

FIXED POINT THEOREMS FOR SEMI-GROUPS OF SELF MAPS OF SEMI-METRIC SPACES

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ABSTRACT. We use selected semi-groups of self maps of a semi-metric space to obtain fixed point theorems for single maps and for families of maps – theorems which generalize results by Browder, Jachymski, Rhoades and Walters, and others. A basic tool in our approach is the concept of commuting maps.

KEY WORDS AND PHRASES: Semi-metric, semi-group, commuting maps, fixed points.

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1. INTRODUCTION. By a semi-group of maps we shall mean a family H of self maps of a set X which is closed with respect to composition of maps. Thus, if $f, g \in H$, then $f \circ g \in H$. Since composition of maps is associative, H is indeed a semi-group with respect to composition. We shall write fg for $f \circ g$ and fx for $f(x)$ when convenient and confusion is not likely.

We shall utilize the following semi-groups of maps in subsequent sections.

- 1.1. Let $g: X \rightarrow X$, and $H = O_g = \{g^n: n \in N \cup \{0\}\}$, where N is the set of positive integers, $g^0 = i_d$ – the identity map, $g^1 = g$ and $g^{n+1} = g \circ g^n$.
- 1.2. Let $g: X \rightarrow X$, and $H = C_g = \{f: X \rightarrow X \mid fg = gf\}$. C_g is a semi-group. For if $f, h \in C_g$, then $(fh)g = f(hg) = f(gh) = (fg)h = (gf)h = g(fh)$, and thus, $fh \in C_g$.
- 1.3. $H = \{i_d\}$, and
- 1.4. If $f, g: X \rightarrow X$ and $fg = gf$, we can let $H = \{f^n g^m: n, m \in N \cup \{0\}\}$.

If H is a semi-group of self maps of a set X and $a \in X$, then $H(a) = \{h(a): h \in H\}$.

Consequently, if $g: X \rightarrow X$ and $H = O_g$, $O_g(a) = \{g^n(a): n \in N \cup \{0\}\}$, and is called the *orbit of g at a* .

Just as we use semi-groups of maps to generalize the concept of orbits, we shall use semi-metric spaces to generalize results pertaining to metric spaces. We need the following definitions.

DEFINITION 1.1 A symmetric on a set X is a function $d: X \times X \rightarrow [0, \infty)$ such that $d(x, y) = 0$ iff $x = y$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.

Given a symmetric d on a set X , we generate an induced topology $t(d)$ for X as follows. For $x \in X$ and $\epsilon > 0$, we let $S_\epsilon(x) = \{y \in X: d(x, y) < \epsilon\}$. Then $t(d)$ consists of all subsets U of X such that for each $p \in U$, $S_\epsilon(p) \subseteq U$ for some $\epsilon > 0$. Just as in the case of a metric, $t(d)$ is a topology on X . However, if d is a symmetric, the sets $S_\epsilon(x)$ need not be neighborhoods of x . A semi-metric is a symmetric d such that all sets $S_\epsilon(x)$ are neighborhoods of x ; i.e., $\exists U \in t(d)$ such that $x \in U \subseteq S_\epsilon(x)$. It

is easy to verify that if d is a semi-metric, then a sequence $\{x_n\}$ in X converges to $x \in X$ in the topology $t(d)$ iff $d(x_n, x) \rightarrow 0$. This is the property we desire. Hence, the following terminology.

DEFINITION 1.2 A semi-metric space is a topological space, denoted by $(X; d)$, with topology $t(d)$ induced on the set X by a semi-metric d .

For further discussion of symmetric and semi-metrics refer to [1] or [2]. In this context, we note that a semi-metric need not be Hausdorff (or T_2) Example 2.2 in [2] gives an instance of such a semi-metric. Since we desire uniqueness of limits, we shall in most instances require that a semi-metric space $(X; d)$ be Hausdorff. Note also that – as in metric spaces – we shall say a semi-metric space $(X; d)$ is *complete* iff every Cauchy sequence in X converges to a point in X . If $g: X \rightarrow X$, then $(X; d)$ is *g-orbitally complete* iff every Cauchy sequence in $O_g(x)$ converges to a point in X for all $x \in X$. A function $F: X \rightarrow [0, \infty)$ is lower semicontinuous iff $F(x) \leq \liminf_{n \rightarrow \infty} F(x_n)$ when $\{x_n\}$ is a sequence in X converging to x .

To produce fixed points we use a contractive function $P: [0, \infty) \rightarrow [0, \infty)$ which is nondecreasing and which satisfies: $\lim_{n \rightarrow \infty} P^n(t) = 0$ for each $t \in [0, \infty)$. Throughout this paper, P will denote such a map, and \mathcal{P} will denote the family of all such maps P .

2. FIXED POINT THEOREMS. The major results in this paper evolve from the following lemma.

LEMMA 2.1. Let X be a set, $g: X \rightarrow X$, and let $d: X \times X \rightarrow [0, \infty)$. Let H be a semi-group of maps $h: X \rightarrow X$ such that $H \subset C_g$. Suppose that for each pair $x, y \in X$ there is a choice of $r = r(x, y)$, $s = s(x, y) \in H$ and $u, v \in \{x, y\}$ for which

$$(i) \quad d(gx, gy) \leq P(d(ru, sv)).$$

Then, if $n \in \mathbb{N}$, for each pair $x, y \in X \exists r_n, s_n \in H$ and $u_n, v_n \in \{x, y\}$ such that

$$(ii) \quad d(g^n x, g^n y) \leq P^n(d(r_n u_n, s_n v_n)).$$

PROOF. (ii) holds for $n=1$ by (i), so suppose $n \in \mathbb{N}$ for which (ii) is true. Then, if $x, y \in X$,

$$d(g^{n+1}x, g^{n+1}y) = d(g(g^n x), g(g^n y)) \leq P(d(r_n u_n, s_n v_n)), \tag{2.1}$$

where $r, s \in H$ and $u, v \in \{g^n x, g^n y\}$, by (i).

Specifically, $u = g^n u_0$, where $u_0 \in \{x, y\}$ and $v = g^n v_0$ with $v_0 \in \{x, y\}$. Thus

$$d(r_n u_n, s_n v_n) = d(r(g^n u_0), s(g^n v_0)) \text{ where } u_0, v_0 \in \{x, y\}. \text{ And since } r, s \in H \subset C_g, \tag{2.2}$$

$$d(r_n u_n, s_n v_n) = d(g^n(r u_0), g^n(s v_0)) \leq P^n(d(r_n u_n, s_n v_n)), \text{ by (ii),}$$

where $r_n, s_n \in H$ and $u_n, v_n \in \{r u_0, s v_0\}$.

Then $r_n u_n \in \{(r_n r)u_0, (r_n s)v_0\}$, where $r_n r, r_n s \in H$ (a semi-group). So, $r_n u_n = r_{n+1} u_{n+1}$, where $r_{n+1} \in H$ (i.e., $r_{n+1} \in \{r_n r, r_n s\}$) and $u_{n+1} \in \{u_0, v_0\} \subset \{x, y\}$. Similarly, $s_n v_n = s_{n+1} v_{n+1}$, where $s_{n+1} \in H$ and $v_{n+1} \in \{x, y\}$. Thus (2.2) implies

$$d(r_n u_n, s_n v_n) \leq P^n(d(r_{n+1} u_{n+1}, s_{n+1} v_{n+1})), \text{ with } r_{n+1}, s_{n+1} \in H \text{ and } u_{n+1}, v_{n+1} \in \{x, y\}. \tag{2.3}$$

But P is nondecreasing; therefore, (2.1) and (2.3) imply

$$d(g^{n+1}x, g^{n+1}y) \leq P(P^n(d(r_{n+1} u_{n+1}, s_{n+1} v_{n+1}))) = P^{n+1}(d(r_{n+1} u_{n+1}, s_{n+1} v_{n+1})),$$

with $r_{n+1}, s_{n+1} \in H$ and $u_{n+1}, v_{n+1} \in \{x, y\}$. Thus, (ii) is true for all n , by induction. \square

THEOREM 2.1 Let $(X; d)$ be a T_2 semi-metric space. Let $g: X \rightarrow X$ and let $(X; d)$ be *g-orbitally complete*. Suppose H is a semi-group of self maps of X such that $H \subset C_g$, and there is an $a \in X$ for which $H(a)$ is bounded and $g(H(a)) \subset H(a)$. If for each $x, y \in X \exists$ a choice of $r, s \in H$ and $u, v \in \{x, y\}$ such that

$$(*) \quad d(gx, gy) \leq P(d(ru, sv)),$$

then $g^n(a) \rightarrow c$ for some $c \in X$. If g is continuous at c , then $g(c) = c$. If d is lower semicontinuous, then c is a fixed point for all $h \in H$ continuous at c . Moreover, if g and each $h \in H$ are continuous at c , then c is the unique common fixed point of g and the family H .

PROOF. We first prove that $\{g^n(a)\}$ is Cauchy. By Lemma 2.1., for each pair $n,k \in \mathbb{N}$, there is a choice of $r_n, s_n \in H$ and $u_n, v_n \in \{a, g^k a\}$ such that

$$d(g^n(a), g^{n+k}(a)) = d(g^n(a), g^n(g^k(a))) \leq P^n(d(r_n u_n, s_n v_n)). \tag{2.4}$$

Now $r_n \in H$ implies $r_n a \in H(a)$ and $r_n g^k(a) = g^k(r_n a) \in H(a)$, since $g(H(a)) \subseteq H(a)$ implies that $g^k(H(a)) \subseteq H(a)$. Thus $r_n u_n \in H(a)$. Similarly, $s_n v_n \in H(a)$. But $H(a)$ is bounded, so $\exists M \geq 0$ such that $d(x, y) \leq M$ for $x, y \in H(a)$. Thus, $d(r_n u_n, s_n v_n) \leq M$ for $n \in \mathbb{N}$. Then (2.4) implies

$$d(g^n(a), g^{n+k}(a)) \leq P^n(M), \text{ for } n, k \in \mathbb{N} \tag{2.5}$$

since P is nondecreasing. But $P^n(M) \rightarrow 0$ as $n \rightarrow \infty$. So given $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that for any $m > n \geq n_0$, (2.5) implies $d(g^n(a), g^m(a)) \leq P^n(M) < \epsilon$, with $m=n+k$. Consequently, $\{g^n(a)\}$ is Cauchy.

Since $(X; d)$ is g -orbitally complete, $g^n(a), g^{n+1}(a) \rightarrow c$ for some $c \in X$. If g is continuous, $g(g^n(a)) = g^{n+1}(a) \rightarrow g(c)$; thus, $c = g(c)$ since X is Hausdorff.

Now suppose that d is lower semicontinuous and that $h \in H$ is continuous at c . Then, since $H \subseteq C_g$ and $g^n(a) \rightarrow c$,

$$g^n(h(a)) = h(g^n(a)) \rightarrow h(c). \tag{2.6}$$

But (*) and Lemma 2.1 tell us that $\exists r_n, s_n \in H$ and $u_n, v_n \in \{a, h(a)\}$ such that

$$d(g^n(a), g^n(h(a))) \leq P^n(d(r_n u_n, s_n v_n)). \tag{2.7}$$

Then $r_n u_n = r_n a \in H(a)$ or $r_n u_n = r_n h(a) \in H(a)$ ($r_n h \in H$, since H is a semi-group.). Thus, in either event, $r_n u_n \in H(a)$. In like manner, we conclude that $s_n v_n \in H(a)$. Then, as above, $d(r_n u_n, s_n v_n) \leq M$ for $n \in \mathbb{N}$, which implies by (2.7)

$$d(g^n(a), g^n(h(a))) \leq P^n(M) \rightarrow 0. \tag{2.8}$$

But since $g^n(a) \rightarrow c$, (2.6) implies that $(g^n(a), g^n(h(a))) \rightarrow (c, h(c))$ in $X \times X$. Since d is lower semicontinuous, $d(c, h(c)) \leq \liminf_{n \rightarrow \infty} d(g^n(a), g^n(h(a))) = 0$ by (2.8), so $h(c) = c$.

To complete the proof we have yet to show that if c is a common fixed point for g and every $h \in H$, then c is the only such point. So suppose that $z \in X$ and that $z = g(z) = h(z)$ for all $h \in H$. Then by (*) and Lemma 3.1, we can write:

$$d(c, z) = d(g^n(c), g^n(z)) \leq P^n(d(r_n u_n, s_n v_n)) \tag{2.9}$$

where $r_n, s_n \in H$ and $u_n, v_n \in \{c, z\}$. But then $r_n u_n \in \{c, z\}$. Similarly, $s_n v_n \in \{c, z\}$. Therefore, $d(r_n u_n, s_n v_n) = 0$ or $d(c, z)$. Thus (2.9) says that $d(c, z) \leq P^n(d(c, z))$. Since $P^n(d(c, z)) \rightarrow 0$, $c = z$; i.e., c is unique. \square

The following example shows that the family H in Theorem 2.1 can have fixed points other than the unique common fixed point of g and H .

EXAMPLE 2.1. Let $X = \{0, 1\}$, $g(x) = 0$ for $x \in X$ $h(x) = x$ for $x \in X$. Let $d(x, y) = |x - y|$ and $H = \{h^n : n \in \mathbb{N}\}$. Since $h^n(x) = x$ for $n \in \mathbb{N}$, $H = \{i_d\}$. Since $d(gx, gy) = 0$ for all $x, y \in X$, it is immediate that g and H satisfy the hypothesis of Theorem 3.1 with $a = 0$, and 1 is a fixed point of H but not of g .

The first corollary provides conditions necessary and sufficient to ensure that a family H of continuous self maps of a semi-metric space has a fixed point.

COROLLARY 2.1. Let $(X; d)$ be a complete Hausdorff semi-metric space with d lower semicontinuous. A semi-group H of continuous self maps of X has a common fixed point iff $H(a)$ is bounded for some $a \in X$, and $\exists P \in \mathcal{P}$ and a continuous self map g of X which satisfies the following.

1. $H \subseteq C_g$ and $g(H(a)) \subseteq H(a)$
2. For any $x, y \in X$, \exists a choice of $r, s \in H$ and $u, v \in \{x, y\}$ such that $d(gx, gy) \leq P(d(ru, sv))$.

PROOF. That the conditions are sufficient follows immediately from Theorem 2.1.. To prove necessity, suppose $a \in X$ and that $h(a) = a$ for $h \in H$. Then $H(a) = \{a\}$ and is thus bounded. Let $g(x) = a$ for $x \in X$. It is immediate that $gh = hg$ for all $h \in H$, so $H \subseteq C_g$. Moreover, $g(h(a)) = a$ for all $h \in H$, so that $g(H(a)) \subseteq H(a)$ and statement 1. of the Corollary holds. Statement 2. follows upon noting that $d(gx, gy) = d(a, a) = 0$ for all $x, y \in X$. (We can let $P(t) = t/2$, e.g.) \square

NOTE 2.1. The next result and proof suggest that the function g of Theorem 2.1 may have an infinitude or unbounded set of fixed points, although H may have just one. Example 3.1 in the next section confirms this.

COROLLARY 2.2. Let $(X; d)$ be a complete Hausdorff semi-metric space with d lower semicontinuous. A semi-group H of continuous self maps of X has a common fixed point provided $\exists a \in X$ such that $H(a)$ is bounded, and for any $x, y \in X \exists r, s \in H$ and $u, v \in \{x, y\}$ such that

$$d(x, y) \leq P(d(ru, sv)).$$

PROOF. Let $g = id$, the identity map. \square

COROLLARY 2.3. Let g be a self map of a metric space (X, d) which is g -orbitally complete. If $\exists a \in X$ such that $O_g(a)$ is bounded and $k \in \mathbb{N}$ such that for each pair $x, y \in X$ there is a choice of $n = n(x, y), m = m(x, y) \in \mathbb{N}$ and $u, v \in \{x, y\}$, for which

$$(*) \quad d(g^k x, g^k y) \leq P(d(g^n u, g^m v))$$

then $g^n(a) \rightarrow c$ for some $c \in X$. Moreover, if $x_0 \in X$ and $O_g(x_0)$ is bounded, then $g^n(x_0) \rightarrow c$. If g is continuous at c , c is the unique fixed point of g .

PROOF. Let $H = O_g$. Note that $H \subseteq C_{g^k}, g(H(a)) = g(O_g(a)) \subseteq O_g(a)$, and that $g^k(H(a)) \subseteq H(a)$. Since d is a metric, d is actually uniformly continuous [3]. Thus, $g^{kn}(a) \rightarrow c$ as $n \rightarrow \infty$ for some $c \in X$, by Theorem 2.1 applied to g^k . If g is continuous at c , each $g^n \in H(= O_g)$ is continuous at c . So, as an element of H , $g(c) = c$ by Theorem 2.1..

We have yet to prove that $g^n(a) \rightarrow c$ and that $g^n(x_0) \rightarrow c$ for x_0 with $O_g(x_0)$ bounded. To see that $g^n(a) \rightarrow c$, let $\epsilon > 0$. Since $(g^k)^m(a) \rightarrow c$ as $m \rightarrow \infty, \exists m_1 \in \mathbb{N}$ such that

$$d(g^{km}(a), c) < \epsilon/2, \text{ for } m > m_1 \tag{2.10}$$

By Lemma 2.1 and $(*)$ of the hypothesis, if $m \in \mathbb{N}$, for each pair $x, y \in X$ there exist $r_m, s_m \in H$ and $u_m, v_m \in \{x, y\}$ such that

$$d((g^k)^m(x), (g^k)^m(y)) \leq P^m(d(r_m u_m, s_m v_m)). \tag{2.11}$$

Since $O_g(a)$ is bounded, $\exists M \geq 0$ such that

$$d(r_m u_m, s_m v_m) \leq M \text{ if } r_m u_m \text{ and } s_m v_m \text{ are in } O_g(a). \tag{2.12}$$

Now $P^m(M) \rightarrow 0$ as $m \rightarrow \infty$, so we can choose $m_0 \in \mathbb{N}$ such that

$$m_0 > m_1 \text{ and } P^{m_0}(M) < \epsilon/2. \tag{2.13}$$

Let $n > km_0$. Then $n = km_0 + t_n$ for some $t_n \in \mathbb{N}$, and (2.11) implies

$$d(g^n(a), g^{km_0}(a)) = d(g^{km_0}(g^{t_n}(a)), g^{km_0}(a)) \leq P^{m_0}(d(r_{m_0} u_{m_0}, s_{m_0} v_{m_0})) \tag{2.14}$$

where $r_{m_0} \in H = O_g$, $u_{m_0} \in \{a, g^{h_1}(a)\}$, so that $r_{m_0}u_{m_0} \in O_g(a)$. Similarly, $s_{m_0}v_{m_0} \in O_g(a)$.

Therefore, (2.12) implies that

$$d(r_{m_0}u_{m_0}, s_{m_0}v_{m_0}) \leq M, \text{ and since } P \text{ is nondecreasing, we have}$$

$$P^{m_0}(d(r_{m_0}u_{m_0}, s_{m_0}v_{m_0})) \leq P^{m_0}(M) < \epsilon/2, \text{ by (2.13). Thus (2.14) implies}$$

$$d(g^n(a), g^{km_0}(a)) < \epsilon/2.$$

But then (2.10) with $m = m_0$ and the triangle inequality imply that $d(g^n(a), c) < \epsilon$, since $m_0 > m_1$. We therefore conclude that $g^n(a) \rightarrow c$

If $x_0 \in X$ such that $O_g(x_0)$ is bounded, the above argument shows us that $\exists p \in X$ such that $g^n(x_0) \rightarrow p$. To see that $p = c$, first observe that $S = O_g(x_0) \cup O_g(a)$ is also bounded since d is a metric; i.e., $\exists M_0 \geq 0$ such that $d(x, y) \leq M_0$ if $x, y \in S$. We can therefore apply (*) and Lemma 2.1 as before to conclude that

$$d(c, p) = \lim_{m \rightarrow \infty} d(g^{km}(a), g^{km}(x_0)) \leq \lim_{m \rightarrow \infty} P^m(M_0) = 0. \quad \square$$

The following example shows that the hypothesis in Corollary 2.3 that the orbit $O_g(a)$ be bounded for at least one $a \in X$ is indeed necessary.

EXAMPLE 2.2. Let $X = [1, \infty)$, $P(t) = t/2$ for $t \in [0, \infty)$, $d(x, y) = |x - y|$ and $g(x) = 3x$ for $x, y \in X$. Then $P^n(t) = t/2^n \rightarrow 0$ as $n \rightarrow \infty$, $g: X \rightarrow X$ and $d(gx, gy) = |gx - gy| = 3|x - y| \leq 9/2|x - y| = 1/2|9x - 9y| = 1/2|g^2x - g^2y| = P(d(g^2x, g^2y))$. But since $g^n(x) = 3^n x \rightarrow \infty$ for each $x \in X$, $O_g(x)$ is bounded for no $x \in X$ and g has no fixed point.

In [4] Rhoades and Watson introduced the concept of a "generalized contraction".

DEFINITION 2.1 Let (X, d) be a metric space. A function $f: X \rightarrow X$ is a *generalized contraction* (with respect to Q) if $\exists p, q \in \mathbb{N}$ such that for all $x, y \in X$

$$(i) \quad d(f^p x, f^q y) \leq Q(M(x, y)),$$

where

$$M(x, y) = \max\{d(f^i x, f^j y), d(f^i x, f^i x), d(f^j y, f^j y) : 0 \leq i, i' \leq p, 0 \leq j, j' \leq q\}.$$

(Q is a nondecreasing function $Q: [0, \infty) \rightarrow [0, \infty)$ such that $Q(s) < s$ for $s > 0$.)

NOTE 2.2. Jachymski [5] studied the relation (i) and observed that it satisfies

$$(ii) \quad d(f^r x, f^r y) \leq Q(\max\{d(f^i u, f^j v) : 0 \leq i, j \leq r \text{ and } u, v \in \{x, y\}\})$$

where $r = \max\{p, q\}$, since Q is nondecreasing. But (ii), and hence (i), satisfy the relation (*) in Corollary 2.3. In fact, the following theorem by Jachymski – except the last sentence therein – is a consequence of Corollary 2.3. This last sentence refers to (9) which is essentially (i) above with the restriction that either $i, i' \in \{0, p\}$ or $j, j' \in \{0, q\}$.

THEOREM 4. ([5]) Let f be a generalized contraction and let (X, d) be f -orbitally complete. If $\lim_{n \rightarrow \infty} Q^n(s) = 0$ for $s \in [0, \infty)$ and there exists a point $x_0 \in X$ with a bounded orbit, then the sequence $\{f^n x_0\}$ converges to some $z \in X$. Moreover, for any $x \in X$ with a bounded orbit, $f^n x \rightarrow z$. Furthermore, if f satisfies (9) z is the unique fixed point of f .

The following example shows that if we use the more general contractive property (*) of Corollary 2.3, continuity at c or restrictions of the ilk found in (9) of Theorem 4 are needed to ensure that the point c (or z) is a fixed point.

EXAMPLE 2.3. Let $X = [0, 1]$ and let $d(x, y) = |x - y|$. Define $g: X \rightarrow X$ by $g(x) = \frac{1}{2}(x+1)$ for $x \in [0, 1)$ and $g(1) = \frac{1}{2}$. Then it is easy to see that $g^n(x) = (x - 1)2^{-n} + 1$ ($x \neq 1$), and $g^n(1) = 1 - 2^{-n}$ for $n \in \mathbb{N}$. Thus, $g^n(a) \rightarrow 1$ for any $a \in X$. Since X is bounded, $O_g(a)$ is bounded for each $a \in X$. Thus, to see that the hypothesis of Corollary 2.3 is satisfied, we need only to verify that (*) holds. A check shows that

$$d(g^2x, g^2y) = \frac{1}{2} d(gx, gy) \text{ for all } x, y \in X. \tag{2.15}$$

Thus, (*) holds trivially with $k=2$, $n = m = 1$, and $u = x, v = y$ for all x, y in $[0, 1]$. However, $g^n(a) \rightarrow 1$ for any a , but g is not continuous at 1 and 1 is not a fixed point of g . Note that property (9) of Jachymski's Theorem 4. does not hold since in this instance $p = q = 2$, and g^2, g^o do not appear in the right member of (2.15).

3. THE BOUNDED CASE. The following is an example of a function g and a family H which satisfy the hypothesis of Theorem 2.1 and for which the set F_g – the set of fixed points of g – is not bounded We then consider the significance of this phenomenon.

EXAMPLE 3.1. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for $x, y \in X$. Let $g(x) = x$ for $x \in X$; i.e., g is the identity map. So $F_g = [0, \infty)$, and is unbounded. Let $P(t) = t/2$ for $t \in [0, \infty)$ and define $h_n(x) = nx$ for $x \in X$ and $n \in \mathbb{N}$. If $H = \{h_n: n \in \mathbb{N}\}$, and $h_n, h_m \in H$, then $h_n h_m(x) = h_n(h_m(x)) = h_n(mx) = (nm)x = h_{nm}(x)$. Thus $h_n h_m = h_{nm} \in H$, so that H is indeed a semi-group. Since g is the identity, the conditions $H \subseteq C_g$ and $g(H(a)) \subseteq H(a)$ (for any $a \in X$) are satisfied trivially. Moreover, $d(gx, gy) = |x - y| \leq \frac{3}{2} |x - y| = \frac{1}{2} |3x - 3y| = P(d(h_3x, h_3y))$, so that (*) and hence the hypothesis of Theorem 2.1 is satisfied. $a=0$ is the unique fixed point for g and H , but g has an infinitude of other fixed points.

In the remainder of the paper, if (X, d) is a T_2 semi-metric space and $g: X \rightarrow X$, we shall say that g has property P relative to a semi-group H of self maps of X iff for each pair $x, y \in X \exists r, s \in H$ and $u, v \in \{x, y\}$ such that

$$(*) \quad d(gx, gy) \leq P(d(ru, sv)).$$

NOTE 3.1. If a function $g: X \rightarrow X$ has property P relative to a semi-group H of self maps of X for which $H \subseteq C_g$, Lemma 2.1 implies that if $n \in \mathbb{N}$, for any pair $x, y \in X$ there exist $r_n, s_n \in H$ and $u_n, v_n \in \{x, y\}$ such that

$$(**) \quad d(g^n x, g^n y) \leq P^n(d(r_n u_n, s_n v_n))$$

PROPOSITION 3.1. Let (X, d) be a T_2 semi-metric space and let $g: X \rightarrow X$. Suppose H is a semigroup of self maps of X such that $H \subseteq C_g$. If g has property P relative to H and F_g is nonempty and bounded, then

- (i) F_g is a singleton $\{c\}$, and
- (ii) $c = g(c) = h(c)$ for all $h \in H$.

PROOF. To prove (i), we first note that $h(F_g) \subseteq F_g$ for all $h \in H$. For if $h \in H$ and $a = g(a)$, then $g(h(a)) = h(g(a)) = h(a)$, so that $h(a) \in F_g$. Moreover, since F_g is bounded, $\exists M \geq 0$ such that $d(a, c) \leq M$ for $a, c \in F_g$.

Now by hypothesis, $\exists c \in F_g$. We assert that c is unique. For suppose $a \in F_g$. Then Note 3.1 says that we can choose $u_n, v_n \in \{a, c\}$ and $r_n, s_n \in H$ such that

$$d(a, c) = d(g^n(a), g^n(c)) \leq P^n(d(r_n u_n, s_n v_n)). \tag{3.1}$$

But since $h(F_g) \subseteq F_g$ for $h \in H$, and since $a, c \in F_g$, $r_n u_n, s_n v_n \in F_g$. So by the above,

$$d(r_n u_n, s_n v_n) \leq M.$$

Therefore, since P and hence P^n is nondecreasing, (3.1) yields:

$$d(a, c) \leq P^n(M) \text{ for } n \in \mathbb{N}. \tag{3.2}$$

Since $P^n(M) \rightarrow 0$ as $n \rightarrow \infty$, (3.2) implies that $a = c$.

(ii) is an immediate consequence of (i), since $(h \in H) \Rightarrow h(F_g) \subseteq F_g$. Therefore, if $h \in H$, $h(c) \in \{c\}$; i.e, $h(c) = c$. \square

COROLLARY 3.1. Let $(X; d)$ be a bounded and complete T_2 semi-metric space. Let $g: X \rightarrow X$ and let H be a semi-group of self maps of X such that $H \subset C_g$ and $g(H(a)) \subset H(a)$ for $a \in X$. If g has property P relative to H , then for each $x \in X \exists$ a point $c_x \in X$ such that $g^n(x) \rightarrow c_x$. If g is continuous at one such c_x , then F_g is a singleton, $\{c\}$, and $c_x = c$ for all x . Moreover, $c = h(c)$ for all $h \in H$.

PROOF. Since X is bounded, $H(x)$ is bounded for each $x \in X$. Therefore, Theorem 2.1 implies that $g^n(x) \rightarrow c_x$ for some $c_x \in X$. If g is continuous at one such c_x , then $g(c_x) = c_x$. But then F_g is bounded and nonempty, so that Proposition 3.1 implies that $F_g = \{c\}$, a singleton, and that $c = h(c)$ for all $h \in H$. \square

Corollary 3.1 has Theorem 1.[6] by Browder and a result by Zitarosa [7] on contractive self maps of a bounded complete metric space as special cases with $H = \{id\}$.

The proof of our next theorem, as did the proof of Corollary 2.3, requires that the union of two bounded sets be bounded. So we again need a metric. Also, observe that in Example 3.1 the set $H(a) = \{na : n \in \mathbb{N}\}$ is unbounded for $a \neq 0$.

THEOREM 3.1. Let g be a self map of a metric space (X, d) which is g -orbitally complete. Suppose that H is a semi-group of self maps of X such that $H \subset C_g$ and that g has property P relative to H . If $g(H(a)) \subset H(a)$ and $H(a)$ is bounded for all $a \in X$, then g has a contractive point c ; i.e., $g^n(x) \rightarrow c$ for all $x \in X$. Moreover, $c = g(c) = h(c)$ for all $h \in H$ if g is continuous at c .

PROOF. Let $a \in X$. By Theorem 2.1, since $H(a)$ is bounded, $g^n(a) \rightarrow c$ for some $c \in X$. But $g^n(x) \rightarrow c_x \in X$ for any $x \in X$ since $H(x)$ is bounded. We show $c_x = c$ for any $x \in X$. To this end, let $x \in X$. Then $H(x) \cup H(a)$ is bounded. Since g has property P relative to H and $H \subset C_g$, Note 3.1 implies that for all $n \in \mathbb{N}$ we have:

$$d(g^n(a), g^n(x)) \leq P^n(d(r_n u_n, s_n v_n)), \tag{3.3}$$

where $r_n, s_n \in H$ and $u_n v_n \in \{a, x\}$; hence, $r_n u_n, s_n v_n \in H(a) \cup H(x)$ for $n \in \mathbb{N}$. But $H(a) \cup H(x)$ is bounded, and so $\exists M \geq 0$ such that $d(r_n u_n, s_n v_n) \leq M$ for all n . Thus, $P^n(d(r_n u_n, s_n v_n)) \leq P^n(M) \rightarrow 0$ as $n \rightarrow \infty$. Hence (3.3) and the above imply:

$$d(c, c_x) = \lim_{n \rightarrow \infty} d(g^n(a), g^n(x)) = 0.$$

Thus $c = c_x$. If g is continuous at c , then $c = g(c)$. But since $g^n(x) \rightarrow c$ for all x , c is the only fixed point of g . Therefore, $c = h(c)$ for all $h \in H$ by Proposition 3.1. \square

COROLLARY 3.2. Let g be a self map of a metric space (X, d) which is g -orbitally complete. Suppose that $O_g(x)$ is bounded for all $x \in X$. If g has property P relative to O_g , then $\exists z \in X$ such that $g^n(x) \rightarrow z$ for all $x \in X$. z is a unique fixed point of g iff the function $F(x) = d(x, g(x))$ is lower semicontinuous at z .

PROOF. Since trivially, $O_g \subset C_g$ and $g(O_g(x)) \subset O_g(x)$ for all $x \in X$, Corollary 3.2 follows immediately from Theorem 3.1 (with the observation that the last statement is a well known consequence of " $g^n(x) \rightarrow z$ ") \square

We conclude with a theorem (rephrased) by Jachymski [5] which generalizes theorems of Rhoades and Watson [4], and which is a consequence of our Corollary 2.3.

THEOREM 2. [5] Assume that f is a generalized contraction, and (X, d) is f -orbitally complete. If $\lim_{n \rightarrow \infty} Q^n(s) = 0$ for $s \in [0, \infty)$ and $\lim_{s \rightarrow \infty} (s - Q(s)) = \infty$, then there exists $z \in X$ such that $f^n x \rightarrow z$ for any $x \in X$. z is a unique fixed point of f if and only if the function $F(x) = d(x, f(x))$ is lower semicontinuous at z .

To see that Theorem 2. [5] does indeed follow from Corollary 2.3, first observe that (as noted before) a generalized contraction satisfies (*) of Corollary 2.3. Moreover, Lemma 3. [5] tells us that if

$\lim_{s \rightarrow \infty} (s - Q(s)) = \infty$, then the orbits $O_f(x)$ are bounded for all $x \in X$. Therefore, Corollary 2.3 assures us that $\exists z \in X$ such that $f^n(x) \rightarrow z$ for all $x \in X$. The assertion that z is the unique fixed point of f follows as in the proof of Corollary 3.2

4. **RETROSPECT.** In closing we emphasize the general nature and utility of the semi-groups H of self maps employed. For example, in Corollary 2.2 H is any family of continuous self maps closed under composition with $H(a)$ bounded at some one point $a \in X$ – no commutativity requirements are imposed. Corollary 2.3 demonstrates the utility of options provided by H in letting $H = O_g$. And Example 1.4 indicates how, when given a map $g: X \rightarrow X$, we can generate semigroups H which satisfy $g(H(a)) \subset H(a)$.

Note also that Hausdorff semi-metric spaces $(X; d)$ generalize metric spaces, even if the semi-metric d is lower semi-continuous. In fact, Cook [8] provides an example of a semi-metric space with a continuous semi-metric which is developable but not normal, and hence not a metric

A final comment. The semi-group C_g has been used to some extent in fixed point research. See, e.g., [9, 10, 11].

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