

# SYMMETRIC GENERATING SET OF THE GROUPS $A_{2n+1}$ AND $S_{2n+1}$ USING $S_n$ AND AN ELEMENT OF ORDER TWO

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**ABSTRACT.** In this paper we will show how to generate in general  $A_{2n+1}$  and  $S_{2n+1}$  using a copy of  $S_n$  and an element of order 2 in  $A_{2n+1}$  or  $S_{2n+1}$  for all positive integers  $n \geq 2$ . We will also show how to generate  $A_{2n+1}$  and  $S_{2n+1}$  symmetrically using  $n$  elements each of order 2.

**KEY WORD AND PHRASES:** Symmetric generators, Involution, Double transitive groups, Group presentation

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## 1. INTRODUCTION

It is shown by Hammam [1] that  $A_{2n+1}$  can be presented as

$$G = A_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, S_{n-1}] = 1 \rangle$$

for  $n=4, 6$ , where  $[T, S_{n-1}]$  means that  $T$  commutes with  $Y$  and with  $X^{-2}YX$ , (the generators of  $S_{n-1}$ ). The relations of the symmetric group  $S_n = \langle X, Y \rangle$  of degree  $n$  are found in Coxeter and Moser[2]. Some relations must be added to the presentation of  $A_{2n+1}$  in order to complete the coset enumeration. Also, it has been shown by Hammam [1] that for  $n = 4, 6$ , the group  $A_{2n+1}$  can be symmetrically generated by  $n$  elements  $T_0, T_1, \dots, T_{n-1}$ , each of order 2, of the form  $T_i = T^{X^i} = X^{-i}TX^i$ , where  $T$  and  $X$  satisfy the relations of the group  $A_{2n+1}$ . The set  $\{T_0, T_1, \dots, T_{n-1}\}$  is called a symmetric generating set of  $A_{2n+1}$  (see section 3).

In this paper, we give a generalization of the results obtained by Hammam [1] for all  $n \geq 2$ . Moreover a proof is given to show that the group

$$G = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, S_{n-1}] = 1 \rangle$$

is either  $A_{2n+1}$  if  $n$  is even or  $S_{2n+1}$  if  $n$  is odd for all  $n \geq 2$ . We give permutations that generate  $A_{2n+1}$  and  $S_{2n+1}$  for all  $n \geq 2$  which satisfy the conditions given in the presentation of the group  $G$ . Further, we prove that  $G$  can be symmetrically generated by  $n$  permutations, each of order 2, satisfying the condition given in remark 2.4.

Our research is motivated by the aim of showing groups in their most "natural" role acting on (or permuting) the members of a symmetric generating set. The author has applied the method to obtain the symmetric generating sets and the presentations of the following finite simple groups:

Tits group  ${}^2F_4(2)'$ , Janco groups  $J_1$  and  $J_2$ , Mathieu groups  $M_{12}$  and  $M_{24}$ , and some of the linear groups  $PSL(2, q)$ . For more details, see Hammam [1].

## 2. PRELIMINARY RESULTS

In this section, we give some of the preliminary results to be used in later sections. The proofs of these results can be found in many references, see for example [2], [4], and [5].

**LEMMA 2.1.** Let  $1 \leq a \neq b \leq n$  be integers where  $n$  is odd. Let  $G$  be the group generated by the  $n$ -cycle  $(1, 2, \dots, n)$  and the 3-cycle  $(n, a, b)$ . If the highest common factor  $\text{hcf}(n, a, b) = 1$ , then  $G = A_n$ .

**LEMMA 2.2.** Let  $n$  be an odd integer and let  $G$  be the group generated by the  $n$ -cycle  $(1, 2, \dots, n)$  and the  $k$ -cycle  $(1, 2, \dots, k)$ . If  $1 < k < n$  and  $k$  is an odd integer, then  $G = A_n$ .

**PROOF.** Let  $\sigma = (1, 2, 3, \dots, n)$ , and  $\tau = (1, 2, \dots, k)$ . Since the commutator  $[\sigma, \tau] = (1, 2, k+1)$ , then by Lemma 2.1,  $G \cong A_n$ .

**LEMMA 2.3.** Let  $G$  be the group generated by  $n$ -cycle  $(1, 2, \dots, n)$  and the involution  $(n, 1)(i, j)$  for  $1 < i \neq j < n$ . If  $n \geq 9$  is an odd integer then  $G \cong A_n$ .

**REMARK 2.4.** The main condition used in Hammam [1], which we are going to use in this paper, is that  $T$  commutes with the generators of the group  $S_{n-1}$ .

## 3. SYMMETRIC GENERATING SETS

Let  $G$  be a group and let  $\Gamma = \{T_0, T_1, \dots, T_{n-1}\}$  be a subset of  $G$ , where  $T_i = T^{X^i} = X^{-i}TX^i$  for all  $i = 0, 1, \dots, n-1$ . Let  $S_n$  be the normalizer of the set  $\Gamma$  in  $G$ , which is a copy of the symmetric group of degree  $n$ . We define  $\Gamma$  to be a symmetric generating set of  $G$  if and only if  $G = \langle \Gamma \rangle$  and  $S_n$  permutes  $\Gamma$  doubly transitively by conjugation. Equivalently,  $\Gamma$  is realizable as an inner automorphism.

## 4. PERMUTATIONAL GENERATING SET OF $A_{2n+1}$ and $S_{2n+1}$

**THEOREM 4.1.**  $A_{2n+1}$  ( $S_{2n+1}$ ) can be generated using a copy of  $S_n$  and an element of order 2 in  $A_{2n+1}$  ( $S_{2n+1}$ ) if  $n$  is even (odd) for all  $n \geq 2$ .

**PROOF.** Let  $X = (1, 2, \dots, n)(n+1, n+2, \dots, 2n)$ ,  $Y = (n-1, n)(2n-1, 2n)$  and  $T = (1, 2n+1)(2, n+2) \dots (n, 2n)$  be three permutations; the first is of order  $n$ , the second and the third are of order 2. Let  $H$  be the group

generated by  $X$  and  $Y$ . By the Burnside and Moore Theorem (see Coxeter and Moser [2]), the group  $H$  is the symmetric group  $S_n$ . Let  $\bar{G}$  be the group generated by  $X, Y$  and  $T$ . Consider the commutator  $\eta = [X, T]$ , which has the form  $\eta = (1, n+1, 2n+1, n+2, 2)$ . Then

$$\eta^3 \eta^X = (1, 2n+1)(2, n+3, 3)(n+1, n+2) = \alpha.$$

Therefore  $\alpha^2 = (2, 3, n+3)$ . Hence

$$X\eta(\alpha^2)^{X^{-1}} = (1, 2, \dots, n, n+1, \dots, 2n, 2n+1).$$

Let  $\beta = X\eta(\alpha^2)^{X^{-1}}$ . Let  $K = \langle \beta, \alpha^2, T \rangle$  be a subgroup of  $\bar{G}$ . Since the highest common factor  $\text{hcf}(2, 3, n+3) = 1$ , then by Lemma 2.1  $\langle \beta, \alpha^2 \rangle = A_{2n+1}$ . Now if  $n$  is an even integer, then  $K = A_{2n+1}$ . Since  $X, Y$  and  $T$  are even permutations then  $K = \bar{G} = A_{2n+1}$ . Also, if  $n$  is an odd integer, then  $T$  is an odd permutation and therefore  $K = \bar{G} = S_{2n+1}$ .

## 5. SYMMETRIC GENERATING SET OF $A_{2n+1}$ and $S_{2n+1}$

**THEOREM 5.1.** Let  $X, Y$  and  $T$  be the permutations described in Theorem 4.1. Let  $\Gamma = \{T_0, T_1, \dots, T_{n-1}\}$ ,

where  $T_i = T^{X^i}$  and  $i = 0, 1, \dots, n-1$ . If  $n$  is an even integer, then the set  $\Gamma$  generates the alternating group  $A_{2n+1}$  symmetrically, while if  $n$  is an odd integer, then the set  $\Gamma$  generates the symmetric group  $S_{2n+1}$  symmetrically.

**PROOF.** Let  $T_0 = (1, 2n+1)(2, n+2) \dots (n, 2n)$ ,  $T_1 = (1, n+1)(2, 2n+1) \dots (n, 2n)$ , ...,  $T_{n-1} = T^{X^{n-1}} = (n, 2n+1) \dots (1, n+1) \dots (n-1, 2n-1)$ . Let  $H = \langle \Gamma \rangle$ . We claim that if  $n$  is an even integer, then  $H \cong A_{2n+1}$  and if  $n$  is an odd integer, then  $H \cong S_{2n+1}$ . To show this, suppose first that  $n$  is an even integer. Consider the element

$$\alpha = \prod_{i=0}^{n-1} T^{X^i}.$$

It is not difficult to show that  $\alpha = (1, 2, n+2, n+3, 3, 4, n+4, n+5, 5, 6, \dots, 2n, 2n+1, n+1)$  and it is a cycle of length  $2n+1$ . Let  $\beta = T_0 T_1$ . It is clear that  $\beta = (1, 2, n+2, 2n+2, n+1)$ . Let  $H_1 = \langle \alpha, \beta \rangle$ . We claim that  $H_1 \cong A_{2n+1}$ . To prove this, let  $\theta$  be the mapping which takes the element in the position  $i$  of the cycle  $\alpha$  into the element  $i$  of the cycle  $(1, 2, \dots, 2n+1)$ . Under this mapping, the group  $H_1$  will be mapped into the group  $\theta(H_1) = \langle (1, 2, \dots, 2n+1), (1, 2, 3, 2n, 2n+1) \rangle$  which is, by Lemma 2.2, the alternating group  $A_{2n+1}$ . Hence  $H \cong H_1 \cong \theta(H_1) \cong A_{2n+1}$ .

Second, suppose that  $n$  is an odd integer. Consider the element

$$\delta = \prod_{i=1}^n T X^i.$$

It is not difficult to show that  $\delta = (1, 2n+1, 2, n+3, 4, n+5, 6, n+7, \dots, 2n)$  and it is a cycle of length  $n+1$ . Let

$\mu = \delta^{T_1} T_0$ . Since  $\delta^{T_1} = (2, 2n+1, 3, n+4, 5, n+6, \dots, n, n+1)$ , then

$$\mu = (1, 2n+1, n+3, 3, 4, n+4, n+5, \dots, n, n+1, n+2, 2)$$

which is a cycle of length  $2n+1$ . Let  $\beta = T_1^{T_2} T_2^{T_3}$ , then  $\beta = (2, n+2, 3)(4, 2n+1)(n+3, n+4)$ . Therefore

$\beta^2 = (2, 3, n+2)$ . Let  $H_2 = \langle \mu, \beta^2, T_0 \rangle$ . We claim that  $H_2 \cong S_{2n+1}$ . To prove this, let  $\theta$  be the mapping which takes the element in the position  $i$  of the cycle  $\mu$  into the element  $i$  of the cycle  $(1, 2, \dots, 2n+1)$ .

Under this mapping the group  $H_2$  will be mapped into the group

$$\theta(H_2) = \langle (1, 2, \dots, 2n+1), (2n+1, 4, 2n), (1, 2)(3, 4) \dots (2n-3, 2n-2)(2n, 2n+1) \rangle.$$

Since  $\text{hcf}(2n+1, 4, 2n) = 1$ , then the group  $\langle (1, 2, \dots, 2n+1), (2n+1, 4, 2n) \rangle$  is the alternating group  $A_{2n+1}$ .

Since  $n$  is an odd integer, then the permutation  $(1, 2)(3, 4) \dots (2n-3, 2n-2)(2n, 2n+1)$  is an odd permutation.

Therefore the group  $\theta(H_2)$  is the symmetric group  $S_{2n+1}$ . Hence  $H \cong H_2 \cong \theta(H_2) \cong S_{2n+1}$ .

The set  $\Gamma$  described above satisfies the conditions of the group  $G$  given in section 1. It is important to note that  $\Gamma$  must have **exactly**  $n$  elements each of order 2 to generate  $A_{2n+1}$  or  $S_{2n+1}$ . The following Theorem characterizes all groups obtained by removing  $m$  elements of the set  $\Gamma$  for some integer  $m$ .

**THEOREM 5.2.** Let  $T$  and  $X$  be the permutations described above and let  $\Gamma = \{T_1, T_2, \dots, T_n\}$ . Then, removing  $m$  elements of the set  $\Gamma$  for all  $1 \leq m \leq n-3$ , the resulting set generates  $S_{2(n-m)+1}$ , removing  $m=(n-2)$  elements of the set  $\Gamma$ , the resulting set generates the dihedral group of order 10 ( $D_{10}$ ), and removing  $m=(n-1)$  elements of the set  $\Gamma$ , the resulting set generates the cyclic group  $C_2$ .

**PROOF.** Using induction on  $n-m$ , if  $n-m=1$ , then  $\Gamma_1 = \{T_1\}$ . Since  $T_1 = (1, n+1)(2, 2n+1)(3, n+3) \dots (n, 2n)$ , then  $\Gamma_1$  generates  $C_2$ . If  $n-m=2$ , then  $\Gamma_2 = \{T_1, T_2\}$ . Since  $T_1$  is the permutation described above,  $T_2 = (1, n+1)(2, n+2)(3, 2n+1) \dots (n, 2n)$ , and  $T_1 T_2 = (2, 3, n+3, 2n+1, n+2)$ , then it is clear that  $\Gamma_2$  generates

$D_{10}$ . Now suppose that  $1 \leq m \leq n-3$ . If  $n-m=k$ , then  $\Gamma_k = \{T_1, \dots, T_k\}$ . Assuming  $\alpha = T_1^{(T_2 T_3 \dots T_{k-1})} T_k$ ,

then for  $k$  an even integer we have

$$\alpha = (2, 3, n+4, 5, n+6, 7, n+8, \dots, k-1, n+k, k+1, n+k+1, 2n+1, k, n+k-1, k-2, n+k-3, \dots, 4, n+3, n+2)$$

which is a permutation of length  $2k+1$ ; while if  $k$  is an odd integer, then

$$\alpha = (2, n+2, 3, n+3, 4, n+5, 6, n+7, 8, n+9, \dots, k-1, n+k, k+1, n+k+1, 2n+1, k, n+k-1, k-2, n+k-3, \dots, 5, n+4, 3),$$

it is also a permutation of length  $2k+1$ . Let  $\beta = T_1^{T_2} T_1 T_2 T_3$ . Since  $\beta = (2, n+3)(3, n+2)(4, n+4, 2n+1)$ , then  $\beta^3 = (2, n+3)(3, n+2)$ . By Lemma 2.3,  $\alpha$  and  $\beta^3$  generate  $A_{2k+1}$ . Hence the group generated by  $\alpha$ ,  $\beta^3$  and  $T_1$  is the Symmetric group  $S_{2k+1}$ . Therefore the Theorem is true for all  $m$ .

**REMARK.** The above results are summarized in the following table

	$n$	$G = \langle X, Y, T \rangle$	$\langle X, T \rangle$	$\langle \Gamma \rangle$
1	even	$A_{2n+1}$	$A_{2n+1}$	$A_{2n+1}$
2	odd	$S_{2n+1}$	$S_{2n+1}$	$S_{2n+1}$

where

$$A_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, Y] = [T, X^{-2} Y X] = (XT)^{2n+1} = (YT_{n-2})^{10} \rangle.$$

$$S_{2n+1} = \langle X, Y, T \mid \langle X, Y \rangle = S_n, T^2 = [T, Y] = [T, X^{-2} Y X] = (XT)^{n(n+1)} = (Y T_{n-2})^{10} \rangle.$$

From the above, we can see that the order of the element  $XT$  is  $n(n+1)$  when  $n$  is an odd integer. As  $n$  gets larger, the order of  $XT$  becomes very large. For this reason, Hammam [1] had been unable to proceed for large odd values of  $n$ .

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