x^{\alpha_{i+2}} (i = 1, 2)$$

with real constants α_i and c_i ($0 \leq i \leq 4$). Then the critical case (4.1) is given by

$$\alpha_4 - \alpha_2 = 1. \quad (5.1)$$

The values of ω_k ($1 \leq k \leq 3$) in (4.1) are given by

$$\omega_1 = \frac{1}{2} \alpha_4 c_2 c_4^{-1}, \quad \omega_2 = \left(\alpha_1 - \frac{1}{2} \alpha_3 \right) c_2 c_4^{-1}, \quad \omega_3 = \frac{1}{2} \alpha_3 c_2 c_4^{-1}, \quad (5.2)$$

where

$$\psi_k(x) = 0 \quad (1 \leq k \leq 4). \quad (5.3)$$

(ii) More general coefficients are

$$p_0 = c_0 x^{\alpha_0} e^{-2x^b}, \quad p_1 = c_1 x_1^{\alpha_1} e^{\frac{1}{2}x^b}, \quad p_2 = c_2 x^{\alpha_2} e^{x^b},$$

$$q_1 = c_3 x^{\alpha_3} e^{-\frac{1}{2}x^b}, \quad q_2 = c_4 x^{\alpha_4} e^{x^b}.$$

with real constants c_i , α_i ($0 \leq i \leq 4$) and b (> 0). Then the critical case (4.1) is given by

$$\alpha_2 - \alpha_4 = b - 1 \quad (5.4)$$

and the values of ω_k ($1 \leq k \leq 4$) are given by

$$\omega_1 = \frac{1}{2} b c_4 c_2^{-1}, \quad \omega_2 = \frac{3}{2} \omega_1, \quad \omega_3 = -\frac{1}{2} \omega_1,$$

with $\psi_1 = \alpha_4 b^{-1} x^{-b}$, $\psi_2 = \frac{4}{3} b^{-1} \left(\alpha_1 - \frac{1}{2} \alpha_3 \right) x^{-b}$, $\psi_3 = -2\alpha_3 b^{-1} x^{-b}$. Here it is clear that $\psi'_k \in L(a, \infty)$ because $b > 0$.

(iii) We note that in both critical cases (5.1) and (5.4) represent an equation of line in the $\alpha_2 \alpha_4$ -plane.

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