

ON STRICT AND SIMPLE TYPE EXTENSIONS

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ABSTRACT. Let (Y, τ) be an extension of a space (X, τ') . $p \in Y$, let $\mathcal{O}_y^p = \{W \cap X : W \in \tau, p \in W\}$. For $U \in \tau'$, let $o(U) = \{p \in Y : U \in \mathcal{O}_y^p\}$. In 1964, Banaschewski introduced the strict extension $Y^\#$, and the simple extension Y^* of X (induced by (Y, τ)) having base $\{o(U) : U \in \tau'\}$ and $\{U \cup \{p\} : p \in Y, \text{ and } U \in \mathcal{O}_y^p\}$, respectively. The extensions $Y^\#$ and Y^* have been extensively used since then. In this paper, the open filters $\mathcal{A}^p = \{W \in \tau' : W \supseteq \text{int}_X \text{cl}_X(U) \text{ for some } U \in \mathcal{O}_y^p\}$, and $\mathcal{U}^p = \{W \in \tau' : \text{int}_X \text{cl}_X(W) \in \mathcal{O}_y^p\} = \{W \in \tau' : \text{int}_X \text{cl}_X(W) \in \mathcal{A}^p\} = \bigcap \{\mathcal{U} : \mathcal{U} \text{ is an open ultrafilter on } X, \mathcal{O}_X^p \subset \mathcal{U}\}$ on X are used to define some new topologies on Y . Some of these topologies produce nice extensions of (X, τ') . We study some interrelationships of these extensions with $Y^\#$, and Y^* respectively.

KEY WORDS AND PHRASES: Extension, simple extension, strict extension, H-closed, s-closed, almost realcompact, near compact.

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1. INTRODUCTION

A topological space Y is an *extension* of a space X if X is a dense subspace of Y . If Y_1 and Y_2 are two extensions of a space X , then Y_2 is said to be *projectively larger* than Y_1 , written $Y_2 \geq Y_1$ (or $Y_1 \leq Y_2$), provided that there exists a continuous map $f: Y_2 \rightarrow Y_1$ such that $f|_X = i_X$, the identity map on X . Two extensions Y_1 and Y_2 of X are called *equivalent* if $Y_1 \leq Y_2$ and $Y_2 \leq Y_1$. We shall identify two equivalent extensions of X . With this convention, the class $E(X)$ of all the Hausdorff extensions of a Hausdorff space X is a set. Let $(Y, \tau) \in E(X)$ and let $p \in Y$. If N_p is the open neighborhood filter of p in Y , the set $\mathcal{O}_Y^p = \{N \cap X : N \in N_p\}$ (called the *trace* of N_p on X) is an open filter on X . If U is open in X , denote

$$o_Y(U) = \{p \in Y : U \in \mathcal{O}_Y^p\}.$$

In 1964 Banaschewski [1] introduced the extensions $Y^\#$ (resp. Y^+) the *strict extension* (resp. the *simple extension*) of X induced by Y satisfying $Y^\# \leq Y \leq Y^+$. The topology $\tau^\#$ on $Y^\#$ (resp. τ^+ on Y^+) has for an open base the collection $\{o_Y(U): U \text{ open in } X\}$ (resp., the collection $\{U \cup \{p\}: p \in Y, \text{ and } U \in \mathcal{O}_Y^p\}$). The extensions $Y^\#$, and Y^+ have been studied extensively and have proved extremely useful regarding some properties weaker than compactness, such as nearly compact, almost realcompact, feebly compact, H -closed, s -closed, etc.. In this paper we introduce new extensions Y^j , Y^u , Y^{j*} , and Y^{u*} , study some of their properties, and compare them with Y , $Y^\#$, and Y^+ . All spaces under consideration are Hausdorff.

2. THE EXTENSIONS Y^j AND Y^u .

In this section, we introduce several topologies on Y , and compare them with τ . Some of these topologies yield interesting extensions of (X, τ') .

DEFINITION 2.1. Let (Y, τ) be an extension of a space (X, τ') . For $p \in Y$ define

$$\mathcal{U}^p = \{W: W \in \tau', \text{int}_X \text{cl}_X W \in \mathcal{O}_Y^p\}, \quad (2.1)$$

$$\mathcal{L}^p = \{W: W \in \tau', W \supseteq \text{int}_X \text{cl}_X U \text{ for some } U \in \mathcal{O}_Y^p\}. \quad (2.2)$$

LEMMA 2.1.

(a) Both \mathcal{U}^p and \mathcal{L}^p are open filters on X such that $\mathcal{L}^p \subseteq \mathcal{O}_Y^p \subseteq \mathcal{U}^p$.

(b) $\mathcal{U}^p = \{W: W \in \tau', \text{int}_X \text{cl}_X W \in \mathcal{L}^p\}$
 $= \cap \{\mathcal{U}: \mathcal{U} \text{ is an open ultrafilter on } X, \mathcal{O}_Y^p \subset \mathcal{U}\}$

PROOF. We prove (b). Let $\mathcal{W} = \{W: W \in \tau', \text{int}_X \text{cl}_X W \in \mathcal{L}^p\}$. If $W \in \mathcal{W}$, then $W \in \tau'$ and $\text{int}_X \text{cl}_X W \supseteq \text{int}_X \text{cl}_X U$ for some $U \in \mathcal{O}_Y^p$. Therefore, $\text{int}_X \text{cl}_X W \in \mathcal{O}_Y^p$, whence $W \in \mathcal{U}^p$. Thus, $\mathcal{W} \subseteq \mathcal{U}^p$. To prove the reverse inequality, let $W \in \mathcal{U}^p$. Then $\text{int}_X \text{cl}_X W \in \mathcal{O}_Y^p$. Since $\text{int}_X \text{cl}_X W \supseteq \text{int}_X \text{cl}_X (\text{int}_X \text{cl}_X W)$ it follows that $\text{int}_X \text{cl}_X W \in \mathcal{L}^p$. Hence $W \in \mathcal{W}$. This proves the first equality in (b). The second equality follows from [9], completing the proof of the lemma.

REMARK 2.1. Since $\mathcal{O}_Y^p = \mathcal{O}_Y^p \# = \mathcal{O}_Y^p + [9, 10, 11]$, it follows that each one of Y , Y^+ and $Y^\#$ yield the same \mathcal{L}^p (resp., \mathcal{U}^p) for all $p \in Y$. Moreover, if $Z \in E(X)$ has the same underlying set as Y , and is such that $Y^\# \leq Z \leq Y^+$, then Y and Z induce the same \mathcal{L}^p (resp., \mathcal{U}^p) for all $p \in Y$. Also, if $p \neq q$ are distinct elements of Y then $\mathcal{L}^p \neq \mathcal{L}^q$ and $\mathcal{U}^p \neq \mathcal{U}^q$. Obviously, if $U \in \mathcal{O}_Y^p$, then $\text{int}_X \text{cl}_X(U) \in \mathcal{L}^p$. Moreover, $U \in \mathcal{U}^p$ if and only if $\text{int}_X \text{cl}_X(U) \in \mathcal{U}^p$.

DEFINITION 2.2. Let (Y, τ) be an extension of (X, τ') . For $G \in \tau'$, define

$$o_i(G) = G \cup \{p: p \in Y \setminus X, G \in \mathcal{L}^p\} \quad (2.3)$$

$$o_i(G) = G \cup \{p: p \in Y \setminus X, G \in \mathcal{U}^p\} \quad (2.4)$$

$$a_i(G) = \{p \in Y: G \in \mathcal{L}^p\} \quad (2.5)$$

$$a_u(G) = \{p \in Y: G \in \mathcal{U}^p\} \quad (2.6)$$

The proof of the Propositions 2.1, and 2.2 is straightforward.

PROPOSITION 2.1. Let (Y, τ) be an extension of (X, τ') . Then for all $U, V \in \tau'$

- (a) $o_i(\emptyset) = \emptyset, o_i(X) = Y$,
- (b) $o_i(U) \cap X = U$,
- (c) $o_i(U \cap V) = o_i(U) \cap o_i(V)$,
- (d) The family $\{o_i(G): G \in \tau'\}$ is an open base for a Hausdorff topology τ_i on Y and (Y, τ_i) is an extension of X .

PROPOSITION 2.2. Let (Y, τ) be an extension of (X, τ') . Then for all $U, V \in \tau'$,

- (a) $o_u(\emptyset) = \emptyset$ and $o_u(X) = Y$,
- (b) $o_u(U) \cap X = U$,
- (c) $o_u(U \cap V) = o_u(U) \cap o_u(V)$,
- (d) The family $\{o_u(G): G \in \tau'\}$ is an open base for a Hausdorff topology τ_u on Y and (Y, τ_u) is an extension of X .

PROPOSITION 2.3. Let (Y, τ) be an extension of (X, τ') . Then for all $U, V \in \tau'$

- (a) $a_i(\emptyset) = \emptyset, a_i(X) = Y$,
- (b) $a_i(U) \cap X \subseteq U$,
- (c) $a_i(U \cap V) = a_i(U) \cap a_i(V)$,
- (d) $a_i(U) = \cup \{W: W \in \tau \text{ and } \text{int}_X \text{cl}_X(W \cap X) \subseteq U\}$
- (e) The family $\{a_i(G): G \in \tau'\}$ is an open base for a coarser Hausdorff topology τ_{ai} on Y , X is dense in (Y, τ_{ai}) , but (Y, τ_{ai}) may not be an extension of X .

PROOF. We prove (d). The rest is straightforward. Let $p \in a_i(U)$. Then $U \in \mathcal{L}^p$. Therefore, $U \supseteq \text{int}_X \text{cl}_X V$ for some $V \in \mathcal{O}_Y^p$. Therefore, there exists $W \in \tau$ such that $p \in W$ and $W \cap X = V$. It follows that $\text{int}_X \text{cl}_X(W \cap X) \subseteq U$. Conversely, if $W \in \tau$ is such that $\text{int}_X \text{cl}_X(W \cap X) \subseteq U$ and $p \in W$, then $W \cap X \in \mathcal{O}_Y^p$. So, $\text{int}_X \text{cl}_X(W \cap X) \in \mathcal{L}^p$. This implies that $U \in \mathcal{L}^p$ and hence $p \in a_i(U)$. The proof of the proposition is now complete.

PROPOSITION 2.4. Let (Y, τ) be an extension of (X, τ') . Then for all $U, V \in \tau'$,

- (a) $a_u(\emptyset) = \emptyset$ and $a_u(X) = Y$,

- (b) $a_u(U) \cap X = \text{int}_X \text{cl}_X(U)$,
- (c) $a_u(U \cap V) = a_u(U) \cap a_u(V)$,
- (d) $a_u(U) = \bigcup \{W : W \in \tau \text{ and } W \cap X \subseteq \text{int}_X \text{cl}_X(U)\}$
- (e) The family $\{a_u(G) : G \in \tau'\}$ is an open base for a coarser Hausdorff topology τ_{au} on Y , X is dense in (Y, τ_{au}) , but (Y, τ_{au}) may not be an extension of X .

PROOF. We prove (d). The rest is straightforward. Let $p \in a_u(U)$. Then $U \in \mathcal{U}^p$. Therefore, $\text{int}_X \text{cl}_X U \in \mathcal{O}_Y^p$. It follows that there exists $W \in \tau$ such that $p \in W$ and $W \cap X \subseteq \text{int}_X \text{cl}_X U$. Conversely, if $W \in \tau$ is such that $W \cap X \subseteq \text{int}_X \text{cl}_X U$ and $p \in W$, then $W \cap X \in \mathcal{O}_Y^p$. So, $\text{int}_X \text{cl}_X U \in \mathcal{O}_Y^p$. Therefore, $U \in \mathcal{U}^p$ and $p \in a_u(U)$.

DEFINITION 2.3. The spaces (Y, τ_l) , (Y, τ_u) , (Y, τ_{al}) , and (Y, τ_{au}) described in propositions 2.1-2.4 will, henceforth, be denoted by Y^l , Y^u , Y^{al} , and Y^{au} respectively. If $A \subseteq Y$, then $\text{int}_{Y^l}(A)$ (resp. $\text{cl}_{Y^l}(A)$) will be denoted by $\text{int}_l(A)$ (resp., $\text{cl}_l(A)$). Likewise, $\text{int}_u(A)$, $\text{cl}_u(A)$, $\text{int}_{al}(A)$, $\text{cl}_{al}(A)$, $\text{int}_{au}(A)$, and $\text{cl}_{au}(A)$ are defined in an analogous manner.

LEMMA 2.2. If $U \in \tau'$, then

- (a) $a_l(U) \subseteq o_l(U) \subseteq o_r(U) \subseteq o_u(U) \subseteq o_u(\text{int}_X \text{cl}_X U) = a_u(U) = a_u(\text{int}_X \text{cl}_X U)$,
- (b) $a_l(U) \setminus X = o_l(U) \setminus X$, and $a_u(U) \setminus X = o_u(U) \setminus X$
- (c) $o_l(\text{int}_X \text{cl}_X U) \setminus X = o_u(U) \setminus X$, and
- (d) if U is regular open (i.e. $U = \text{int}_X \text{cl}_X U$), then $a_u(U) = a_l(U)$, and the equality holds in (a).

PROOF. Part (a): We show that $o_u(\text{int}_X \text{cl}_X U) = a_u(U)$, the rest being straightforward. Certainly, $o_u(\text{int}_X \text{cl}_X U) \cap X = \text{int}_X \text{cl}_X U = a_u(U) \cap X$. Let $p \in o_u(\text{int}_X \text{cl}_X U) \setminus X$. Then $\text{int}_X \text{cl}_X U \in \mathcal{U}^p$. Therefore, $U \in \mathcal{U}^p$, and $p \in a_u(U) \setminus X$. Conversely, let $p \in a_u(U) \setminus X$. Then, $U \in \mathcal{U}^p$. So, $p \in o_u(U) \setminus X \subseteq o_u(\text{int}_X \text{cl}_X U) \setminus X$. The above arguments prove (a).

To prove (c), let $q \in o_l(\text{int}_X \text{cl}_X G) \setminus X$. Then, $\text{int}_X \text{cl}_X G \in \mathcal{L}^q$ whence, $G \in \mathcal{U}^q$. Therefore, $q \in o_u(G) \setminus X$. Thus, $o_l(\text{int}_X \text{cl}_X G) \setminus X \subseteq o_u(G) \setminus X$. To prove the reverse inequality, let $q \in o_u(G) \setminus X$. Then, $G \in \mathcal{U}^q$, whence $\text{int}_X \text{cl}_X G \in \mathcal{L}^q$. Therefore, $q \in o_l(\text{int}_X \text{cl}_X G) \setminus X$ and $o_u(G) \setminus X \subseteq o_l(\text{int}_X \text{cl}_X G) \setminus X$. Hence, $o_l(\text{int}_X \text{cl}_X G) \setminus X = o_u(G) \setminus X$. The rest of the lemma is straightforward.

Given a space (X, τ') , the family $\{\text{int}_X \text{cl}_X U : U \in \tau'\}$ forms an open base for a coarser Hausdorff topology τ'_s on X . The space $X_s = (X, \tau'_s)$ is called the *semiregularization* of X . A space (X, τ') is called *semiregular* if $(X, \tau') = X_s$.

THEOREM 2.1. If X is semiregular, and (Y, τ) (not necessarily semiregular) is an extension of X , then Y^l is an extension of X such that $Y^l \leq Y$.

PROOF. If X is semiregular, then $o_l(U) = a_l(U)$ for all $U \in \tau'$. Hence, Y^l is an extension of X such that $Y^l = Y^{al} \leq Y$.

THEOREM 2.2. The spaces Y^{al} and Y^{au} are homeomorphic.

PROOF. For all $U \in \tau'$, $a_l(\text{int}_X \text{cl}_X U) = o_u(\text{int}_X \text{cl}_X U) = a_u(U)$ implies that $\tau_{au} \subseteq \tau_{al}$. Also, if $G \in \tau'$ and $p \in a_l(G)$, then $G \supseteq \text{int}_X \text{cl}_X(U)$ for some $U \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$. Now, if $q \in a_u(U)$, then $\text{int}_X \text{cl}_X U \in \mathcal{L}^q$ which implies that $G \in \mathcal{L}^q$, or $q \in a_l(G)$. Therefore, $p \in a_u(U) \subseteq a_l(G)$. Hence, $\tau_{al} \subseteq \tau_{au}$. This proves the theorem.

LEMMA 2.3. Let (Y, τ) be an extension of (X, τ') . Then, for all $G \in \tau'$ the following are true.

- (a) $\text{cl}_{al}(G) \subseteq \text{cl}_l(G) = \text{cl}_l(\text{int}_X \text{cl}_X(G))$,
- (b) $\text{cl}_u(G) = \text{cl}_{au}(G) = \text{cl}_u(\text{int}_X \text{cl}_X(G))$,
- (c) $\text{cl}_u(G) = \text{cl}_l(G)$,
- (d) $\text{cl}_Y(G) = \text{cl}_{au}(G) = \text{cl}_{al}(\text{int}_X \text{cl}_X(G))$, and
- (e) $\text{cl}_u(o_u(G)) = \text{cl}_{au}(a_u(\text{int}_X \text{cl}_X(G)))$

PROOF. Part (a): Let $p \in \text{cl}_{al}(G)$, and let $o_l(U)$ be a basic open neighborhood of p in Y^l . If $p \in o_l(U) \cap X$, then $p \in U \subseteq \text{int}_X \text{cl}_X U \in \mathcal{L}^p$. Therefore, $a_l(\text{int}_X \text{cl}_X U)$ is an open neighborhood of p in Y^{al} . Consequently, $a_l(\text{int}_X \text{cl}_X U) \cap G \neq \emptyset$. By Proposition (2.7) (b), $\text{int}_X \text{cl}_X U \cap G \neq \emptyset$. Hence $U \cap G \neq \emptyset$. This in turn implies that $o_l(U) \cap G \neq \emptyset$, and $p \in \text{cl}_l(G)$. If $p \in o_l(U) \setminus X$, then $U \in \mathcal{L}^p$. Now, $a_l(U)$ is an open neighborhood of p in Y^{al} . Consequently, $a_l(U) \cap G \neq \emptyset$. Therefore, $o_l(U) \cap G \neq \emptyset$ whence $p \in \text{cl}_l(G)$.

Part (b): Let $p \in \text{cl}_{au}(G)$, and let $o_u(U)$ be a basic open neighborhood of p in Y^u . Since $o_u(U) \subseteq a_u(U)$, $a_u(U)$ is an open neighborhood of p in Y^{au} . Hence, $a_u(U) \cap G \neq \emptyset$. Therefore, $\text{int}_X \text{cl}_X U \cap G \neq \emptyset$, whence $U \cap G \neq \emptyset$. Consequently, $o_u(U) \cap G \neq \emptyset$. Therefore, $p \in \text{cl}_u(G)$. Therefore, $\text{cl}_{au}(G) \subseteq \text{cl}_u(G)$. Conversely, let $p \in \text{cl}_u(G)$, and let $a_u(U)$ be a basic open neighborhood of p in Y^{au} . If $p \in a_u(U) \cap X = \text{int}_X \text{cl}_X U$, then $o_u(\text{int}_X \text{cl}_X U)$ is an open neighborhood of p in Y^u . Therefore, $a_u(U) \cap X = \text{int}_X \text{cl}_X U \cap G = o_u(\text{int}_X \text{cl}_X U) \cap G \neq \emptyset$. Hence, $p \in \text{cl}_{au}(G)$. Now, if $p \in a_u(U) \setminus X$, then $U \in \mathcal{U}^p$ and $o_u(U)$ is an open neighborhood of p in Y^u . Therefore, $o_u(U) \cap G \neq \emptyset$. Consequently, $a_u(U) \cap G \neq \emptyset$, and $p \in \text{cl}_{au}(G)$. Therefore, $\text{cl}_u(G) \subseteq \text{cl}_{au}(G)$. Hence, $\text{cl}_u(G) = \text{cl}_{au}(G)$. The other half of (b) is straightforward.

The proof of (c) is straightforward.

Part (d): Let $p \in \text{cl}_{au}(G)$, and let W be an open neighborhood of p in Y . Then, $W \cap X \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$ shows that $o_u(W \cap X)$ is an open neighborhood of p in Y^{au} . Therefore, $a_u(W \cap X) \neq \emptyset$. This shows that $W \cap G \neq \emptyset$, whence $p \in \text{cl}_Y(G)$. Conversely, let $p \in \text{cl}_Y(G)$, and let $a_u(U)$ be a basic open neighborhood of p in Y^{au} . Then, $U \in \mathcal{U}^p$. So, $o_Y(\text{int}_X \text{cl}_X U)$ is an open neighborhood of p in Y such that $o_Y(\text{int}_X \text{cl}_X U) \cap G \neq \emptyset$. This implies that $a_u(U) \cap G \neq \emptyset$. Hence, $p \in \text{cl}_{au}(G)$. The rest follows from (c).

THEOREM 2.3. The spaces $Y^l \setminus X$, $Y^{al} \setminus X$, and $Y^u \setminus X$ are pairwise homeomorphic.

PROOF. To prove the continuity of the identity map $i: Y^u \setminus X \rightarrow Y^l \setminus X$, let $o_l(G) \setminus X$ be a basic open neighborhood of p in $Y^l \setminus X$. Then, $G \in \mathcal{A}^p$. Hence $G \supseteq \text{int}_X \text{cl}_X U$ for some $U \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$. Therefore, $o_u(U) \setminus X$ is an open neighborhood of p in Y^u such that $o_u(U) \setminus X \subseteq o_l(G) \setminus X$. To prove that the identity map $i: Y^l \setminus X \rightarrow Y^u \setminus X$ is continuous, let $o_u(G) \setminus X$ be a basic open neighborhood of p in $Y^u \setminus X$. Then $o_l(\text{int}_X \text{cl}_X G) \setminus X$ is an open neighborhood of p in $Y^l \setminus X$ such that $o_l(\text{int}_X \text{cl}_X G) \setminus X = o_u(G) \setminus X$. Hence, the spaces $Y^l \setminus X$, and $Y^u \setminus X$ are homeomorphic. The rest of the theorem follows directly from Lemma 2.2.

Let Z_1 and Z_2 be spaces. A map $f: Z_1 \rightarrow Z_2$ is called θ -continuous [3] if for every $p \in Z_1$ and for every open neighborhood V of $f(p)$ in Z_2 , there exists an open neighborhood U of p in Z_1 such that $f(\text{cl}_{Z_1} U) \subseteq \text{cl}_{Z_2}(V)$. f is called *perfect* if f is a closed map (not necessarily continuous) such that $f^{-1}(z)$ is compact in Z_1 for every $z \in Z_2$. Also, f is called *irreducible* if f is closed and there is no proper closed subset K of Z_1 for which $f(K) = Z_2$. Two extensions Z_1 , and Z_2 of a space X are called θ -equivalent if there exists a θ -homeomorphism f from Z_1 onto Z_2 such that $f|_X = i_X$, the identity map on X .

The next theorem depicts some of the several interrelationships between the spaces Y , $Y^\#$, Y^l , Y^u , and Y^{al} .

THEOREM 2.4. Let (Y, τ) be an extension of a space (X, τ') . The following statements are true.

- (a) The identity map $i: Y^{al} \rightarrow Y$ is perfect, irreducible and θ -continuous.
- (b) The identity map $i: Y^{au} \rightarrow Y^u$ is perfect, irreducible and θ -continuous.
- (c) The identity map $i: Y^{al} \rightarrow Y^\#$ is θ -continuous.
- (d) The identity map $i: Y^\# \rightarrow Y^l$ is θ -continuous.
- (e) The identity map $i: Y^\# \rightarrow Y^u$ is θ -continuous.
- (f) The identity map $i: Y^l \rightarrow Y^\#$ is θ -continuous.
- (g) The identity map $i: Y^u \rightarrow Y^\#$ is θ -continuous.
- (h) The identity map $i: Y^l \rightarrow Y^u$ is θ -continuous.
- (i) The identity map $i: Y^u \rightarrow Y^l$ is θ -continuous.
- (j) The identity map $i: Y^l \rightarrow Y$ is θ -continuous.
- (k) The identity map $i: Y^u \rightarrow Y$ is θ -continuous.
- (l) The identity map $i: Y^\# \rightarrow Y^{al}$ is θ -continuous.

PROOF. Below, we outline the proofs of some parts of the theorem. The rest of the proofs are analogous.

Part (a) Since $\tau_{al} \subseteq \tau$, $i: Y \rightarrow Y^{al}$ is continuous. Hence, $i: Y \rightarrow Y^{al}$ is irreducible and perfect. To prove the θ -continuity of $i: Y^{al} \rightarrow Y$, let V be an open neighborhood of p in Y . Then $V \cap X \in \mathcal{O}_X^p$ and $\text{int}_X \text{cl}_X (V \cap X) \in \mathcal{A}^p$. Therefore, $a_l(\text{int}_X \text{cl}_X (V \cap X))$ is an open neighborhood of p in Y^{al} such that

$\text{cl}_{a_l}(a_l(\text{int}_X \text{cl}_X(V \cap X))) = \text{cl}_Y(a_l(\text{int}_X \text{cl}_X(V \cap X))) = \text{cl}_Y[a_l(\text{int}_X \text{cl}_X(V \cap X)) \cap X]$
 $= \text{cl}_Y(\text{int}_X \text{cl}_X(V \cap X)) \subseteq \text{cl}_Y(V)$. Hence $i: Y^{al} \rightarrow Y$ is θ -continuous.

Part (b): For all $G \in \tau', a_u(G) = o_u(\text{int}_X \text{cl}_X G) \in \tau_u$ shows that $i: Y^u \rightarrow Y^{au}$ is continuous. Therefore, $i: Y^{au} \rightarrow Y$ is irreducible and perfect. Let $o_u(G)$ be a basic open neighborhood of p in Y^u . Since $o_u(G) \subseteq a_u(G), a_u(G)$ is an open neighborhood of p in Y^{au} such that $\text{cl}_{au}(a_u(G)) = \text{cl}_u(o_u(G))$, establishing the θ -continuity of $i: Y^{au} \rightarrow Y^u$.

Part (c): To prove the θ -continuity of $i: Y^{al} \rightarrow Y^{\#}$, let $p \in Y$ and let $o_Y(G), G \in \tau'$ be a basic open neighborhood of p in $Y^{\#}$. Then, $G \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$ implies that $a_l(\text{int}_X \text{cl}_X G)$ is an open neighborhood of p in Y^{al} such that $\text{cl}_{a_l}(a_l(\text{int}_X \text{cl}_X G)) \subseteq \text{cl}_Y \#(o_Y(G))$.

Part (d): Let $o_l(G)$ be a basic open neighborhood of p in Y^l . Then, $o_Y(G)$ is an open neighborhood of p in $Y^{\#}$ such that $\text{cl}_Y \#(o_Y(G)) \subseteq \text{cl}_l(o_l(G))$, establishing the θ -continuity of $i: Y^{\#} \rightarrow Y^l$.

Part (h): Let $o_u(G)$ be a basic open neighborhood of p in Y^u . Then, $o_l(\text{int}_X \text{cl}_X G)$ is an open neighborhood of p in Y^l satisfying $\text{cl}_l(o_l(\text{int}_X \text{cl}_X G)) \subseteq \text{cl}_u(o_u(G))$.

Part (i): Let $p \in Y$ and let $a_l(G), G \in \tau'$ be a basic open neighborhood of p in Y^{al} . Then, $G \supseteq \text{int}_X \text{cl}_X U$ for some $U \in \mathcal{O}_Y^p$. So, $p \in o_Y(U)$. Now, $\text{cl}_Y(o_Y(U)) = \text{cl}_Y(o_Y(U) \cap X) = \text{cl}_Y(U) \subseteq \text{cl}_{a_l}(U) \subseteq \text{cl}_{a_l}(a_l(G))$.

We now summarize the results proved above in the following theorem.

THEOREM 2.5. The spaces $Y, Y^{\#}, Y^l, Y^u$, and Y^{al} , are pairwise θ -homeomorphic. The spaces Y^l , and Y^u are θ -equivalent extensions of X with homeomorphic remainders.

It is well known that spaces Y and Z are θ -homeomorphic if and only if their semiregularizations are homeomorphic. [11] Hence, we have the following corollary.

COROLLARY 2.1. Let (Y, τ) be an extension of a space (X, τ) . Then, the spaces $Y_s, Y_s^{\#}, Y_s^l, Y_s^u$, and Y_s^{al} are pairwise homeomorphic. Moreover, Y_s, Y_s^l , and Y_s^u are equivalent extensions of X_s .

3. THE EXTENSIONS Y^{l*} , AND Y^{u*} .

In this section, we define extensions Y^{l*} , and Y^{u*} , analogous to the simple extension Y^+ of (X, τ') induced by an extension (Y, τ) of X . The spaces Y^{l*}, Y^{al*}, Y^{u*} , and Y^{au*} all have the same underlying set as the set Y . An open base for the topology τ_{l*} on Y^{l*} (respectively, τ_{al*} on Y^{al*}) is the family $\tau' \cup \{G \cup \{p\} : p \in Y \setminus X, G \in \mathcal{A}^p\}$ (respectively, $\tau' \cup \{G \cup \{p\} : G \in \mathcal{A}^p\}$). An open base for the topology τ_{u*} on Y^{u*} (respectively, τ_{au*} on Y^{au*}) is the family $\tau' \cup \{G \cup \{p\} : p \in Y \setminus X, G \in \mathcal{U}^p\}$ (respectively, $\tau' \cup \{G \cup \{p\} : G \in \mathcal{U}^p\}$). For any $A \subset Y, \text{cl}_{l*}(A)$ will denote the closure of A in Y^{l*} , with analogous notations in other cases. The proofs of the following statements are straightforward, and we omit the details. Obviously, the spaces $Y^{l*} \setminus X, Y^{u*} \setminus X, Y^{al*} \setminus X$, and $Y^{au*} \setminus X$ are all discrete.

THEOREM 3.1. The spaces Y^{l*} , and Y^{u*} are extensions of (X, τ') such that $Y^{u*} \geq Y^+ \geq Y^{l*}$. The set X is dense in the spaces Y^{al*} , and Y^{au*} . But, Y^{al*} and Y^{au*} may not be extensions of X .

LEMMA 3.1. For each $G \in \tau'$, $\text{cl}_{l_*}(o_l(G)) = \text{cl}_l(o_l(G))$, and $\text{cl}_{u_*}(o_u(G)) = \text{cl}_u(o_u(G))$.

THEOREM 3.2. Each one of the identity maps $i: Y^+ \rightarrow Y^{u*}$, and $i: Y^{l*} \rightarrow Y^+$ is θ -continuous.

THEOREM 3.3. The spaces Y^+ , Y^{l*} , Y^{u*} , Y^{al*} , and Y^{au*} are θ -homeomorphic. Moreover, Y^+ , Y^{l*} , and Y^{u*} are θ -equivalent extensions of X with homeomorphic remainders.

COROLLARY 3.1. If (Y, τ) is an extension of a space (X, τ') , then the spaces Y_s^+ , Y_s^{l*} , Y_s^{u*} , Y_s^{al*} , and Y_s^{au*} are homeomorphic in pairs. Moreover, the spaces Y_s^+ , Y_s^{l*} , and Y_s^{u*} are equivalent extensions of X_s .

REMARKS 3.1. (a) If \mathbf{P} is any property of topological spaces which is preserved under θ -continuous surjections, and if (Y, τ) is a \mathbf{P} -extension of (X, τ') , then Y^l , Y^u , Y^{l*} , and Y^{u*} are also \mathbf{P} -extensions of X .

(b) The extensions Y^l , Y^u , Y^{l*} , and Y^{u*} introduced above are, in general, all distinct from Y , $Y^\#$, and Y^* . It would be interesting to find a characterization of spaces Y for which $Y^\# = Y^l$. A space Z is called H -closed if it is closed in every Hausdorff space in which it is embedded [see 11 for more details]. The Katetov (respectively, Fomin) extension of a space (X, τ') is the space κX (respectively, σX) whose underlying set is the set $X \cup \{p: p \text{ is a free open ultrafilter on } X\}$, and whose topology has for an open base the family $\tau' \cup \{U \cup \{p\}: U \in p, \text{ and } p \in \kappa X \setminus X\}$ (respectively, the family $\{o_{\kappa X}(U): U \in \tau'\}$). The spaces κX , and σX are H -closed extensions of X such that $(\sigma X)^+ = \kappa X$, and $(\kappa X)^* = \sigma X$ [3, 6, 11]. In general $(\sigma X)^l \neq \sigma X$, $(\kappa X)^u \neq \kappa X$, $(\sigma X)^{u*} \neq \sigma X$, and $(\kappa X)^{l*} \neq \kappa X$. Analogous remarks apply to the Banaschewski-Fomin-Shanin extension μX [13] of a Hausdorff space X .

(c) A space Z is called *compact like*, or *nearly compact* if every regular open cover of Z is reducible to a finite subcover. A space X has a compactlike extension if and only if X_s is Tychonoff [14]. Compactlike extensions ($=$ near compactifications) of Hausdorff almost completely regular spaces X (whence, X_s is Tychonoff) have been constructed in [2] via EF-Proximities. For a Hausdorff space X whose semiregularization X_s is Tychonoff, a maximal compactlike extension BX of X , satisfying $(BX)_s = \beta X_s$, is constructed in [14]. If (X, τ') is any Hausdorff almost completely regular space, and if (Y, τ) is any near compactification of (X, τ') , then so are Y^l , Y^u , Y^{l*} , and Y^{u*} .

(d) A space Z is called *almost real compact* if every open ultrafilter on Z with countable closed intersection property in Z converges in Z [4]. A space Z is *almost realcompact* if and only if Z_s is *almost realcompact* [12]. Almost realcompactifications of a Hausdorff space have been constructed (among others) in [7], and [12]. If (X, τ') is any Hausdorff space, and if (Y, τ) is any almost realcompactification of (X, τ') , then so are Y^l , Y^u , Y^{l*} , and Y^{u*} .

(e) A Hausdorff space Z is called *extremally disconnected* if for each open subset U of Z , $\text{cl}_2(U)$ is open. A space Z is *extremally disconnected* if and only if each dense subspace of Z [respectively, if and only if Z_s] is *extremally disconnected* [see 11 for more details]. A Hausdorff space Z is called *s-closed* if it is H -closed and *extremally disconnected* [8]. A Hausdorff space Z is *s-closed* if and only if Z_s is *s-closed*. It is shown in [8] that every *extremally disconnected* space X admits an *s-closed* extension, viz.

κX ; moreover, an extension Y of X is s -closed if and only if X is C^* -embedded in Y . If (X, τ') is any extremally disconnected Hausdorff space, and if (Y, τ) is any s -closed extension of (X, τ') , then so are Y', Y'', Y'^* , and Y''^* .

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