α -DERIVATIONS AND THEIR NORM IN PROJECTIVE TENSOR PRODUCTS OF Γ-BANACH ALGEBRAS

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ABSTRACT. Let (V,Γ) and (V',Γ') be Gamma-Banach algebras over the fields F_1 and F_2 isomorphic to a field F which possesses a real valued valuation, and $(V,\Gamma) \otimes_p (V',\Gamma')$, their projective tensor product. It is shown that if D_1 and D_2 are α - derivation and α'- derivation on (V,Γ) and (V',Γ') respectively and $u = \sum_i x_i \otimes y_i$ is an arbitrary element of $(V,\Gamma) \otimes_p (V',\Gamma')$, then there exists an $\alpha \otimes \alpha'$- derivation D on $(V,\Gamma) \otimes_p (V',\Gamma')$ satisfying the relation

$$D(u) = \sum_i \left[(D_1 x_i) \otimes y_i + x_i \otimes (D_2 y_i) \right]$$

and possessing many enlightening properties. The converse is also true under a certain restriction. Furthermore, the validity of the results $\| D \| = \| D_1 \| + \| D_2 \|$ and $sp(D) = sp(D_1) + sp(D_2)$ are fruitfully investigated.

KEY WORDS AND PHRASES : Γ - Banach algebras , projective tensor products , α - derivations.
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1. INTRODUCTION

$\Gamma-$ Banach algebras and α – derivations are generalisation of ordinary Banach algebras and derivations respectively. The set of all m x n rectangular matrices and the set of all bounded linear transformations from an infinite dimensional normed linear space X into a Banach space Y are nice examples of Γ- Banach algebras which are not general Banach algebras. Similarly a derivation can't be defined on these spaces as there appears to be no natural way of introducing an algebraic multiplication into these. So, a new concept of derivation known as α- derivation is introduced on a Γ- Banach algebra. Bhattacharya and Maity have defined a Γ- Banach algebra in their paper [1] and have discussed in their another paper [2] various tensor products of Γ- Banach algebras over fields which are isomorphic to another field with a real valued valuation by using semilinear transformations, [3]. Derivations and tensor products of general Banach algebras are discussed in many papers, [4,5,6,7,8]. Now there are some natural questions : Does every pair of derivations D_1 and D_2 on Gamma Banach algebras (V,Γ) and (V',Γ') respectively give rise to a derivation D on their projective tensor product? If yes, then can one estimate the norm of D with the help of norms of D_1 and D_2 ? Can one evaluate the spectrum of D with the help

of those of D_1 and D_2 ? Are the converses of the above problems true? We give affirmative answers to some of these questions. The useful terminologies are forwarded below :

DEFINITION 1.1. Let X (F_1) and Y (F_2) be given normed linear spaces over fields F_1 and F_2, which are isomorphic to a field F with a real valued valuation, (refer to Backman's book [9]). If $u= \sum_i (x_i \otimes y_i)$ is an element of the algebraic tensor product X $\otimes$ Y, then the projective norm p is defined by

$$p(u) = \inf \left\{ \sum_i \| x_i \| \| y_i \| : x_i \, \varepsilon \, X , y_i \, \varepsilon \, Y \right\},$$

where the infimum is taken over all finite representations of u. Further the weak norm w on u is defined by

$$w(u)= \sup \left\{ | \sum_i \zeta_1 (f (x_i)). \zeta_2(g(y_i)) | : f \, \varepsilon \, X^*, g \, \varepsilon \, Y^*, \| f \| \le 1, \| g \| \le 1 \right\}.$$

[Here X^* and Y^* are respective dual spaces of X and Y; and F_1, F_2 are isomorphic to F under isormorphisms ζ_1 and ζ_2]. The projective tensor product X $\otimes_p$ Y and the weak tensor product X $\otimes_w$ Y are the completions of X $\otimes$ Y with their respective norms. For details, see Bonsall and Duncan's book [10].

DEFINITION 1.2. Let (V,Γ) be a Γ- Banach algebra and α, a fixed element of Γ. Then α-**identity**, 1_α, is an element of V satisfying the conditions $x\alpha 1_\alpha = x$ and $1_\alpha \alpha x= x$ for every x in V.

DEFINITION 1.3. A linear operator D of (V,Γ) into itself is called an α - **derivation** if

$$D (x \, \alpha \, y) = (Dx) \, \alpha y + x\alpha \, (Dy), \qquad\qquad x, y \, \varepsilon \, V.$$

Every x ε V gives rise to an α- derivation D_x defined by $D_x(y) = x\alpha y - y\alpha x$. Such a derivation is called an α-**inner derivation**. Further, if (V,Γ) is an involutive Γ- Banach algebra with an involution * , then an α- derivation D is called an α- **star-derivation** if $Dx^* = - (Dx)^*$, x^* being the adjoint of x. Again, we define an operation o by $xoy = x\alpha y + y\alpha x$, x,y ε V. A linear map D on (V, Γ) is called an α-**Jordan derivation** if $D (xoy) = (Dx) oy+xo (Dy)$ for all x and y in V.

2.   THE MAIN RESULTS

Throughout our discussion, unless stated otherwise, (V,Γ) and (V',Γ') are Gamma-Banach algebras over F_1 and F_2, isomorphic to F which possesses a real valued valuation ; α and α' are fixed elements of Γ and Γ' ; and $1_\alpha, 1_{\alpha'}$ are α- identity and α'-identity of V and V' respectively. Moreover, suppose that $\| 1_\alpha \| = k_1 \ne 0$ and $\| 1_{\alpha'} \| = k_2 \ne 0$.

The following proposition is fundamental for our purpose, and we refer to Bhattacharya and Maity [2] for its proof.

PROPOSITION 2.1. The projective tensor product $(V,\Gamma) \otimes_p (V',\Gamma')$ with the projective norm is a $\Gamma \otimes \Gamma'$- Banach algebra over the field F , where multiplication is defined by the formula

$$(x \otimes y) (\gamma \otimes \delta) (x' \otimes y') = (x \gamma x') \otimes (y \delta y') , \text{where x,y } \varepsilon \, V; x', y' \, \varepsilon \, V'; \gamma \, \varepsilon \, \Gamma ; \delta \, \varepsilon \, \Gamma'.$$

THEOREM 2.1. Let D_1 and D_2 be bounded α- derivation and α'- derivation on (V,Γ) and (V',Γ') respectively. Then

(i) there exists a bounded $\alpha \otimes \alpha'$- derivation D on the projective tensor product $(V, \Gamma) \otimes_p (V',\Gamma')$ defined

by the relation

$$D(u) = \sum_i \left[(D_1 x_i) \otimes y_i + x_i \otimes (D_2 y_i) \right] \text{, for each vector } u = \sum_i x_i \otimes y_i \, \varepsilon \, (V, \Gamma) \otimes_p (V', \Gamma').$$

(ii) If D_1 and D_2 are α- and α'- inner derivations implemented by the elements $r_o \, \varepsilon \, V$ and $s_o \, \varepsilon \, V'$ respectively then D is an $\alpha \otimes \alpha'$ - inner derivation implemented by $r_o \otimes 1_\alpha + 1_\alpha \otimes s_o$.

(iii) If D_1 and D_2 are α- and α'- Jordan derivations, then D is an $\alpha \otimes \alpha'$- Jordan derivation.

(iv) If (V, Γ) and (V', Γ) are involutive Gamma -Banach algebras, and if D_1 and D_2 are α- and α'- star derivations, then D is $\alpha \otimes \alpha'$ - star derivation.

PROOF. (i) We define a map $D : (V, \Gamma) \otimes_p (V', \Gamma') \rightarrow (V, \Gamma) \otimes_p (V', \Gamma')$ by the rule

$$D(u) = \sum_i \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right] \text{, for each vector } u = \sum_i x_i \otimes y_i.$$

Clearly, D is well - defined. Before establishing the linearity of D, we first aim at proving the boundedness of D. For any arbitrary element $u \, \varepsilon \, (V, \Gamma) \otimes_p (V', \Gamma')$ and $\varepsilon > o$, the definition of the projective norm provides a finite representation $\sum_{i=1}^{n} x'_i \otimes y'_i$ such that $\| u \|_p + \varepsilon \geq \sum_{i=1}^{n} \| x'_i \| \| y'_i \|$. Therefore , for this representation of u, we obtain

$$\| Du \|_p = \| \sum_i \left[D_1 x'_i \otimes y'_i + x'_i \otimes D_2 y'_i) \right] \|_p$$

$$\leq \sum_i \left[\| D_1 x'_i \otimes y'_i \|_p + \| x'_i \otimes D_2 y'_i \|_p \right]$$

$$= \sum_i \left[\| D_1 x'_i \| \| y'_i \| + \| x'_i \| \| D_2 y'_i \| \right] \text{, (because a projective norm is a cross norm).}$$

$$\leq (\| D_1 \| + \| D_2 \|) \sum_i \| x'_i \| \| y'_i \| \text{, (because } D_1 \text{ and } D_2 \text{ are bounded)}$$

$$\leq K (\| u \|_p + \varepsilon) \text{, where } K = \| D_1 \| + \| D_2 \|.$$

Thus, $\| Du \|_p \leq K (\| u \|_p + \varepsilon)$. Since the left hand side is independent of ε, and ε was arbitrary, it follows that $\| Du \|_p \leq K \| u \|_p$ for every $u \, \varepsilon \, (V, \Gamma) \otimes_p (V', \Gamma')$. Consequently, D is bounded.

Next to establish the linearity, let $u = \sum_{i=1}^{n} x_i \otimes y_i$ and $v = \sum_{j=1}^{m} r_j \otimes s_j$ be any two elements of

$(V, \Gamma) \otimes_p (V', \Gamma')$. Then $u + v = \sum_{i=1}^{n+m} x_i \otimes y_i$, where $x_{n+j} = r_j$ and $y_{n+j} = s_j, j = 1, 2, \ldots \ldots m$.

Now, $D(u + v) = D(\sum_{i=1}^{n+m} x_i \otimes y_i)$

$$= \sum_{i=1}^{n+m} \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right]$$

$$= \sum_{i=1}^{n} \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right] + \sum_{i=n+1}^{m+n} \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right]$$

$$= \sum_{i=1}^{n} \left[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right] + \sum_{j=1}^{m} \left[D_1 r_j \otimes s_j + r_j \otimes D_2 s_j \right] = D(u) + D(v).$$

The boundedness of D implies that the rusult, $D(u + v) = D(u) + D(v)$, is also true for any infinite

representations of u and v. Similarly it can be shown easily that $D(au) = aD(u)$ for any scalar a. Consequently D is a bounded linear map.

To show that D is an $\alpha \otimes \alpha'$ - derivation, we suppose that $u = x \otimes y$ and $v = r \otimes s$ are any two elementary tensors of $(V, \Gamma) \otimes_p (V', \Gamma')$. Then $u \, \alpha \otimes \alpha' \, v = x \, \alpha \, r \otimes y \, \alpha's$. Now

$$D(u \, \alpha \otimes \alpha' \, v) = (D_1 \, x \, \alpha \, r) \otimes y \, \alpha's + x \, \alpha \, r \otimes (D_2 y \, \alpha' s)$$

$$= \left[(D_1 \, x) \, \alpha \, r + x \, \alpha (D_1 \, r) \right] \otimes y \alpha' s + x \, \alpha \, r \otimes \left[(D_2 y) \, \alpha' s + y \, \alpha' (D_2 s) \right]$$

$$= \left[(D_1 \, x) \, \alpha \, r \otimes y \, \alpha' s + x \, \alpha r \otimes (D_2 y) \, \alpha' s \right] + \left[x \, \alpha (D_1 r) \otimes y \, \alpha' s + x \, \alpha r \otimes y \, \alpha' (D_2 s) \right]$$

$$= (Du) \, \alpha \otimes \alpha' \, v + u \, \alpha \otimes \alpha' \, (Dv).$$

Similarly, if $u = \sum_i x_i \otimes y_i$ and $v = \sum_j r_j \otimes s_j$ be any two elements of $(V, \Gamma) \otimes_p (V', \Gamma')$, then summing over i and j we can prove easily that $D(u \, \alpha \otimes \alpha' \, v) = (Du) \, \alpha \otimes \alpha' v + u \, \alpha \otimes \alpha'(Dv)$. so D is an $\alpha \otimes \alpha'$- derivation.

(ii) Let D_1 and D_2 be α- and α'- inner derivations implemented by the vectors r_0 and s_0 respectively.

So, $D_1(x) = r_0 \alpha \, x - x \alpha r_0$, $\forall x \varepsilon V$ and $D_2(y) = s_0 \alpha' \, y - y \alpha' s_0$, $\forall y \, \varepsilon \, V'$.

Now, $D(u) = \sum_i \left[D_1 \, x_i \otimes y_i + x_i \otimes D_2 y_i \right]$

$$= \sum_i \left[(r_0 \alpha \, x_i - x_i \, \alpha r_0) \otimes y_i + x_i \otimes (s_0 \alpha' \, y_i - y_i \alpha' s_0) \right]$$

$$= \sum_i \left[r_0 \alpha x_i \otimes y_i - x_i \, \alpha r_0 \otimes y_i + x_i \otimes s_0 \alpha' \, y_i - x_i \otimes y_i \, \alpha' s_0 \right]$$

$$= \sum_i \left[(r_0 \otimes 1_{\alpha'})(\alpha \otimes \alpha')(x_i \otimes y_i) - (x_i \otimes y_i)(\alpha \otimes \alpha')(r_0 \otimes 1_{\alpha'}) \right. $$
$$\left. + (1_\alpha \otimes s_0)(\alpha \otimes \alpha')(x_i \otimes y_i) - (x_i \otimes y_i)(\alpha \otimes \alpha')(1_\alpha \otimes s_0) \right]$$

$$= \sum_i \left[(r_0 \otimes 1_{\alpha'} + 1_\alpha \otimes s_0)(\alpha \otimes \alpha')(x_i \otimes y_i) - (x_i \otimes y_i)(\alpha \otimes \alpha')(r_0 \otimes 1_{\alpha'} + 1_\alpha \otimes s_0) \right]$$

$$= D_{t_0}(u), \quad \text{where } t_0 = r_0 \otimes 1_{\alpha'} + 1_\alpha \otimes s_0.$$

Consequently, D is an $\alpha \otimes \alpha'$ -inner derivation implemented by t_0.

(iii) The proof is routine.

(iv) Let D_1 and D_2 be star derivations. If $u = \sum_i x_i \otimes y_i$ is an element of $(V, \Gamma) \otimes_p (V', \Gamma')$, then the adjoint of u is given by $u^* = \sum_i x_i^* \otimes y_i^*$ Now,

$$Du^* = D(\sum_i x_i^* \otimes y_i^*)$$

$$= \sum_i \left[D_1 x_i^* \otimes y_i^* + x_i^* \otimes D_2 y_i^*) \right]$$

$$= \sum_i \left[-(D_1 x_i)^* \otimes y_i^* + x_i^* \otimes \{-(D_2 y_i)^*\} \right], \text{ because } D_1 \text{ and } D_2 \text{ are star derivation.}$$

$$= -\sum_i \left[(D_1 x_i)^* \otimes y_i^* + x_i^* \otimes (D_2 y_i)^* \right] = - (Du)^*. \quad \text{So, D is a star-derivation. Q.E.D.}$$

REMARK 2.1. (i) The above theorem can be extended to the projective tensor product of n number of Γ- Banach algebras.

(ii) If $u = x \otimes 1_{\alpha'} \varepsilon (V, \Gamma) \otimes_p (V', \Gamma')$, then from the definition of D, we get

$$Du = D_1 x \otimes 1_{\alpha'}, \text{ because } D_2 1_{\alpha'} = 0 \qquad \dots \qquad (2.1)$$

From this result, we can ascertain that for each derivation D on $(V, \Gamma) \otimes_p (V', \Gamma')$, there may **not** exist derivations D_1 and D_2 on (V, Γ) and (V', Γ') respectivey such that D, D_1 and D_2 are connected by the relation given in Theorem 2.1. For example, let D' be an $\alpha \otimes \alpha'$ - inner derivation implemented by an element $r_0 \otimes s_0$, where s_0 is not a scalar multiple of the identity element $1_{\alpha'}$. Then

$D' u = (r_0 \otimes s_0) (\alpha \otimes \alpha') u - u (\alpha \otimes \alpha') (r_0 \otimes s_0)$, for every $u \varepsilon (V, \Gamma) \otimes_p (V', \Gamma')$. Now if $u = x \otimes 1_{\alpha'}$, then

$$D'u = (r_0 \otimes s_0) (\alpha \otimes \alpha') (x \otimes 1_{\alpha'}) - (x \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (r_0 \otimes s_0)$$

$$= r_0 \alpha x \otimes s_0 \alpha' 1_{\alpha'} - x \alpha r_0 \otimes 1_{\alpha'} \alpha' s_0 = (r_0 \alpha x - x \alpha r_0) \otimes s_0$$

$$= (D_{1_{r_0}} x) \otimes s_0, \text{ where } D_1 \text{ is a derivation on } (V, \Gamma) \text{ implemented by } r_0 \quad \dots \quad (2.2)$$

From the results (2.1) and (2.2) we can conclude that unless s_0 is a scalar multiple of the identity element $1_{\alpha'}$, D' $(x \otimes 1_{\alpha'})$ may not be of the form $x_1 \otimes 1_{\alpha'}$, where $x_1 \varepsilon V$, [x_1 may be different from x]. This implies that D' may not equal D in general. However, we have a converse of Theorem 2.1 as follows. Recall that an element $x \varepsilon V$ is called an α - **idempotent element** if $x \alpha x = x$.

THEOREM 2.2. The following results are true :

(i) If D is a derivation on $(V, \Gamma) \otimes_p (V', \Gamma')$ such that $D (\sum_i x_i \otimes y_i) = \sum_i z_i \otimes y_i$, $z_i \varepsilon V$ and y_i's are α'- idempotent elements of V', then there exists an α'-derivation D_1 on V defined by the rule $D_1 x \otimes y = D (x \otimes y)$ for all $x \varepsilon V$ and for every α'- idempotent element $y \varepsilon V'$;

(ii) If D is bounded, so is D_1;

(iii) If D is an $\alpha \otimes \alpha'$-inner derivation implemented by an element w of the form $w = \sum_i x_i \otimes y_i$, where y_i's are α'- idempotent elements, then D_1 is also an α- inner derivation implemented by the element $\sum_i x_i$;

(iv) If (V, Γ) and (V', Γ') are involutive Gamma-Banach algebras, and D is a star derivation, then so is D_1;

(v) If D is an $\alpha \otimes \alpha'$ - Jordan derivation then D_1 is an α- Jordan derivation;

(vi) If D is an $\alpha \otimes \alpha'$- derivation on $(V, \Gamma) \otimes_p (V', \Gamma')$ such that $D (\sum_i x_i \otimes y_i) = \sum_i x_i \otimes s_i$ for α- idempotent elements x_i's in V, and $s_i \varepsilon V'$, then there exists an α'- derivation D_2 on (V', Γ') given by the relation $x \otimes D_2 y = D(x \otimes y)$ for every α- idempotent element $x \varepsilon V$ and for all elements $y \varepsilon V'$. The above results (ii), (iii), (iv) and (v) are also true for D_2.

PROOF. (i) We define a map $D_1 : V \to V$ by

$$D_1 x \otimes y = D(x \otimes y), \text{ for all } x \varepsilon V \text{ and for every } \alpha'\text{-idempotent element } y \varepsilon V'.$$

Clearly, D_1 is well-defined. In particular, we have $D_1 x \otimes 1_{\alpha'} = D (x \otimes 1_{\alpha'})$, $\forall x \varepsilon V$. We first establish the linearity of D_1. Let $x_1, x_2 \varepsilon V$.

Then $D_1 (x_1 + x_2) \otimes 1_{\alpha'} = D((x_1 + x_2) \otimes 1_{\alpha'})$

$$= D (x_1 \otimes 1_{\alpha'} + x_2 \otimes 1_{\alpha'})$$

$$= D (x_1 \otimes 1_{\alpha'}) + D (x_2 \otimes 1_{\alpha'})$$

$$= (D_1 x_1 \otimes 1_{\alpha'} + D_1 x_2 \otimes 1_{\alpha'})$$

$$= (D_1 x_1 + D_1 x_2) \otimes 1_{\alpha'}$$

So, $(D_1(x_1+x_2) \otimes 1_{\alpha'})(f,g) = ((D_1 x_1+ D_1 x_2)\otimes 1_{\alpha'})(f,g)$, $\forall f \varepsilon V^*, \forall g \varepsilon V''$.

This gives, $f (D_1 (x_1+x_2)) g (1_{\alpha'}) = f (D_1 x_1 + D_1 X_2) g (1_{\alpha'})$, $\forall f \varepsilon V^*, \forall g \varepsilon V''$.
The Hahn-Banach theorem provides a functional $g_0 \varepsilon V''$ in such a way that $g_0(1_{\alpha'}) = \| 1_{\alpha'}\| = k_2$.

Then, $f (D_1 (x_1 + x_2)) = f (D_1 x_1 + D_1 x_2), \forall f \varepsilon V^*$. This yields, $D_1 (x_1+x_2) = D_1 x_1 + D_1 x_2$.

By appealing to the same mechanism, we can show that $D_1 (ax) = aD_1 (x)$ for any scalar a. So D_1 is linear.
Next, to show that D_1 is an α- derivation.

$$D_1 (x_1 \alpha x_2) \otimes 1_{\alpha'} = D (x_1 \alpha x_2 \otimes 1_{\alpha'})   (x_1, x_2 \varepsilon V)$$

$$= D \left[(x_1 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (x_2 \otimes 1_{\alpha'}) \right]$$

$$= (D (x_1 \otimes 1_{\alpha'})) (\alpha \otimes \alpha') (x_2 \otimes 1_{\alpha'}) + (x_1 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') D (x_2 \otimes 1_{\alpha'})$$
$$\text{(because D is an } \alpha \otimes \alpha'\text{-derivation)}$$

$$= (D_1 x_1 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (x_2 \otimes 1_{\alpha'}) + (x_1 \otimes 1_{\alpha'}) (\alpha \otimes \alpha') (D_1 x_2 \otimes 1_{\alpha'})$$

$$= (D_1 x_1) \alpha x_2 \otimes 1_{\alpha'} + (x_1 \alpha (D_1 x_2)) \otimes 1_{\alpha'} = \left[(D_1 x_1) \alpha x_2 + x_1 \alpha (D_1 x_2) \right] \otimes 1_{\alpha'}$$

So, $D_1 (x_1 \alpha x_2) = (D_1 x_1) \alpha x_2 + x_1 \alpha (D_1 x_2)$. Therefore, D_1 is an α- derivation. The rest of the results are routine.

3.   THE NORM OF D

We now shift our attention to study the possibility of the result , $\| D \| = \| D_1 \| + \| D_2 \|$, when D, D_1 and D_2 are related as in Theorem 2.1.

THEOREM 3.1. If D, D_1 and D_2 are related as in Theorem 2.1, then
$$\| D \| \le \| D_1 \| + \| D_2 \| \le 2 \| D \|.$$

PROOF. For each $u \varepsilon (V, \Gamma) \otimes_p (V', \Gamma')$ with $\| u \|_p = 1$ and for each $\varepsilon > 0$, $\exists$ a (finite) representation

$$u = \sum_i x_i \otimes y_i \text{ such that } \| u \|_p + \varepsilon \ge \sum_i \| x_i \| \| y_i \|.$$

Now, $\|D\| = \sup_u \left\{ \| Du \|_p : \| u \|_p = 1 \right\}$

$$= \sup_{u} \left\{ \| \sum_i [D_1 x_i \otimes y_i + x_i \otimes D_2 y_i] \|_p : \| u \|_p = 1 \right\}$$

$$\leq \sup_{u} \left\{ \sum_i [\| D_1 x_i \otimes y_i \|_p + \| x_i \otimes D_2 y_i \|_p] : \| u \|_p = 1 \right\}$$

$$= \sup_{u} \left\{ \sum_i [\| D_1 x_i \| \| y_i \| + \| x_i \| \| D_2 y_i \|] : \| u \|_p = 1 \right\}$$

$$\leq \sup_{u} \left\{ \sum_i [\| D_1 \| \| x_i \| \| y_i \| + \| x_i \| \| D_2 \| \| y_i \|] : \| u \|_p = 1 \right\}$$

$$\leq (\| D_1 \| + \| D_2 \|) \sup_{u} \left\{ 1 + \varepsilon : \| u \|_p = 1 \right\}$$

$$= (\| D_1 \| + \| D_2 \|) (1 + \varepsilon)$$

Since ε was arbitrary, it follows that $\| D \| \leq \| D_1 \| + \| D_2 \|$ $\qquad$ (3 1)

Next, let $x \varepsilon V$ be such that $\| x \| = 1$. Then $\| x/k_2 \otimes 1_{\alpha'} \| = \| x/k_2 \| \| 1_{\alpha'} \| = 1$

Now, $\qquad \| D \| = \sup_{u} \left\{ \| Du \|_p : \| u \|_p = 1 \right\}$

$$\geq \| D (x/k_2 \otimes 1_{\alpha'}) \|_p = \| D_1 (x/k_2) \otimes 1_{\alpha'} \|_p , (\text{Since } D_2 (1_{\alpha'}) = 0) = \| D_1 x \|$$

Thus, $\| D_1 x \| \leq \| D \|$ for every $x \varepsilon V$ with $\| x \| = 1$. This gives $\| D_1 \| \leq \| D \|$. Similarly, we can prove that $\| D_2 \| \leq \| D \|$. Hence, we have $\| D_1 \| + \| D_2 \| \leq 2 \| D \|$ $\qquad$ (3.2)

The inequalities (3.1) ard (3.2) together imply $\quad \| D \| \leq \| D_1 \| + \| D_2 \| \leq 2 \| D \|.$ $\qquad$ Q.E.D.

Our next question is - can one improve the above result - ? We illustrate the possibility with the help of examples :

Let V be the set of 2×3 rectangular matrices and Γ be the set of all 3×2 rectangular matrices with real (or complex) entries. Then V and Γ are Banach spaces under usual matrix addition, scalar multiplication, and the norm defined by $\| A \|_\infty = \max_{ij} | a_{ij} |$, where $A = (a_{ij})$. Then (V, Γ) is a Γ- Banach algebra Now the following result is true :

THEOREM 3.2. For a fixed $\alpha \varepsilon \Gamma$, each α- derivation on V is inner.

Since α -derivations on a finite dimensional Γ-Banach algebra are all inner, the result follows immediately, see [10] .

We show below with an exampe in the Γ- Banach algebra of 2×3 rectangular matrices that the equality $\qquad \| D \| = \| D_1 \| + \| D_2 \|$ holds.

AN EXAMPLE 3.1.

Let $\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}$ be a fixed element in Γ, and let $D_{1\alpha}$ and $D_{2\alpha}$ be two α- derivations on V

implemented by A_o and B_o respectively, where $A_o = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ and $B_o = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix}$

Now $\quad \| A_o \| = 2$ and $\| B_o \| = 3$. and $D_{1\alpha} (A) = A_o \alpha A - A \alpha A_o , \forall A \varepsilon V$.

Then $\| D_{1\alpha} A \| \leq 2 \| A_0 \| \| \alpha \| \| A \| = 2 \| A_0 \| \| A \|$, because $\| \alpha \| = 1$.

Hence , $\quad \| D_{1\alpha} \| \leq 2 \| A_0 \| = 2.2 = 4$. Next , suppose that $X_o = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Then $\| X_o \| = 1$.

Also $\| A_o \alpha X_o - X_o \alpha A_o \| = \| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \| = 4.$ Hence $\| D_{1\alpha} \| = 4$

Similarly we can show that $\| D_{2\alpha} \| = 6.$ So $\| D_{1\alpha} \| + \| D_{2\alpha} \| = 4 + 6 = 10.$

If D is the derivation defined by the relation as in Theorem 3.1, then we always have

$$\| D \| \leq \| D_{1\alpha} \| + \| D_{2\alpha} \| = 10 \qquad (3.1)$$

Next, consider the element $u_o = e_1 \otimes e_1$, where $e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$ Then $\| u_o \|_p = 1.$

Now, $\| D \| \geq \| Du_o \|_p$

$$= \| D_{1\alpha} e_1 \otimes e_1 + e_1 \otimes D_{2\alpha} e_1 \|_p$$

$$\geq D_{1\alpha} e_1 \otimes e_1 + e_1 \otimes D_{2\alpha} e_1 \|_w$$

(because the projective norm is always greater than or equal to the weak norm)

$$= \sup \left\{ | f(D_{1\alpha} e_1) g(e_1) + f(e_1) g(D_{2\alpha} e_1) | : f, g \in V^*, \| f \| = \| g \| = 1 \right\} . \quad (3.2)$$

Again $D_{1\alpha} e_1 = A_o \alpha e_1 - e_1 \alpha A_o$

$$= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$D_{2\alpha} e_1 = B_o \alpha e_1 - e_1 \alpha B_o$

$$= \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

We know that if we define
$f_i(e_j) = 1$ if $i = j$ and $= 0$ if $i \neq j$, then $\left\{ f_1 , f_2 , f_3 , f_4 , f_5 , f_6 \right\}$ is a basis for V^*
In (3.2) put $f = g = f_1$. Then we find that $\| D \| \geq 10.$ ...   (3.3)
The inequalities (3.1) and (3.3) combinedly give $\| D \| = 10.$ Hence $\| D \| = \| D_{1\alpha} \| + D_{2\alpha} \|$

ANOTHER EXAMPLE 3.2.

Next we wish to illustrate that the result in Theorem 3 1 cannot be improved in general. If we assume
V and Γ represent the same set of all 2 x 2 real matrices, then (V, Γ) is a particular Γ - Banach

algebra with the usual operations. The ordinary identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity of (V, Γ) under
multiplication.

If $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $\beta = \{ e_1, e_2, e_3, e_4 \}$ is the standard basis for (V , Γ). For a simple example, let D_1 and D_2 be derivations on (V , Γ) implemented by the matrices $A_o = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and $B_o = \begin{pmatrix} 4 & -7 \\ 0 & 2 \end{pmatrix}$ respectively. Then the matrix representations of D_1 and D_2 with respect to the basis β are respectively

$$[D_1]_\beta = \begin{pmatrix} 0 & 0 & 3 & 0 \\ -3 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \quad \text{and} \quad [D_2]_\beta = \begin{pmatrix} 0 & 0 & -7 & 0 \\ 7 & 2 & 0 & -7 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 7 & 0 \end{pmatrix}$$

So, $\| D_1 \| = 3$ and $\| D_2 \| = 7$. Again, $\gamma = \{ e_i \otimes e_j \mid i, j = 1, 2, 3, 4 \}$ is a basis for $(V, \Gamma) \otimes_p (V, \Gamma)$ and the matrix representation of D with respect to the basis γ is

$$[D]_\gamma = \begin{bmatrix}
0 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 2 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & 1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 7 & 3 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 & -7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 7 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 7 & 0
\end{bmatrix}$$

Hence $\| D \| = 7$. Thus the strict inequality $\| D \| < \| D_1 \| + \| D_2 \| < 2 \| D \|$ holds.

4.   THE SPECTRUM OF D

We next devote to studying the validity of the result $\text{sp} (D) = \text{sp} (D_1) + \text{sp} (D_2)$. Recall that $\text{sp} (D_1)$ consists of all scalars λ_1 such that $D_1 - \lambda_1 I_1$ is singular. Analogous definitions apply to $\text{sp} (D_2)$ and $\text{sp} (D)$. Further, for the singularity and invertibility of a rectangular matrix, see . Joshi [11].

THEOREM 4.1. The derivations D, D_1 and D_2 are defined as in Theorem 2.1. Then

$$\text{sp} (D_1) + \text{sp} (D_2) \subsetneq \text{sp} (D)$$

PROOF. Let $\lambda_1 \, \varepsilon \, \text{sp} (D_1)$ and $\lambda_2 \, \varepsilon \, \text{sp} (D_2)$.

$\Rightarrow D_1 - \lambda_1 I_1$ and $D_2 - \lambda_2 I_2$ are singular

$\Rightarrow \exists$ nonzero vectors $x_o \, \varepsilon V$ and $y_o \, \varepsilon V'$ such that $(D_1 - \lambda_1 I_1) \, x_o = o$ and $(D_2 - \lambda_2 I_2) \, y_o = 0$

Now, $x_o \otimes y_o$ is a non-zero element in $(V, \Gamma) \otimes_p (V', \Gamma')$.

Again, $[D - (\lambda_1 + \lambda_2) I] (x_0 \otimes y_0) = D (x_0 \otimes y_0) - (\lambda_1 + \lambda_2) (x_0 \otimes y_0)$

$$= D_1 x_0 \otimes y_0 + x_0 \otimes D_2 y_0 - (\lambda_1 + \lambda_2) x_0 \otimes y_0$$

$$= (D_1 - \lambda_1 I_1) x_0 \otimes y_0 + x_0 \otimes (D_2 - \lambda_2 I_2) y_0 = 0$$

So, $D - (\lambda_1 + \lambda_2) I$ is singular and hence $\lambda_1 + \lambda_2 \varepsilon$ sp (D). Thus, we obtain sp $(D_1) +$ sp $(D_2) \subseteq$ sp (D). Q.E.D

REMARK 4.1. (i) We conjecture that the above result cannot be improved in general.

(ii) However, the equality holds in finite dimensional Γ- Banach algebras. For, if dim (V, Γ) = m, dim (V',Γ') = n, then dim $((V, \Gamma) \otimes_p (V', \Gamma')) = mn$. So, sp (D_1), sp (D_2) and sp (D) have m,n and mn eigenvalues respectively. Again, sp $(D_1) +$ sp (D_2) gives mn values which are precisely the eigenvalues of D.

Further, we have the following illuminating result.

THEOREM 4.2. As uusal, let D_1, D_2 and D be derivations connected by the relation as in Theorem 2.1(i). If (V, Γ) and (V,' Γ) are finite dimensional Gamma-Banach algebras, D_1 and D_2 are implemented by $r \varepsilon$ V and $s \varepsilon$ V' respectively, then

$$sp (D_1) = \{ a = \lambda - \mu \mid \lambda, \mu \varepsilon sp (r) \},$$

$$sp (D_2) = \{ b = \lambda' - \mu' \mid \lambda', \mu' \varepsilon sp (s) \}$$

and   sp $(D) = \{ a + b \mid a \varepsilon$ sp $(D_1),$ b ε sp $(D_2) \}$.

PROOF. The first two results will follow from Propostion 9,§18, Ch2 in [10], and the last result will follow from Remark 4.1 (ii).   Q.E.D.

REFERENCES

[1] BHATTACHARYA, D.K. and MAITY, A.K., Regular representation of Γ-Banach Algebra, *J. of Pure Mathematics, Calcutta University, India,* (To appear).

[2.] BHATTACHARYA, D.K. and MAITY, A.K., Semilinear tensor product of Γ-Banach algebras, *Ganita* Vol. **40.** No. 2, (1989), 75-80.

[3] GREUB, W.H., *Multilinear algebra,* Springer Verlag, 1967.

[4.] BADE, W.G. and DALES, H.G., Discontinuous derivations from algebras of power series, *Proc. London Math. Soc.* (3), **69** No. 1 (1989), 133 - 152.

[5.] CARNE, T.K , Tensor products of Banach algebras, *J. London Math. Soc.* (2), **17** (1978), 480-88

[6] CHUANKUN SUN, The essential norm of the generalized derivation, *Chinese Anna. Math.,* **13A :2** (1992), 211-221.

[7] KYLE, J., Norms of derivations, *J. London Math. Soc.* (2), **16** (1977), 297-312

[8] VUKMAN, J., A result concerning derivations in Banach algebras, *Proc. Amer. Math. Soc.,* Vol **116,** No. 4 (December 1992), 971-975.

[9] BACKMAN, G., *Introduction to p-adic numbers and valuation theory,* Academic Press, 1964

[10]. BONSALL, F.F. and DUNCAN, J., *Complete normed algebras,* Springer Verlag, 1973

[11]. JOSHI, V.N.,A determinant for rectangular matrices, *Bull. Austral. Math. Soc.,*(Series A), Vol **21** (1980), 137-146

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