

A REMARK ON Θ -REGULAR SPACES

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ABSTRACT. In this paper we give an embedding characterization of θ -regularity using the Wallman-type compactification. The productivity of θ -regularity and a slight generalization of Nagami's Product Theorem to non-Hausdorff paracompact Σ -spaces we obtain as a corollary.

KEY WORDS AND PHRASES. θ -regularity, Wallman-type compactification, paracompact Σ -space

AMS SUBJECT CLASSIFICATION CODES: Primary 54A20, 54D35. Secondary 54B10, 54D20.

1. PRELIMINARIES

A filter base Φ in a topological space X has a θ -cluster point $x \in X$ if every closed neighborhood H of x and every $F \in \Phi$ have a nonempty intersection. The filter base Φ θ -converges to its θ -limit x if for every closed neighborhood H of x there is $F \in \Phi$ such that $F \subseteq H$. Recall that a topological space X is said to be θ -regular [3] if every filter base in X with a θ -cluster point has a cluster point. A topological space is said to be a Σ -space [1] if there exist locally finite closed collections Φ_i , $i = 1, 2, \dots$ and a cover Γ which consists of closed countably compact sets such that if $C \in \Gamma$ and $C \subseteq U$, where U is open in X , then $C \subseteq F \subseteq U$ for some $i \in \mathbb{N}$, $F \in \Phi_i$. A topological space is called (semi-) paracompact, if every its open cover has an open (σ -) locally finite refinement. Paracompact spaces are θ -regular [4].

Let X be a topological space with \mathfrak{G} its closed base which is a lattice (that means $\emptyset, X \in \mathfrak{G}$ and \mathfrak{G} contains all its finite unions and intersections). Recall that the Wallman-type [2] or Šanin [6] compactification is defined as the set $\omega(X, \mathfrak{G}) = X \cup \{y \mid y \text{ is an ultra-}\mathfrak{G} \text{ filter in } X \text{ with no cluster point}\}$, where the term "ultra- \mathfrak{G} " means maximal among all filters with a base consisting of elements from \mathfrak{G} . The set $\omega(X, \mathfrak{G})$ can be topologized by the open base consisting of the sets $S(U) = U \cup \{y \mid y \in \omega(X, \mathfrak{G}) \setminus X, U \in y\}$ where $X \setminus U \in \mathfrak{G}$. If \mathfrak{C} is the collection of all closed sets in X then $\omega(X, \mathfrak{C}) = \omega X$ is the Wallman compactification of X .

2. MAIN RESULTS

Let X be a topological space with a closed base \mathfrak{G} . We say that \mathfrak{G} is *balanced* if \mathfrak{G} is a lattice and every $x \in X$ has a neighborhood base, say δ_x , such that $\text{cl } U \in \mathfrak{G}$ for every $U \in \delta_x$. Trivially, the collection \mathfrak{C} of all closed sets of X is balanced. Two disjoint sets $A, B \subseteq X$ are said to be

point-wise separated in X if every $x \in A$, $y \in B$ have open disjoint neighborhoods. Now, we can state the theorem.

Theorem 1. *Let X be a topological space with a balanced closed base \mathfrak{B} . The following statements are equivalent.*

- (i) X is θ -regular
- (ii) *The sets X , $\omega(X, \mathfrak{B}) \setminus X$ are point-wise separated in $\omega(X, \mathfrak{B})$.*
- (iii) *There exists a compact space K containing X as a subspace such that the sets X , $K \setminus X$ are point-wise separated in K .*

Proof. Suppose (i). Let $x \in X$ and $y \in \omega(X, \mathfrak{B}) \setminus X$. Since X is θ -regular the filter y has no θ -cluster point. It follows that x has an open neighborhood U with $\text{cl} U \in \mathfrak{B}$ such that $F \cap \text{cl} U = \emptyset$ for some $F \in y$. Then $V = X \setminus \text{cl} U \in y$ and, consequently, $y \in S(V)$. Now, let $W \subseteq U$ be an open neighborhood of x with $X \setminus W \in \mathfrak{B}$. One can easily check that $S(W)$, $S(V)$ are disjoint neighborhoods of the points x, y . It follows (ii).

(ii) \implies (iii) is trivial. Suppose (iii). Let Φ be a filter base with a θ -cluster point $x \in X$. There exists a filter base Φ' finer than Φ which θ -converges to x . Since K is compact, Φ' has some cluster point $y \in K$. But a θ -limit and a cluster point of the same filter base cannot have disjoint neighborhoods; hence $y \in X$. Finally, y is a cluster point of Φ which implies (i).

Corollary 1. *The product of θ -regular topological spaces is θ -regular.*

Proof. Let X_α , $\alpha \in A$ be θ -regular topological spaces. It follows from the Theorem 1 that there are compact spaces $K_\alpha \supseteq X_\alpha$ such that for every $\alpha \in A$ the sets X_α , $K_\alpha \setminus X_\alpha$ are point-wise separated. Let $K = \prod_{\alpha \in A} K_\alpha$, $X = \prod_{\alpha \in A} X_\alpha$. Then K is compact and, evidently, the sets X , $K \setminus X$ are point-wise separated. Hence, the space X is θ -regular.

K. Nagami in [5] proved that a countable product of paracompact Hausdorff Σ -spaces is paracompact. Nagami uses Hausdorff separation axiom for upgrading semiparacompactness to paracompactness. However, Nagami's proof essentially contains the result that a countable product of paracompact Σ -spaces is semiparacompact which needs no separation axioms. The following result now follows from the fact that θ -regular semiparacompact spaces are paracompact ([4], Theorem 6).

Corollary 2. *A countable product of paracompact (not necessarily regular or Hausdorff) Σ -spaces is paracompact.*

It is easy to check that a second countable space has a countable balanced closed base. Theorem 1 (with Theorem 6, [4]) also yields the following.

Corollary 3. *A topological space X is paracompact second countable if and only if there exists a compact second countable space K containing X as a subspace such that the sets X , $K \setminus X$ are point-wise separated in K .*

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