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PROBABILISTIC CONVERGENCE SPACES AND GENERALIZED METRIC SPACES

PAUL BROCK
Washington State University
Pullman, WA. 99164-3113

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ABSTRACT. The category $PPRS(\Delta)$, whose objects are probabilistic pretopological spaces which satisfy an axiom (Δ) and whose morphisms are continuous mappings, is introduced. Categories consisting of generalized metric spaces as objects and contraction mappings as morphisms are embedded as full subcategories of $PPRS(\Delta)$. The embeddings yield a description of metric spaces and their most natural generalizations entirely in terms of convergence criteria.

KEY WORDS AND PHRASES. Generalized metric space, probabilistic convergence space, t-norm, diagonal axiom, functor, isomorphism

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1 Introduction

The category $pqs\text{-}MET^{\infty}$, with extended pseudo-quasi-semi-metric spaces as objects and contraction mappings as morphisms, is the most general metric category we shall consider. An object (X,d) in $pqs\text{-}MET^{\infty}$ consists of a set X and a distance function $d : X \times X \to [0,\infty]$ which satisfies the single axiom $d(x,x) = 0$, for all $x \in X$. By allowing d to assume the value ∞, we obtain a well-behaved category; i.e., the category $pqs\text{-}MET^{\infty}$ is topological.

In this paper, we study $pqs\text{-}MET^{\infty}$ as a full subcategory of $PPRS$, the category of probabilistic pretopological spaces. The latter spaces were first introduced by G. Richardson and D. Kent [14] as generalizations of probabilistic metric spaces (see [15], [16]). The existence of an isomorphism between $pqs\text{-}MET^{\infty}$ and a subcategory of PPRS follows from results established in [3] and [10]. Among other important full subcategories of PPRS are the categories PRTOP of pretopological spaces and TOP of topological spaces, each of which is embedded in PPRS in an obvious way.

Our goal is to find simple axioms for objects in $PPRS$, based entirely on convergence criteria, which characterize $pqs\text{-}MET^{\infty}$ (considered as a subcategory of $PPRS$) along with its most important full subcategories: $pq\text{-}MET^{\infty}$, $p\text{-}MET^{\infty}$, and MET^{∞}.

It follows from the results of [2] that there is a family of isomorphic embeddings of $pqs\text{-}MET^{\infty}$ into $PPRS$ which depend on the choice of an order reversing homeomorphism S. However the $pqs\text{-}MET^{\infty}$ objects in PPRS are characterized by means of an axiom (Δ) which is independent the t-norm derived from S. On the other hand, the axioms F_T and R_T which, along with (Δ), characterize the subcategories $pq\text{-}MET^{\infty}$ and $p\text{-}MET^{\infty}$ infinity in $PPRS$ do depend on T.

It is interesting to observe that the notion of "t-norm," whose original purpose was to define the "triangle inequality" for probabilistic metric spaces, also arises naturally when characterizing those "metric objects" in PPRS which depend on the triangle inequality, notably the pseudo-quasi metric objects.

2 Convergence Spaces

Let X be a set, $\mathbf{F}(X)$ the set of all (proper) filters on X, $\mathbf{U}(X)$ the set of all ultrafilters filters on X, and 2^X the set of all subsets of X. For $x \in X$, denote by $\dot{x}$ the fixed ultrafilter generated by $\{x\}$.

Definition 2.1 A *convergence structure* q on a set X is a function $q : \mathbf{F}(X) \to 2^X$ satisfying:
(C_1) $x \in q(\dot{x})$, for all $x \in X$;
(C_2) $\mathcal{F} \leq \mathcal{G} \Rightarrow q(\mathcal{F}) \subseteq q(\mathcal{G})$;
(C_3) $x \in q(\mathcal{F}) \Rightarrow x \in q(\mathcal{F} \cap \dot{x})$.

We interpret $x \in q(\mathcal{F})$ to mean "$\mathcal{F}$ q-converges to x." The idea of "convergence" may be more intuitively conveyed by writing "$\mathcal{F} \xrightarrow{q} x$." If q is a convergence structure on X, then the pair (X, q) is called a *convergence space*. A function $f : (X, q) \to (Y, p)$ between convergence spaces is *continuous* if $f(\mathcal{F}) \xrightarrow{p} f(x)$ whenever $\mathcal{F} \xrightarrow{q} x$. If p and q are convergence structures on X and $f : (X, q) \to (X, p)$ is continuous, where f is the identity map on X, then we write $p \leq q$ (p is *coarser* than q, or q is *finer* than p).

There is associated with each convergence space (X, q), a closure operator cl_q and an interior operator I_q. These are defined for each $A \in 2^X$ by the following:
$cl_q A = \{x \in X : \exists\, \mathcal{F} \xrightarrow{q} x \text{ such that } A \in \mathcal{F}\}$;
$I_q A = \{x \in A : \mathcal{F} \xrightarrow{q} x \Rightarrow A \in \mathcal{F}\}$.
The q-closure and q-interior can also be described using only ultrafilters in the above definitions.

If $\mathcal{F}$ is a filter on X, $cl_q\mathcal{F}$ denotes the filter generated by $\{cl_q F : F \in \mathcal{F}\}$. A convergence space (X, q) is defined to be *regular* if $\mathcal{F} \xrightarrow{q} x$ implies $cl_q\mathcal{F} \xrightarrow{q} x$.

For every $x \in X$, let

$$\mathcal{V}_q(x) = \{V \subseteq X : x \in I_q V\};$$

$\mathcal{V}_q(x)$ is called the *q-neighborhood filter at x*. The q-neighborhood filter can also be described as the intersection of all ultrafilters which q-converge to x.

We have three additional convergence axioms to consider:
(C_4) $q(\mathcal{F} \cap \mathcal{G}) = q(\mathcal{F}) \cap q(\mathcal{G})$, for all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$;
(C_5) For each $\mathcal{F} \in \mathbf{F}(X), x \in q(\mathcal{F})$ iff $x \in q(\mathcal{G})$, for every ultrafilter $\mathcal{G} \geq \mathcal{F}$;
(C_6) $x \in q(\mathcal{V}_q(x))$, for all $x \in X$.

A convergence structure q which satisfies (C_4) (respectively, (C_5), (C_6)) is called a *limit structure* (respectively, *pseudo-topology*, *pretopology*). Observe that pretopology $\Rightarrow$ pseudo-topology $\Rightarrow$ limit structure $\Rightarrow$ convergence structure. A set $A \subseteq X$ is *q-open* if $I_q A = A$. A pretopology q is a *topology* if every neighborhood filter $\mathcal{V}_q(x)$ has a filter base consisting of sets which are *q-open*.

It is interesting that the convergence properties "regular" and "topological" are in a very natural sense dual to each other, since they can be characterized by means of dual axioms, which we call F and R, due to C.H. Cook and H.R. Fischer [3]. Let X and J be non-empty sets, $\mathcal{F} \in \mathbf{F}(J)$, and $\sigma : J \to \mathbf{F}(X)$. We define

$$\kappa\sigma\mathcal{F} = \bigcup_{F\in\mathcal{F}} \bigcap_{y\in F} \sigma(y);$$

κ is called the "compression operator for $\mathcal{F}$ relative to σ." Note that if $\mathcal{F} \in \mathbf{U}(J)$, and $\sigma(y) \in \mathbf{U}(X)$ for all $y \in J$, then $\kappa\sigma\mathcal{F} \in \mathbf{U}(X)$. We can now define the axioms F and R.

F : Let J be a non-empty set, $\psi : J \to X$, and let $\sigma : J \to \mathbf{F}(X)$ have the property that $\sigma(y) \xrightarrow{q} \psi(y)$, for all $y \in J$. If $\mathcal{F} \in \mathbf{F}(J)$ is such that $\psi(\mathcal{F}) \xrightarrow{q} x$, then $\kappa\sigma\mathcal{F} \xrightarrow{q} x$.

R : Let J be a non-empty set, $\psi : J \to X$, and let $\sigma : J \to \mathbf{F}(X)$ have the property that $\sigma(y) \xrightarrow{q} \psi(y)$, for all $y \in J$. If $\mathcal{F} \in \mathbf{F}(J)$ is such that $\kappa\sigma\mathcal{F} \xrightarrow{q} x$, then $\psi(\mathcal{F}) \xrightarrow{q} x$.

The next proposition summarizes previously mentioned results pertaining to these axioms. The first assertion is proved in [14], the second in [1] and [5].

Proposition 2.2 Let (X, q) be a convergence space.
(1) (X, q) is topological if and only if it satisfies F.
(2) (X, q) is regular if and only if it satisfies R.

Let $\tilde{F}$ and $\tilde{R}$ denote the axioms obtained when "$\mathbf{F}(X)$" and "$\mathbf{F}(J)$" are replaced by "$\mathbf{U}(X)$" and "$\mathbf{U}(J)$" in F and R, respectively. Obviously, $F \Rightarrow \tilde{F}$ and $R \Rightarrow \tilde{R}$. Part (1) of the next proposition is proved in [17]. Part (2) is proved in [4]

Proposition 2.3 Let (X, q) be a pseudotopological convergence space.
(1) (X, q) is topological if and only if it satisfies $\tilde{F}$.
(2) (X, q) is regular if and only if it satisfies $\tilde{R}$.

3 The (β) Axiom for Convergence Spaces.

In this section, we introduce the convergence space axiom (β), and give equivalent characterizations of $\tilde{F}$ and $\tilde{R}$ for convergence spaces (X, q) which satisfy (β). It should be mentioned that (β) is a very strong axiom; the only $\mathbf{T}_1$ space which satisfies (β) is the discrete topology. Some additional terminology will be useful.

Definitions 3.1 Let (X, q) be a convergence space, and let $x, y, z \in X$. (X, q) is said to be
symmetric, if $\dot{y} \xrightarrow{q} x$ whenever $\dot{x} \xrightarrow{q} y$;
transitive, if $\dot{x} \xrightarrow{q} z$ whenever $\dot{x} \xrightarrow{q} y$ and $\dot{y} \xrightarrow{q} z$;
skew transitive, if $\dot{x} \xrightarrow{q} z$ whenever $\dot{y} \xrightarrow{q} x$ and $\dot{y} \xrightarrow{q} z$.

Our first proposition provides an equivalence for skew transitivity. The proof is easy.

Proposition 3.2 A convergence space (X,q) is skew transitive iff it is both transitive and symmetric.

(β) : For all $\mathcal{U} \in \mathrm{U}(X)$, $\mathcal{U} \xrightarrow{q} x \iff$ for all $U \in \mathcal{U}$, there exists $y \in U$ such that $\dot{y} \xrightarrow{q} x$.

In the next section, we will extend both (β) and the following results to the setting of probabilistic convergence spaces.

Proposition 3.3 Let (X,q) be a convergence space which satisfies (β). Then (X,q) satisfies $\tilde{F}$ iff (X,q) is transitive.

Proof: Assume $\tilde{F}$ holds, let $J = X$, and $\psi = \mathrm{id}_X$. Let $x,y,z \in X$, such that $\dot{x} \xrightarrow{q} y$ and $\dot{y} \xrightarrow{q} z$. Define $\sigma : X \to \mathrm{U}(X)$ as follows : $\sigma(w) = \dot{w}$ for all $w \neq y$, and $\sigma(y) = \dot{x}$. Thus, $\kappa\sigma(\dot{y}) = \dot{x} \xrightarrow{q} z$.

To show the converse, let J, σ, ψ, and $\mathcal{F}$ be as in the statement of $\tilde{F}$, $\sigma(w) \xrightarrow{q} \dot{\psi}(w)$ for all $w \in J$, and $\psi\mathcal{F} \xrightarrow{q} z$ for some $z \in X$. Let $D \in \kappa\sigma\mathcal{F}$. Recall that a filter base for $\kappa\sigma\mathcal{F}$ is given by

$$\{\bigcup_{y\in F} U_y : F \in \mathcal{F} \text{ and } U_y \in \sigma(y)\}.$$

Thus, $\bigcup_{y\in F} U_y \subseteq D$ for some $F \in \mathcal{F}$. By axiom (β), there exists a $b \in F$ such that $\dot{\psi(b)} \xrightarrow{q} z$. Since $\sigma(b) \xrightarrow{q} \dot{\psi}(b)$, (β) further implies that there exists a $x \in U_b \subseteq D$ such that $\dot{x} \xrightarrow{q} \psi(b)$. Hence, $\dot{x} \xrightarrow{q} z$ by transitivity. Since D was chosen arbitrarily, (β) implies $\kappa\sigma\mathcal{F} \xrightarrow{q} z$. ∎

Proposition 3.4 Let (X,q) be a convergence space which satisfies (β). Then (X,q) satisfies $\tilde{R}$ iff (X,q) is skew transitive.

Proof: Assume $\tilde{R}$ holds, let $J = X$, and $\psi = \mathrm{id}_X$. Let $x,y,z \in X$, such that $\dot{y} \xrightarrow{q} x$ and $\dot{y} \xrightarrow{q} z$. Define $\sigma : X \to \mathrm{U}(X)$ as follows : $\sigma(w) = \dot{w}$ for all $w \neq x$ and $\sigma(x) = \dot{y}$; hence, $\kappa\sigma(\dot{x}) = \dot{y} \xrightarrow{q} z$. Therefore, $\psi\dot{x} = \dot{x} \xrightarrow{q} z$.

For the converse, let J, ψ, σ, and $\mathcal{F}$ be as in $\tilde{R}$, $\sigma(w) \xrightarrow{q} \dot{\psi}(w)$ for all $w \in J$, and $\kappa\sigma\mathcal{F} \xrightarrow{q} z$. Let $F \in \mathcal{F}$ and let $A_F = \{x \in X : \dot{x} \xrightarrow{q} \psi(j), \text{ for some } j \in F\}$. If $X \backslash A_F \in \kappa\sigma\mathcal{F}$, then there exists $G \in \mathcal{F}$ such that $\bigcup_{w\in G} U_w \subseteq X\backslash A_F$, where $U_w \in \sigma(w)$. Let $b \in F \cap G$. Since $\sigma(b) \xrightarrow{q} \dot{\psi}(b)$, there exists a $x_b \in U_b \subseteq X\backslash A_F$ such that $\dot{x}_b \xrightarrow{q} \psi(b)$, by axiom (β). But this implies $x_b \in A_F$, which is a contradiction; therefore, $A_F \in \kappa\sigma\mathcal{F}$. Hence, there exists a $y \in A_F$ such that $\dot{y} \xrightarrow{q} z$, by (β); but $\dot{y} \xrightarrow{q} \psi(j)$ for some $j \in F$, by definition of A_F. It follows that $\dot{\psi(j)} \xrightarrow{q} z$, by skew transitivity. Since F was chosen arbitrarily, $\psi\mathcal{F} \xrightarrow{q} z$, by (β). ∎

Corollary 3.5 Let (X,q) be a convergence space which satisfies (β). Then (X,q) satisfies $\tilde{R}$ iff (X,q) is transitive and symmetric.

Proof: Use Proposition 3.2. and Proposition 3.4. ∎

4 Probabilistic Convergence Spaces

Probabilistic convergence spaces have evolved from the study of probabilistic metric spaces and their generalizations (see [6], [7], [12], [15], [16]). A filter-based theory for such spaces was introduced in [14].

Let I denote the unit interval $[0,1]$ in R.

Definition 3.1 A *probabilistic convergence structure* $\mathbf{q}$ on X is a function

$$\mathbf{q} : \mathbf{F}(X) \times I \rightarrow 2^X$$

satisfying:

(PCS1) : For each $\mu \in I$, $\mathbf{q}(\mathcal{F}, \mu) = q_\mu(\mathcal{F})$, where $q_\mu \in \mathbf{C}(X)$;

(PCS2) : If $\mu = 0$, q_μ is the indiscrete topology;

(PCS3) : If $\mu \leq \nu \in I$, then $q_\mu \leq q_\nu$;

(PCS4) : For each $\mu \in I$, $q_\mu = \sup\{q_\nu : \nu < \mu\}$.

The condition (PCS4) of Definition 3.1 is called *left-continuity.*

We will generally write $\mathbf{q} = (q_\mu)$, where μ is assumed to range through I. If $\mathbf{q}$ is a probabilistic convergence structure on X, then $(X, \mathbf{q})$ is called a probabilistic convergence space. Essentially, a probabilistic convergence space may be regarded as a family of convergence spaces $\{(X, q_\mu) : \mu \in I\}$. If a filter $\mathcal{F}$ q_μ-converges to a point x, we say that "the probability that $\mathcal{F}$ $\mathbf{q}$-converges to x is at least μ." So $\mathbf{q}$ gives a rule for determining the probability that any given filter on X converges to any given point in X. The *probability that a filter $\mathcal{F}$ $\mathbf{q}$-converge to x* is defined to be $\lambda = \sup\{\mu \in I : \mathcal{F} \xrightarrow{q_\mu} x\}$.

If $(X, \mathbf{q})$ and $(Y, \mathbf{p})$ are probabilistic convergence spaces and $f : X \rightarrow Y$ is a mapping, then $f : (X, \mathbf{q}) \rightarrow (Y, \mathbf{p})$ is said to be *continuous* if $f : (X, q_\mu) \rightarrow (Y, p_\mu)$ is continuous for all $\mu \in I$. The category with probabilistic convergence spaces as objects and continuous functions as morphisms is denoted by PCS. If $(X, \mathbf{q}) \in |PCS|$ and each q_μ is a limit structure (respectively, pseudotopology, pretopology), then $(X, \mathbf{q})$ is called a *probabilistic limit space* (respectively, *probabilistic pseudotopological space, probabilistic pretopological space*), and the corresponding full subcategory of PCS is denoted by PLS (respectively, $PPSS$, $PPRS$). Note that $PPRS \subseteq PPSS \subseteq PLS \subseteq PCS$. A probabilistic convergence space $(X, \mathbf{q})$ is defined to be $\mathbf{T_1}$ (respectively, $\mathbf{T_2}$) if the convergence space (X, q_1) is $\mathbf{T_1}$ (respectively, $\mathbf{T_2}$).

We next define "t-norm," a notion which is fundamental in the study of probabilistic metric spaces and their generalizations. For further information about this topic, the reader is referred to [16].

Definition 4.1 A *t-norm* is a binary operation $T : I^2 \rightarrow I$ which is associative, commutative, increasing in each variable, and satisfies: $T(\mu, 1) = \mu$, for all $\mu \in I$.

A t-norm T is said to be *left-continuous* if $T(x, y) = \sup\{T(u, v) : 0 < u < x, 0 < v < y\}$, for all $x, y \in (0, 1]$.

Let $\mathcal{T}$ be the set of all t-norms. A partial order on $\mathcal{T}$ is defined as follows:

$$T \leq T' \text{ iff } T(\mu,\nu) \leq T'(\mu,\nu) \text{ for all } (\mu,\nu) \in I^2.$$

The smallest t-norm, $\hat{T}$ is defined by

$$\hat{T}(\mu,\nu) = \begin{cases} \mu, & if\ \nu = 1 \\ \nu, & if\ \mu = 1 \\ 0, & otherwise. \end{cases}$$

The largest t-norm is $\check{T}$, defined by

$$\check{T}(\mu,\nu) = \min\{\mu,\nu\}.$$

for all $(\mu,\nu) \in I^2$.

Let $(X,\mathbf{q})$ be a probabilistic convergence space, and $T \in \mathcal{T}$. We define two axioms for $(X,\mathbf{q})$ relative to T which are derived in an obvious way from the axioms F and R of Section 2.

F_T : Let $\mu,\nu \in I$. Let J be any non-empty set, $\psi : J \to X$ and $\sigma : J \to \mathbf{F}(X)$ be such that $\sigma(y) \xrightarrow{q_\nu} \psi(y)$, for each $y \in J$. If $\mathcal{F} \in \mathbf{F}(J)$ and $\psi\mathcal{F} \xrightarrow{q_\mu} x$, then $\kappa\sigma\mathcal{F} \xrightarrow{q_{T(\mu,\nu)}} x$.

$R_{T'}$: Let $\mu,\nu \in I$. Let J be any non-empty set, $\psi : J \to X$ and $\sigma : J \to \mathbf{F}(X)$ be such that $\sigma(y) \xrightarrow{q_\nu} \psi(y)$, for each $y \in J$. If $\mathcal{F} \in \mathbf{F}(J)$ and $\kappa\sigma\mathcal{F} \xrightarrow{q_\mu} x$, then $\psi\mathcal{F} \xrightarrow{q_{T(\mu,\nu)}} x$.

Definitions 4.2 Let T be a t-norm. If a probabilistic convergence space (X,q) satisfies F_T, it is called *T-topological.* If (X,q) satisfies R_T, it is called *T-regular.*

The proof of the following proposition appears in [2].

Proposition 4.3 Let T be a t-norm, and let $(X,\mathbf{q})$ be a probabilistic convergence space. Then $(X,\mathbf{q})$ is T-regular iff, for all $\mu,\nu \in I$, $\mathcal{F} \xrightarrow{q_\nu} x$ implies $cl_{q_\mu}\mathcal{F} \xrightarrow{q_{T(\mu,\nu)}} x$.

Similarly, for any t-norm T, we may derive the axioms $\tilde{F}_T$ and $\tilde{R}_T$ from the axioms $\tilde{F}$ and $\tilde{R}$, respectively. The next proposition is proved in [4].

Proposition 4.4 Let $(X,\mathbf{q})$ be a probabilistic pseudotopological space.

(1) $(X,\mathbf{q})$ satisfies $\tilde{F}_T$ iff $(X,\mathbf{q})$ satisfies F_T.

(2) $(X,\mathbf{q})$ satisfies $\tilde{R}_T$ iff $(X,\mathbf{q})$ is R_T.

For a fixed t-norm T, the full subcategory of PCS whose objects are T-topological is denoted by F_TPCS. The categories R_TPCS, $\tilde{F}_TPCS$, and $\tilde{R}_TPCS$ are defined analogously.

We conclude this section with the following simple result.

Proposition 4.5 Let T be a t-norm, and let $(X,\mathbf{q})$ be a T-regular probabilistic convergence space. Then $(X,\mathbf{q})$ is $\mathbf{T}_1$ iff $(X,\mathbf{q})$ is $\mathbf{T}_2$.

Proof: Assume $(X,\mathbf{q})$ is $\mathbf{T}_1$. Then (X,q_1) is $\mathbf{T}_1$, and by Proposition 4.3, $\mathcal{F} \xrightarrow{q_1} x$ implies $cl_{q_1}\mathcal{F} \xrightarrow{q_{T(1,1)}} x$. But $T(1,1) = 1$, by Definition 4.1; hence, (X,q_1) is regular, and so (X,q_1) is $\mathbf{T}_2$. Thus, $(X,\mathbf{q})$ is $\mathbf{T}_2$.

The converse is clear. ∎

5 The (Δ) Axiom for Probabilistic Convergence Spaces

We now extend the results of Section 3 to the setting of probabilistic convergence spaces.

Definitions 5.1 Let $(X, \mathbf{q})$ be a probabilistic convergence space, let T be a t-norm and let $x, y, z \in X$. $(X, \mathbf{q})$ is said to be
symmetric : if, for every $\mu \in I$, $\dot{x} \xrightarrow{q_\mu} y$ implies $\dot{y} \xrightarrow{q_\mu} x$;
T-transitive : if $\dot{x} \xrightarrow{q_{T(\mu,\nu)}} z$ whenever $\dot{x} \xrightarrow{q_\nu} y$ and $\dot{y} \xrightarrow{q_\mu} z$;
skew T-transitive : if $\dot{x} \xrightarrow{q_{T(\mu,\nu)}} z$ whenever $\dot{y} \xrightarrow{q_\nu} x$ and $\dot{y} \xrightarrow{q_\mu} z$.

Proposition 5.2 A probabilistic convergence space $(X, \mathbf{q})$ is skew T-transitive iff it is both T-transitive and component-wise symmetric.

Proof: Let $\mu \in I$, and assume $(X, \mathbf{q})$ is skew T-transitive. Then $\dot{z} \xrightarrow{q_1} z$ and $\dot{z} \xrightarrow{q_\mu} x$ implies $\dot{x} \xrightarrow{q_{T(1,\mu)}} z$, or more simply, $\dot{z} \xrightarrow{q_\mu} x$ implies $\dot{x} \xrightarrow{q_\mu} z$. Hence, $(X, \mathbf{q})$ is symmetric, and transitivity now follows easily.

The converse is straightforward. ∎

The following axiom, denoted (Δ), extends the (β) axiom of Section 3 to the setting of probabilistic convergence spaces. The (Δ) axiom will hold in those probabilistic convergence spaces which have a natural correspondence with pseudo-quasi-semi metric spaces.

(Δ) : For all $\mathcal{U} \in \mathbf{U}(X)$ and $\nu \in I$, $\mathcal{U} \xrightarrow{q_\nu} x \iff$ for all $U \in \mathcal{U}$, and all $\mu < \nu$, there exists a $y \in U$ such that $\dot{y} \xrightarrow{q_\mu} x$.

Propositions 5.3 and 5.4 extend the results of Propositions 3.3 and 3.4, repectively.

Proposition 5.3 Let $(X, \mathbf{q})$ be a probabilistic convergence space which satisfies (Δ). Then, for any left-continuous t-norm T, $(X, \mathbf{q})$ satisfies $\tilde{F}_T$ iff $(X, \mathbf{q})$ is T-transitive.

Proof: Assume $\tilde{F}_T$ holds, let $J = X$, and $\psi = \mathrm{id}_X$. Let $x, y, z \in X$, such that $\dot{x} \xrightarrow{q_\mu} y$ and $\dot{y} \xrightarrow{q_\nu} z$. Define $\sigma : X \to \mathbf{U}(X)$ as follows : $\sigma(w) = \dot{w}$ for all $w \neq y$, and $\sigma(y) = \dot{x}$. Thus, $\kappa\sigma(\dot{y}) = \dot{x} \xrightarrow{q_{T(\mu,\nu)}} z$.

For the converse, let J, σ, ψ, and $\mathcal{F}$ be as in the statement of $\tilde{F}_T$, $\sigma(w) \xrightarrow{q_\nu} \psi(w)$ for all $w \in J$, and $\psi\mathcal{F} \xrightarrow{q_\mu} z$ for some $z \in X$. Recall that a filter base for $\kappa\sigma\mathcal{F}$ is given by

$$\{\bigcup_{y \in F} U_y : F \in \mathcal{F} \text{ and } U_y \in \sigma(y)\}.$$

Let $\epsilon < T(\mu, \nu)$, and choose $\gamma < \mu$, $\theta < \nu$ such that $\epsilon < T(\gamma, \theta) < T(\mu, \nu)$. Let $D \in \kappa\sigma\mathcal{F}$; thus, $\bigcup_{y \in F} U_y \subseteq D$ for some $F \in \mathcal{F}$. Since $\psi\mathcal{F} \xrightarrow{q_\mu} z$ by hypothesis, there exists $b \in F$ such that $\dot{\psi(b)} \xrightarrow{q_\gamma} z$, by ($\Delta$). Since $\sigma(b) \xrightarrow{q_\nu} \psi(b)$, ($\Delta$) further implies that there exists $x \in U_b \subseteq D$ such that $\dot{x} \xrightarrow{q_\theta} \psi(b)$. Hence, $\dot{x} \xrightarrow{q_{T(\gamma,\theta)}} z$ by T-transitivity, which implies $\dot{x} \xrightarrow{q_\epsilon} z$. Since D and ϵ were chosen arbitrarily, (Δ) implies $\kappa\sigma\mathcal{F} \xrightarrow{q_{T(\mu,\nu)}} z$. ∎

Proposition 5.4 Let $(X, \mathbf{q})$ be a probabilistic convergence space which satisfies (Δ). Then, for any left-continuous t-norm T, $(X, \mathbf{q})$ satisfies $\tilde{R}_T$ iff $(X, \mathbf{q})$ is skew T-transitive.

Proof: Assume $\tilde{R}_T$ holds, let $J = X$, and $\psi = \mathrm{id}_X$. Let $x, y, z \in X$, such that $\dot{y} \xrightarrow{q_\mu} x$ and $\dot{y} \xrightarrow{q_\nu} z$. Define $\sigma : X \to \mathbf{U}(X)$ as follows : $\sigma(w) = \dot{w}$ for all $w \neq x$ and $\sigma(x) = \dot{y}$; hence, $\kappa\sigma(\dot{x}) = \dot{y} \xrightarrow{q_\nu} z$. Therefore, $\psi\dot{x} = \dot{x} \xrightarrow{q_{T(\mu,\nu)}} z$.

For the converse, let J, ψ, σ, and $\mathcal{F}$ be as in $\tilde{R}_T$, $\sigma(w) \xrightarrow{q_\nu} \psi(w)$ for all $w \in J$, and $\kappa\sigma\mathcal{F} \xrightarrow{q_\mu} z$. Let $\epsilon < T(\mu, \nu)$, and choose $\gamma < \mu$, $\theta < \nu$ such that $\epsilon < T(\gamma, \theta) < T(\mu, \nu)$. Let $F \in \mathcal{F}$ and let $A_F = \{x \in X : \dot{x} \xrightarrow{q_\theta} \psi(j),\ \text{for some } j \in F\}$. If $X \backslash A_F \in \kappa\sigma\mathcal{F}$, then there exists $G \in \mathcal{F}$ such that $\bigcup_{w \in G} U_w \subseteq X \backslash A_F$, where $U_w \in \sigma(w)$. Let $b \in F \cap G$. Since $\sigma(b) \xrightarrow{q_\nu} \psi(b)$, there exists $x \in U_b \subseteq X \backslash A_F$ such that $\dot{x} \xrightarrow{q_\theta} \psi(b)$, by axiom (Δ). But this implies $x \in A_F$, which is a contradiction; therefore, $A_F \in \kappa\sigma\mathcal{F}$. Since $A_F \in \kappa\sigma\mathcal{F}$, there exists a $y \in A_F$ such that $\dot{y} \xrightarrow{q_\gamma} z$, by (Δ); but, also, $\dot{y} \xrightarrow{q_\theta} \psi(j)$ for some $j \in F$, by definition of A_F. $\psi(\dot{j}) \xrightarrow{q_{T(\theta,\gamma)}} z$ follows by skew T-transitivity, and thus, $\psi(\dot{j}) \xrightarrow{q_\epsilon} z$. Since F and ϵ were chosen arbitrarily, $\psi\mathcal{F} \xrightarrow{q_{T(\mu,\nu)}} z$, by (Δ). ∎

Corollary 5.5 Let $(X, \mathbf{q})$ be a probabilistic convergence space which satisfies (Δ). Then, for any left-continuous t-norm T, $(X, \mathbf{q})$ satisfies $\tilde{R}_T$ iff $(X, \mathbf{q})$ is symmetric and T-transitive.

Proof: Use Proposition 5.2 and Proposition 5.4 ∎

6 Generalized Metric Spaces as Probabilistic Convergence Spaces

Definition 6.1 Let X be a set and let $d : X \times X \to [0, \infty]$, and consider the following "metric" axioms:

($d1$) $d(x, x) = 0$, for all $x \in X$;

($d2$) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z in X;

($d3$) $d(x, y) = d(y, x)$, for all x, y in X;

($d4$) $d(x, y) = 0$ implies $x = y$.

($d2$) is known as the *triangle inequality*, ($d3$) as *symmetry*, and ($d4$) as *separation*. If d is required to satisfy ($d1$), ($d2$), and ($d3$) only (respectively ($d1$) and ($d2$) only, ($d1$) only), then d is known as an *extended pseudo-metric* (respectively, *extended pseudo-quasi-metric*, *extended pseudo-quasi-semi-metric*). A pair (X, d), where d is an extended pseudo-quasi-semi-metric on X is called an *extended pseudo-quasi-semi-metric space* or, more briefly, *extended pqs-metric space*. A mapping $f : (X, d) \to (Y, d')$ between extended pqs-metric spaces is called a *contraction map* if for all $x, y \in X$,

$$d'(f(x), f(y)) \leq d(x, y).$$

The category consisting of extended pseudo-quasi-semi-metric spaces as objects and contraction maps as morphisms will be denoted by *pqs-MET*$^\infty$. The other cases are defined analagously.

We now identify those probabilistic convergence spaces which correspond to generalized metric spaces in *pqs-MET*$^\infty$, *pq-MET*$^\infty$, *p-MET*$^\infty$ and *MET*$^\infty$. Let *PPRS* be the category of all probabilistic pretopological spaces. We denote by $PPRS(\Delta)$, the full subcategory of $PPRS$ whose objects satisfy the (Δ) axiom of Section 5. We introduce three additional full subcategories of PCS:

$F_TPPRS(\Delta)$: The T-topological objects in $PPRS(\Delta)$;

$R_TPPRS(\Delta)$: The T-regular objects in $PPRS(\Delta)$;

$R_TPPRS(\Delta)^*$: The $\mathbf{T}_2$ objects in $R_TPPRS(\Delta)$.

Proposition 6.2 Let T be a left-continuous t-norm, then the following hold:

(a) $R_TPPRS(\Delta) \subseteq F_TPPRS(\Delta)$.

(b) $\tilde{F}_TPCS \cap PPRS(\Delta) = F_TPPRS(\Delta)$.

(c) $\tilde{R}_TPCS \cap PPRS(\Delta) = R_TPPRS(\Delta)$.

Proof: To prove Part (a), apply Proposition 4.4, Proposition 5.3 and Corollary 5.5. For both Part (b) and Part (c), Proposition 4.4 will suffice. ∎

Let I denote the unit interval $[0,1]$, and let $S : I \to [0,\infty]$ be an order reversing homeomorphism; i.e., S is a homeomorphism, and $S(\mu) > S(\nu)$ whenever $\mu,\nu \in I$ and $\mu < \nu$. The set of all such order reversing homeomorphisms S is denoted by $\mathbf{S}$.

We now construct an isomorphism, based on a given $S \in \mathbf{S}$, between $pqs\text{-}MET^\infty$ and $PPRS(\Delta)$. Let S be in $\mathbf{S}$. If (X,d) is in $pqs\text{-}MET^\infty$, let $\phi_S(d) : \mathbf{F}(X) \times I \to 2^X$ be defined as follows: For each $\mathcal{F} \in \mathbf{F}(X)$ and $\mu \in I$, $\phi_S(d)(\mathcal{F},\mu) = \phi_S(d)_\mu(\mathcal{F})$, where $\phi_S(d)_\mu : \mathbf{F}(X) \to 2^X$ is defined by

$$x \in \phi_S(d)_\mu(\mathcal{F}) \text{ iff } \inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) \le S(\mu).$$

Alternatively, $\phi_S(d)_\mu$ can be described in an equivalent fashion by

$$\mathcal{F} \xrightarrow{\phi_S(d)_\mu} x \text{ iff } \inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) \le S(\mu).$$

Proposition 6.3 If (X,d) is in $|pqs\text{-}MET^\infty|$, then $(X,\phi_S(d))$ is in $|PPRS(\Delta)|$.

Proof: We first verify that, for all $\mu \in I$, $\phi_S(d)_\mu$ is a pretopology. (i) If $\mathcal{F} = \dot{x}$, then $\inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) = d(x,x) = 0$. Hence, $\dot{x} \xrightarrow{\phi_S(d)_\mu} x$ for all $\mu \in I$. (ii) If $\mathcal{F} \xrightarrow{\phi_S(d)_\mu} x$, and $\mathcal{G} \ge \mathcal{F}$, then $\inf_{G\in\mathcal{G}} \sup_{y\in G} d(x,y) \le \inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) \le S(\mu)$. Thus, $\mathcal{G} \xrightarrow{\phi_S(d)_\mu} x$. (iii) Given $x \in X$, let $\{\mathcal{F}_j : j \in J\}$ be the set of all filters on X such that $\mathcal{F}_j \xrightarrow{\phi_S(d)_\mu} x$. Then, $\inf_{F\in\mathcal{F}_j} \sup_{y\in F} d(x,y) \le S(\mu)$ for all $j \in J$. Let an arbitrary element of $\bigcap_{j\in J} \mathcal{F}_j$ be denoted $\bigcup_{j\in J} F_j$, where each F_j is in $\mathcal{F}_j$. It follows that $\inf_{\cup F_j\in\cap\mathcal{F}_j} \sup_{y\in\cup F_j} d(x,y) = \sup_{j\in J} \inf_{F_j\in\mathcal{F}_j} \sup_{y\in F_j} d(x,y) \le S(\mu)$. Thus, $\bigcap_{j\in J} \mathcal{F}_j \xrightarrow{\phi_S(d)_\mu} x$, establishing that $\phi_S(d)$ is pretopological. $(PCS2)$: If $\mu = 0$, then $\inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) \le S(\mu) = \infty$ for all $\mathcal{F} \in \mathbf{F}(X)$ and all $x \in X$; hence, $\mathcal{F} \xrightarrow{q_0} x$, for all $\mathcal{F} \in \mathbf{F}(X)$ and all $x \in X$, and so q_0 is the indiscrete topology. $(PCS3)$: Let $\mu \le \nu$, and $\mathcal{F} \xrightarrow{\phi_S(d)_\nu} x$; then $\inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) \le S(\nu) \le S(\mu)$. Hence, $\mathcal{F} \xrightarrow{\phi_S(d)_\mu} x$. $(PCS4)$: To show $\phi_S(d)$ is left continuous, let $\mathcal{F} \overset{\phi_S(d)_\mu}{\rightarrow} x$ for all $\mu < \nu$. Then, $\inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) \le S(\mu)$ for all $\mu < \nu$ implies $\inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) \le S(\nu)$, and thus, $\mathcal{F} \overset{\phi_S(d)_\nu}{\rightarrow} x$.

It remains to show that $\phi_S(d)$ satisfies (Δ). Let $\mathcal{U}$ be an ultrafilter on X and let $\nu \in I$. Assume $\mathcal{U} \overset{\phi_S(d)_\nu}{\rightarrow} x$; then $\inf_{U\in\mathcal{U}} \sup_{y\in U} d(x,y) \le S(\nu)$. Hence, if $\mu < \nu$, there exists a $V \in \mathcal{U}$ such that $\sup_{y\in V} d(x,y) \le S(\mu)$, which implies $d(x,y) \le S(\mu)$ for all $y \in V$. It follows that if U is any element of $\mathcal{U}$, there exists a $y \in U \cap V$, and hence a $y \in U$, such that $\dot{y} \overset{\phi_S(d)_\mu}{\rightarrow} x$.

Conversely, let $\mathcal{U}$ be an ultrafilter on X and let $\nu \in I$. Assume that for all $U \in \mathcal{U}$ and all $\mu < \nu$, there exists a $y \in U$ such that $\dot{y} \overset{\phi_S(d)_\mu}{\rightarrow} x$. For each $\mu < \nu$, put $A_\mu = \{y \in X : \dot{y} \overset{\phi_S(d)_\mu}{\rightarrow} x\}$; then $X - A_\mu \notin \mathcal{U}$, and so $A_\mu \in \mathcal{U}$ for each $\mu < \nu$. It follows that $\inf_{U\in\mathcal{U}} \sup_{y\in U} d(x,y) \le \inf_{\mu<\nu} \sup_{y\in A_\mu} d(x,y) \le S(\nu)$. Therefore, $\mathcal{U} \overset{\phi_S(d)_\nu}{\rightarrow} x$, and consequently, $\phi_S(d)$ satisfies (Δ). ∎

Let ϕ_S be defined for objects by $\phi_S(X,d) = (X, \phi_S(d))$ and for morphisms by $\phi_S(f) = f$.

Proposition 6.4 $\phi_S : pqs\text{-}MET^\infty \rightarrow PPRS(\Delta)$ is a functor.

Proof: Let (X,d) and (X',d') be objects in $pqs\text{-}MET^\infty$, $f : (X,d) \rightarrow (X',d')$ a contraction map and $\mathcal{F} \in \mathbf{F}(X)$. If $\mu \in I$, and $\mathcal{F} \overset{\phi_S(d)_\mu}{\rightarrow} x$, then

$$\inf_{H\in f(\mathcal{F})} \sup_{w\in H} d'(f(x),w) \le \inf_{F\in\mathcal{F}} \sup_{y\in F} d'(f(x),f(y)) \le \inf_{F\in\mathcal{F}} \sup_{y\in F} d(x,y) \le S(\mu).$$

Hence, $f(\mathcal{F}) \overset{\phi_S(d')_\mu}{\rightarrow} f(x)$, establishing that $f : (X,\phi_S(d)) \rightarrow (X',\phi_S(d'))$ is continuous. ∎

We next construct the inverse functor $\psi_S : PPRS(\Delta) \rightarrow pqs\text{-}MET^\infty$. If $(X,\mathbf{q})$ is in $|PPRS(\Delta)|$, define $\psi_S(X,\mathbf{q}) = (X,\psi_S(\mathbf{q}))$, where $\psi_S(\mathbf{q}) : X \times X \rightarrow [0,\infty]$ is given by

$$\psi_S(\mathbf{q})(x,y) = \inf\{S(\nu) : \dot{y} \overset{q_\nu}{\rightarrow} x\}.$$

If f is a morphism in $PPRS(\Delta)$, define $\psi_S(f) = f$.

Proposition 6.5 If $(X,\mathbf{q}) \in |PPRS(\Delta)|$, then $(X,\psi_S\mathbf{q}) \in |pqs\text{-}MET^\infty|$.

Proof: It is sufficient to show that $\psi_S(\mathbf{q})$ satisfies $(d1)$ of Definition 6.1. Since $\dot{x} \overset{q_1}{\rightarrow} x$, for all $x \in X$, $\psi_S(\mathbf{q})(x,x) = \inf\{S(\nu) : \dot{x} \overset{q_\nu}{\rightarrow} x\} = S(1) = 0$, for all $x \in X$. ∎

Proposition 6.6 $\psi_S : PPRS(\Delta) \rightarrow pqs\text{-}MET^\infty$ is a functor.

Proof: Let $(X,\mathbf{q})$ and $(X',\mathbf{q}')$ be objects in $PPRS(\Delta)$, $f : (X,\mathbf{q}) \rightarrow (X',\mathbf{q}')$ a continuous function, and $x,y \in X$. Select $\mu \in I$ such that $\psi_S(\mathbf{q})(x,y) = S(\mu)$; then $\dot{y} \overset{q_\mu}{\rightarrow} x$, which implies $\dot{f(y)} \overset{q'_\mu}{\rightarrow} f(x)$, by continuity of f. Hence, $\psi_S(\mathbf{q}')(f(x),f(y)) \le S(\mu) = \psi_S(\mathbf{q})(x,y)$, and thus, $f : (X,\psi_S(\mathbf{q})) \rightarrow (X',\psi_S(\mathbf{q}'))$ is a contraction map. ∎

Theorem 6.7 $\phi_S : pqs\text{-}MET^\infty \rightarrow PPRS(\Delta)$ is an isomorphism.

Proof: It will be sufficient to show $\psi_S \circ \phi_S(X,d) = (X,d)$ for each $(X,d) \in pqs\text{-}MET^\infty$, and $\phi_S \circ \psi_S(X,\mathbf{q}) = (X,\mathbf{q})$ for each $(X,\mathbf{q}) \in PPRS(\Delta)$. Let (X,d) be in $pqs\text{-}MET^\infty$ and $x,y \in X$. $\psi_S(\phi_S(d))(x,y) = \inf\{S(\nu) : \dot{y} \overset{\phi_S(d)_\nu}{\rightarrow} x\} = \inf\{S(\nu) : d(x,y) \le S(\nu)\} = d(x,y)$. Hence, $\psi_S \circ \phi_S(X,d) = (X,d)$.

Now let $(X,\mathbf{q})$ be in $PPRS(\Delta)$, $\nu \in I$ and $\mathcal{U}$ an ultrafilter on X. Let $\mathcal{U} \overset{q_\nu}{\rightarrow} x$; then by axiom (Δ), for all $\mu < \nu$ and all $U \in \mathcal{U}$, there exists a $y \in U$ such that $\dot{y} \overset{q_\mu}{\rightarrow} x$. For each $\mu < \nu$, put $A_\mu = \{y \in X : \dot{y} \overset{q_\mu}{\rightarrow} x\}$. Then $X - A_\mu \notin \mathcal{U}$, and so $A_\mu \in \mathcal{U}$, for all $\mu < \nu$. It follows that $\inf_{U\in\mathcal{U}} \sup_{y\in U} \psi_S(\mathbf{q})(x,y) \le \inf_{\mu<\nu} \sup_{y\in A_\mu} \psi_S(\mathbf{q})(x,y) \le S(\nu)$. Therefore, $\mathcal{U} \overset{\phi_S(\psi_S(\mathbf{q}))_\nu}{\rightarrow} x$, and thus $\phi_S(\psi_S(\mathbf{q})) \le \mathbf{q}$.

To establish the reverse inequality, let $(X, \phi_S(\psi_S(\mathbf{q})))$ be in $PPRS(\Delta)$, $\nu \in I$ and $\mathcal{U}$ an ultrafilter on X. Let $\mathcal{U} \overset{\phi_S(\psi_S(\mathbf{q}))_\nu}{\rightarrow} x$; then $\inf_{U\in\mathcal{U}} \sup_{y\in U} \psi_S(\mathbf{q})(x,y) \le S(\nu)$. Hence, if $\mu < \nu$, there exists a $V \in \mathcal{U}$ such that $\sup_{y\in V} \psi_S(\mathbf{q})(x,y) \le S(\mu)$, which implies $\psi_S(\mathbf{q})(x,y) \le S(\mu)$ for all $y \in V$. It follows that if U is any element of $\mathcal{U}$, there exists a $y \in U \cap V$ such that $\dot{y} \overset{q_\mu}{\rightarrow} x$. By axiom (Δ), it follows that $\mathcal{U} \overset{q_\nu}{\rightarrow} x$, and consequently, $\mathbf{q} \le \phi_S(\psi_S(\mathbf{q}))$. Hence, $\mathbf{q} = \phi_S \circ \psi_S(\mathbf{q})$. ∎

A useful class of continuous t-norms may be obtained via the next proposition. See Chapter 5 of [16] for additional information.

Proposition 6.8 Let S be in $\mathbf{S}$. Then the mapping $T_S : I^2 \to I$, defined by $T_S(\mu,\nu) = S^{-1}(S(\mu) + S(\nu))$ is a continuous t-norm.

Any continuous t-norm T of the form $T = T_S$ for some $S \in \mathbf{S}$ is called the *strict t-norm derived from S*.

In the results that follow, let S be in $\mathbf{S}$, and let T be the strict t-norm derived from S. Let $\tilde{\phi}_S$ be the restriction of ϕ_S to $pq\text{-}MET^\infty$. We now show that $\tilde{\phi}_S : pq\text{-}MET^\infty \to F_T PPRS(\Delta)$ is an isomorphism.

Proposition 6.9 If $(X,d) \in |pq\text{-}MET^\infty|$, then $(X, \phi_S(d))$ is T-transitive.

Proof: Let $x,y,z \in X$, $\dot{x} \overset{\phi_S(d)_\mu}{\rightarrow} y$, and $\dot{y} \overset{\phi_S(d)_\nu}{\rightarrow} z$. Then, $d(x,y) \le S(\mu)$, and $d(y,z) \le S(\nu)$. By the triangle inequality, $d(x,z) \le S(\mu) + S(\nu) = S(T(\mu,\nu))$; hence, $\dot{z} \overset{\phi_S(d)_{T(\mu,\nu)}}{\longrightarrow} x$. ∎

Corollary 6.10 If $(X,d) \in |pq\text{-}MET^\infty|$, then $(X, \phi_S(d)) \in |F_T PPRS(\Delta)|$.

Proof: $(X, \phi_S(d)) \in |PPRS(\Delta)|$, by Proposition 6.3, and $(X, \phi_S(d))$ satisfies $\tilde{F}_T$ by Proposition 5.3. Hence, $(X, \phi_S(d)) \in |F_T PPRS(\Delta)|$, by Proposition 6.2(b). ∎

Proposition 6.11 If $(X,\mathbf{q}) \in |F_T PPRS(\Delta)|$, then $(X, \psi_S(\mathbf{q})) \in |pq\text{-}MET^\infty|$.

Proof: It is sufficient to show $(X, \psi_S(\mathbf{q}))$ satisfies $(d2)$ of Definition 6.1 (the triangle inequality). Let $x,y,z \in X$. Select $\mu,\nu \in I$ such that $d(x,y) = S(\mu)$, and $d(y,z) = S(\nu)$; thus, $\dot{y} \overset{q_\mu}{\rightarrow} x$ and $\dot{z} \overset{q_\nu}{\rightarrow} y$. It follows that $\dot{z} \overset{q_{T(\mu,\nu)}}{\rightarrow} x$, by Proposition 5.3, and therefore,

$$d(x,z) \le S(T(\mu,\nu)) = S(\mu) + S(\nu) = d(x,y) + d(y,z).$$

∎

Combining the three previous results we obtain the next theorem.

Theorem 6.12 $\tilde{\phi}_S : pq\text{-}MET^\infty \to F_T PPRS(\Delta)$ is an isomorphism.

Now let $\hat{\phi}_S$ be the restriction of ϕ_S to $p\text{-}MET^\infty$. Following a similar procedure, we show that the mapping $\hat{\phi}_S : p\text{-}MET^\infty \to R_T PPRS(\Delta)$ is an isomorphism.

Proposition 6.13 If $(X, d) \in |p\text{-}MET^\infty|$, then $(X, \phi_S(d))$ is symmetric.

Proof: Let $x, y \in X$, $\mu \in I$, such that $\dot{x} \overset{\phi_S(d)_\mu}{\to} y$. Then, using ($d3$) of Definition 6.1, $d(x, y) = d(y, x) \leq S(\mu)$, which implies $\dot{y} \overset{\phi_S(d)_\mu}{\to} x$. ∎

Corollary 6.14 If $(X, d) \in |p\text{-}MET^\infty|$, then $(X, \phi_S(d)) \in |R_T PPRS(\Delta)|$.

Proof: $(X, \phi_S(d)) \in |PPRS(\Delta)|$, by Proposition 6.3. Furthermore, $(X, \phi_S(d))$ is T-transitive by Proposition 6.9, and thus, $(X, \phi_S(d))$ satisfies $\tilde{R}_T$ by Proposition 6.13 and Corollary 5.5. Hence, $(X, \phi_S(d)) \in |R_T PPRS(\Delta)|$ by Proposition 6.2(c). ∎

Proposition 6.15 If $(X, \mathbf{q}) \in |R_T PPRS(\Delta)|$, then $(X, \psi_S(\mathbf{q})) \in |p\text{-}MET^\infty|$.

Proof: We must prove that $(X, \psi_S \mathbf{q})$ satisfies ($d3$) of Definition 6.1. $(X, \mathbf{q})$ is symmetric by Corollary 5.5. It follows that, for all $x, y \in X$, $\psi_S(\mathbf{q})(x, y) = S(\mu) \Rightarrow \dot{y} \overset{q_\mu}{\to} x \Rightarrow \dot{x} \overset{q_\mu}{\to} y \Rightarrow \psi_S(\mathbf{q})(y, x) \leq S(\mu)$. Hence, $\psi_S(\mathbf{q})(y, x) \leq \psi_S(\mathbf{q})(x, y)$, and by a similar argument, $\psi_S(\mathbf{q})(y, x) \leq \psi_S(\mathbf{q})(x, y)$. Thus, $\psi_S(\mathbf{q})(y, x) = \psi_S(\mathbf{q})(x, y)$. ∎

Theorem 6.16 $\hat{\phi}_S : p\text{-}MET^\infty \to R_T PPRS(\Delta)$ is an isomorphism.

Recall that $R_T PPRS(\Delta)^*$ is the full subcategory of $R_T PCS(\Delta)$ consisting of those objects which satisfy the $\mathbf{T_2}$ property. Let $\check{\phi}_S$ be the restriction of ϕ_S to MET^∞. The next results show that $\check{\phi}_S : MET^\infty \to R_T PPRS(\Delta)^*$ is an isomorphism.

Proposition 6.17 If $(X, d) \in |MET^\infty|$, then $(X, \phi_S(d))$ is $\mathbf{T}_2$.

Proof: Let $x, y \in X$, $\dot{x} \overset{\phi_S(d)_1}{\to} y$. Then, $d(x, y) = S(1) = 0$, which implies $x = y$, by ($d4$) of Definition 6.1. Thus, $(X, \phi_S(d))$ is $\mathbf{T}_1$, and so $(X, \phi_S(d))$ is $\mathbf{T}_2$ by Proposition 4.5. ∎

Corollary 6.18 If $(X, d) \in |MET^\infty|$, then $(X, \phi_S(d)) \in |R_T PPRS(\Delta)^*|$.

Proposition 6.19 If $(X, \mathbf{q}) \in |R_T PPRS(\Delta)^*|$, then $(X, \psi_S(\mathbf{q})) \in |MET^\infty|$.

Proof: Let $x, y \in X$, $\psi_S \mathbf{q}(x, y) = 0$. Then, $\dot{y} \overset{q_1}{\to} x$, but, since $(X, \mathbf{q})$ is $\mathbf{T}_1$, we must have $x = y$ since (X, q_1) is $\mathbf{T}_1$. Therefore, $(X, \psi_S \mathbf{q})$ satisfies ($d4$) of Definition 6.1. ∎

Theorem 6.20 If $S \in \mathbf{S}$ and T the t-norm derived from S, then

$$\check{\phi}_S : MET^\infty \to R_T PPRS(\Delta)^*$$

is an isomorphism.

The following list summarizes the four main results of this section. The symbol "$\cong$" means "is isomorphic to."

(1) $pqs\text{-}MET^\infty \cong PPRS(\Delta)$;

(2) $pq\text{-}MET^\infty \cong F_T PPRS(\Delta)$;

(3) $p\text{-}MET^\infty \cong R_T PPRS(\Delta)$;

(4) $MET^\infty \cong R_T PPRS(\Delta)^*$.

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São Josè dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk