

*Research Article*

**Behavior of the Trinomial Arcs  $B(n, k, r)$  When  $0 < \alpha < 1$**

Kaoutar Lamrini Uahabi and Mohammed Zaoui

Received 12 April 2007; Revised 14 July 2007; Accepted 12 August 2007

Recommended by Teodor Bulboaca

We deal with the family  $B(n, k, r)$  of trinomial arcs defined as the set of roots of the trinomial equation  $z^n = \alpha z^k + (1 - \alpha)$ , where  $z = \rho e^{i\theta}$  is a complex number,  $n$  and  $k$  are two integers such that  $0 < k < n$ , and  $\alpha$  is a real number between 0 and 1. These arcs  $B(n, k, r)$  are continuous arcs inside the unit disk, expressed in polar coordinates  $(\rho, \theta)$ . The question is to prove that  $\rho(\theta)$  is a decreasing function, for each trinomial arc  $B(n, k, r)$ .

Copyright © 2007 K. L. Uahabi and M. Zaoui. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## 1. Introduction

Consider the trinomial equation

$$z^n = \alpha z^k + (1 - \alpha), \quad (1.1)$$

where  $z$  is a complex number,  $n$  and  $k$  are two integers such that  $k = 1, 2, \dots, n - 1$ , and  $\alpha$  is a real number. The first discussion of the behavior of the roots of trinomial equations was in Fell [1]. She presented a description of the trajectories of these roots, called *trinomial arcs*. These arcs can be expressed in polar coordinates  $(\rho, \theta)$  by a function  $\rho(\theta)$  and are continuous in functions of  $\alpha$  as  $\alpha$  varies between 0 and 1, or between 1 and  $+\infty$ , or between  $-\infty$  and 0. Fell [1] also studied the monotonicity of the function  $\alpha(\theta)$  and gave a bound for the modulus of roots. However, she did not establish the monotonicity of  $\rho$  as a function of  $\theta$ . Though the descriptive results of Fell [1] give us the information about the form and location of the trinomial arcs, nevertheless, these types of arcs are not suitably designed for further study. In this paper, we will restrict our attention to a family of trinomial arcs, solutions of (1.1) with  $0 < \alpha < 1$ , inside the unit disk  $D_u = \{z : |z| \leq 1\}$ . We begin by defining this family of trinomial curves denoted by  $B(n, k, r)$ , where  $n, k$ , and

$r$  satisfy some conditions. Note that Dubuc and Zaoui [2] studied trinomial arcs denoted by  $B_m$  and which are part of this family of arcs  $B(n, k, r)$ . Next, we prove in this work that  $\rho(\theta)$  is a differentiable function for these arcs. With a view to solving the problem of monotonicity of  $\rho(\theta)$  for the trinomial arcs  $B(n, k, r)$ , two important intermediate results are shown. At last, this study allows us to prove that  $\rho(\theta)$  is a decreasing function.

## 2. Study of the trinomial equation

In (1.1), fix  $n$  and  $k$ . For  $z = \rho e^{i\theta}$  in (1.1), one has  $\rho^n e^{in\theta} = \alpha \rho^k e^{ik\theta} + (1 - \alpha)$ . Separating real and imaginary parts, one gets  $\rho^n \sin n\theta = \alpha \rho^k \sin k\theta$  and  $\rho^n \cos n\theta = \alpha \rho^k \cos k\theta + (1 - \alpha)$ . So, when  $\theta \neq l\pi/n$ , where  $l$  is an integer, we get

$$\rho^{n-k} = \alpha \frac{\sin k\theta}{\sin n\theta}. \quad (2.1)$$

On the other hand, divide (1.1) by  $z^n$  and consider the imaginary part. When  $\alpha \neq 0$  and  $\theta \neq l\pi/(n-k)$ , where  $l$  is an integer, we obtain that

$$\rho^k = \frac{(\alpha - 1)}{\alpha} \frac{\sin n\theta}{\sin(n - k)\theta}. \quad (2.2)$$

Therefore, we have the next equation of the trajectories of roots of (1.1):

$$\rho^{n-k} \sin n\theta - \rho^n \sin(n - k)\theta = \sin k\theta. \quad (2.3)$$

In fact, Fell has studied in [1] the trinomial equation

$$\lambda z^n + (1 - \lambda)z^k - 1 = 0, \quad (2.4)$$

where  $z$  is a complex number,  $n$  and  $k$  are two integers such that  $k = 1, 2, \dots, n - 1$ , and  $\lambda$  is a real number. Substituting into (2.4) the expression given for  $z^n$  by (1.1), we get  $(z^k - 1)[1 - \lambda(1 - \alpha)] = 0$ . So,  $z^k = 1$  or  $\lambda(1 - \alpha) = 1$ . When  $z$  is not a  $k$ th root of unity, it follows that  $\alpha = 1 - 1/\lambda$ . Hence, in order to pass from (1.1) to (2.4), we can set  $\alpha = 1 - 1/\lambda$ . It stems easily from this equality that the case  $0 \leq \alpha \leq 1$  of (1.1) corresponds to the case  $1 \leq \lambda < +\infty$  of (2.4).

In this work, we are interested in the case  $0 \leq \alpha \leq 1$ , so we have

$$\text{sign}(\sin n\theta) = \text{sign}(\sin k\theta) = -\text{sign}(\sin(n - k)\theta). \quad (2.5)$$

*Definition 2.1.* An angle  $\theta$  which fulfills (2.5) will be called an  $(n, k)$ -feasible angle for the trinomial equation (1.1) with  $0 \leq \alpha \leq 1$ .

Moreover, in view of the next lemma of [3], the trajectories of roots of (1.1) with  $0 \leq \alpha \leq 1$  are inside the unit disk.

*LEMMA 2.2.* For any  $(n, k)$ -feasible angle  $\theta$  for (1.1) with  $0 \leq \alpha \leq 1$ , the function of  $\rho$ ,  $-\rho^n \{\sin(n - k)\theta / \sin k\theta\} + \rho^{n-k} \{\sin n\theta / \sin k\theta\} - 1$ , is increasing and vanishes for one and only one positive value of  $\rho$ , which is not larger than 1.

*Remark 2.3.* The upper and lower half-planes are symmetrical. So, we will restrict our study of trinomial arcs to the upper half-plane.

### 3. Description and definition of trinomial arcs $B(n, k, r)$

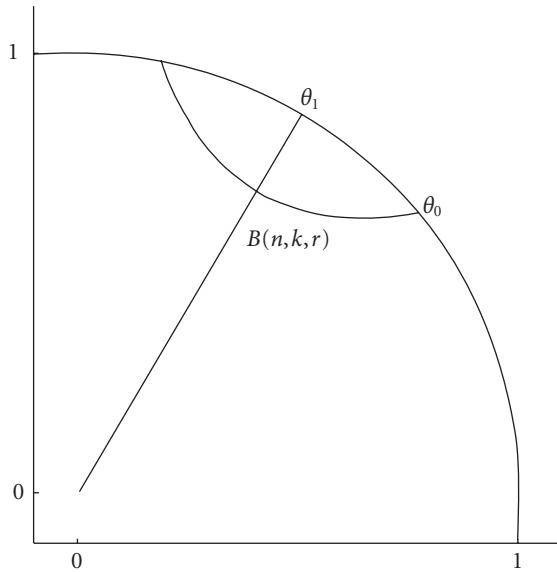
Notice that for  $\alpha = 0$ , (1.1) has  $n$  roots: the  $n$ th roots of unity. Fell [1], in her Descriptive Claim II, pages 314-315, tells us that the trajectories of the  $n$  roots can be described as trajectories of particles starting at these  $n$  roots. As  $\alpha$  changes from 0 to 1, they move continuously until  $\alpha = 1$ ,  $(n - k)$  of them have moved into  $(n - k)$ th roots of unity, and  $k$  of them have collapsed to 0. There are  $k$  trajectories going to 0, the  $k$  tangents being lines going through 0 and one  $k$ th root of  $-1$ . Consider  $C = \{n$ th roots of unity $\}$ ,  $D = \{(n - k)$ th roots of unity $\}$ , and  $E = \{k$ th roots of  $-1$  $\}$ . Let  $\gamma$  be in  $C$  and let  $\delta$  be the unique nearest neighbor of  $\gamma$  in  $D \cap E$ . Fell [1] asserts that, in the case  $\delta \in D \cap E$  with  $0 \leq \alpha \leq 1$ , there exists  $\gamma'$  in  $C$  such that  $\delta$  is equidistant from  $\gamma$  and from  $\gamma'$ . There exists also  $\alpha_0$  in  $[0, 1]$  such that the trajectories of two particles starting at  $\gamma$  and  $\gamma'$  when  $\alpha = 0$  are continuous arcs until the point of their meeting on the line segment  $\theta = \arg(\delta)$  when  $\alpha = \alpha_0$ . When  $\alpha$  moves from  $\alpha_0$  to 1, the two roots remain on the segment  $\theta = \arg(\delta)$ , one of them goes to 0 and the other tends to  $\delta$ . Fell shows in [1] that all the trinomial arcs solutions of (1.1) in the case  $0 \leq \alpha \leq 1$  with  $\delta \in D \cap E$  are such that the feasible angles  $\theta$  belong to intervals of length less than or equal to  $\pi/n$  and bounded on the one side by  $\arg(\delta)$  where  $\delta$  is both a  $k$ th root of  $-1$  and an  $(n - k)$ th root of unity, and on the other side by  $\arg(\gamma)$  where  $\gamma$  is an  $n$ th root of unity. There are so two types of arcs in this case; the first type is such that  $\theta$  belongs to  $[\arg(\gamma), \arg(\delta)]$  where  $\gamma \in C$  and the second type is such that  $\theta$  belongs to  $[\arg(\delta), \arg(\gamma')]$  where  $\gamma' \in C$ , such that  $\delta$  is equidistant from  $\gamma$  and from  $\gamma'$ .

In [2], Dubuc and Zaoui studied a class of trinomial arcs denoted by  $B_m$  and defined as the set of roots of (1.1) with  $0 \leq \alpha \leq 1$ ,  $n = m$ ,  $k = m - 2$ , where  $m$  is an odd integer larger than 2 and the feasible angles belong to the interval  $[\pi - \pi/m, \pi]$ . They showed in [2] that  $\rho(\theta)$  is a decreasing function on  $[\pi - \pi/m, \pi]$  for the arcs  $B_m$ . Because  $m$  is an odd integer, we can say that  $\gamma$  such that  $\arg(\gamma) = \pi - \pi/m$  is an  $n$ th root of unity and  $\delta$  such that  $\arg(\delta) = \pi$  is both a  $k$ th root of  $-1$  and an  $(n - k)$ th root of unity. Dubuc and Zaoui have thus solved the problem of monotonicity of  $\rho(\theta)$ , pointed out in [1?], for some particular trinomial arcs, namely  $B_m$ , solutions of (1.1) in the case  $0 \leq \alpha \leq 1$  with  $\delta \in D \cap E$  and  $\theta \in [\arg(\gamma), \arg(\delta)]$ . In this paper, our objective is to study the monotonicity of  $\rho(\theta)$  for all trinomial arcs corresponding to this case. In fact, these arcs, denoted by  $B(n, k, r)$ , will be defined as follows.

Let  $d$  be the greatest common divider of  $k$  and  $n$ , we assume that  $k/d$  and  $n/d$  are odd numbers, then we define  $d$  trinomial arcs,  $B(n, k, 0), B(n, k, 1), \dots, B(n, k, d - 1)$ . Let  $r \in \{0, 1, \dots, d - 1\}$ , then for any angle  $\theta \in (\theta_0, \theta_1)$  with  $\theta_1 = (2r + 1)\pi/d$  and  $\theta_1 - \theta_0 = \pi/n$ , there exists a unique solution  $\rho = \rho(\theta) \in (0, 1)$  to the equation  $\rho^n \sin(n - k)\theta - \rho^{n-k} \sin n\theta + \sin k\theta = 0$  given by (2.3). Any solution of (2.3) induces the solution  $z = \rho e^{i\theta}$  of (1.1) with  $\alpha = \rho^{n-k} \sin n\theta / \sin k\theta$ .

Figure 3.1 provides such an arc.

*Remark 3.1.* When  $\alpha = 0$ , (1.1) becomes  $z^n = 1$ . So, its solutions are the  $n$ th roots of unity. In the case  $\alpha = 1$ , (1.1) becomes  $z^k[z^{n-k} - 1] = 0$ . Then, the  $n$  roots of (1.1) are the  $(n - k)$ th roots of unity, which are simple roots and 0, a root of multiplicity  $k$ .

FIGURE 3.1. Trinomial arc defined for feasible angles  $\theta \in (\theta_0, \theta_1)$ .

When  $n = 2$ , the trajectories of roots of (1.1) with  $0 < \alpha < 1$  are linear, then we define the continuous arcs  $B(n, k, r)$  as follows.

*Definition 3.2.* Let  $n$  and  $k$  be two integers such that  $n$  is greater than or equal to 3 and  $0 < k < n$ . Let  $d = \gcd(k, n)$  and  $r \in \{0, 1, \dots, d - 1\}$ , one assumes that  $k/d$  and  $n/d$  are odd numbers, then the continuous arc

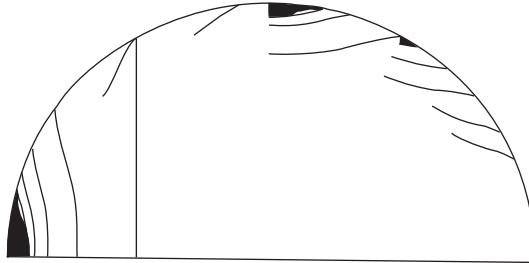
$$B(n, k, r) = \{z = \rho(\theta)e^{i\theta} : 0 < (2r + 1)\pi/d - \theta < \pi/n\} \quad (3.1)$$

is the set of roots of (1.1) with  $0 < \alpha < 1$ .

This family of arcs  $B(n, k, r)$  exists in view of the following lemma.

**LEMMA 3.3.** *Let  $n$  and  $k$  be two integers such that  $n$  is greater than or equal to 3 and  $0 < k < n$ . Let  $d = \gcd(k, n)$  and  $r \in \{0, 1, \dots, d - 1\}$ , one assumes that  $k/d$  and  $n/d$  are odd numbers, then in the trinomial equation (1.1) with  $0 < \alpha < 1$ , any angle of the interval  $(\theta_0, \theta_1)$ , with  $\theta_1 = (2r + 1)\pi/d$  and  $\theta_0 = \theta_1 - \pi/n$  is feasible. In particular, for each trinomial arc  $B(n, k, r)$ , one has  $\sin n\theta > 0$ ,  $\sin k\theta > 0$ , and  $\sin(n - k)\theta < 0$  for any  $\theta$  in the interval  $(\theta_0, \theta_1)$ .*

*Proof.* Let  $\theta$  be an angle such that  $(2r + 1)\pi/d - \pi/n < \theta < (2r + 1)\pi/d$ . So, we have  $(2r + 1)n\pi/d - \pi < n\theta < (2r + 1)n\pi/d$ . Because the number  $n/d$  is odd, it follows that  $\sin n\theta > 0$ . On the other side, we have  $(2r + 1)k\pi/d - k\pi/n < k\theta < (2r + 1)k\pi/d$ . Since  $k/d$  is an odd number and  $k < n$ , it yields that  $\sin k\theta > 0$ . Finally, we have  $(2r + 1)(n - k)\pi/d - (n - k)\pi/n < (n - k)\theta < (2r + 1)(n - k)\pi/d$ . As the number  $(n - k)/d$  is even and  $k > 0$ , we obtain that  $\sin(n - k)\theta < 0$ . The conditions (2.5) are fulfilled.  $\square$

FIGURE 3.2. Trinomial arcs  $B(n, k, r)$  inside the upper half unit disk.

#### 4. Differentiability of the function $\rho(\theta)$ for the arcs $B(n, k, r)$

Now, we will prove that the derivative  $d\rho/d\theta$  exists and it is well defined for the trinomial arcs  $B(n, k, r)$ .

**PROPOSITION 4.1.** *For each trinomial arc  $B(n, k, r)$ , the function  $\rho(\theta)$  is differentiable for any feasible angle  $\theta$  in the interval  $(\theta_0, \theta_1)$ .*

*Proof.* Let  $B(n, k, r)$  be a trinomial arc. By (2.2), we have  $\rho^k(\theta) = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta$ . According to Lemma 3.3, the feasible angles  $\theta$  are such that  $\sin n\theta > 0$  and  $\sin(n - k)\theta < 0$ . If we put  $f(\theta) = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta$  and as  $0 < \alpha = \alpha(\theta) < 1$ , the denominator of  $f(\theta)$  is never zero. The function  $f(\theta)$  is well defined. In addition,  $f$  is differentiable and positive. So, the function  $\rho(\theta) = [f(\theta)]^{1/k}$  is differentiable (since  $\alpha(\theta)$  is differentiable, see Fell [1, pages 326–327]). Therefore, its derivative  $d\rho/d\theta$  exists and it is well defined.  $\square$

#### 5. Monotonicity of the function $\rho(\theta)$ for the arcs $B(n, k, r)$

In this section, our main interest is to show that  $\rho(\theta)$  is a monotonic function, that is, the derivative  $d\rho/d\theta$  is never zero, for each trinomial arc  $B(n, k, r)$ . Differentiating both sides of (2.3) with respect to  $\theta$ , we obtain

$$\begin{aligned} & [(n - k)\rho^{n-k-1} \sin n\theta - n\rho^{n-1} \sin(n - k)\theta] \frac{d\rho}{d\theta} \\ &= k \cos k\theta + (n - k)\rho^n \cos(n - k)\theta - n\rho^{n-k} \cos n\theta. \end{aligned} \tag{5.1}$$

Supposing that  $d\rho/d\theta = 0$ , we will consider  $\rho^n$  and  $\rho^{n-k}$  as solutions of the system,

$$\begin{aligned} & k \cos k\theta + (n - k)\rho^n \cos(n - k)\theta - n\rho^{n-k} \cos n\theta = 0, \\ & \rho^{n-k} \sin n\theta - \rho^n \sin(n - k)\theta - \sin k\theta = 0. \end{aligned} \tag{5.2}$$

This system is equivalent to the following system:

$$\begin{aligned} R(\theta) \cdot \rho^{n-k} &= N_1(\theta), \\ R(\theta) \cdot \rho^n &= N_2(\theta), \end{aligned} \quad (5.3)$$

where

$$\begin{aligned} R(\theta) &= (n-k) \sin k\theta - k \cos n\theta \sin(n-k)\theta, \\ N_1(\theta) &= (n-k) \sin n\theta - n \sin(n-k)\theta \cos k\theta, \\ N_2(\theta) &= (n-k) \sin n\theta \cos k\theta - n \sin(n-k)\theta. \end{aligned} \quad (5.4)$$

The difference of the two equalities of (5.3) leads to the equation:

$$R(\theta)[\rho^n - \rho^{n-k}] = U(\theta)[1 - \cos k\theta] \quad (5.5)$$

with

$$U(\theta) = -[n \sin(n-k)\theta + (n-k) \sin n\theta]. \quad (5.6)$$

We now show that the hypothesis  $d\rho/d\theta = 0$  leads to a contradiction. For that, we need the two following lemmas.

**LEMMA 5.1.** *Let  $d = \gcd(k, n)$  and  $r \in \{0, 1, \dots, d-1\}$ , one assumes that  $0 < k < n$ ,  $k/d$  and  $n/d$  are odd. Then, one assumes  $R(\theta) = (n-k) \sin k\theta - k \sin(n-k)\theta \cos n\theta > 0$  for any feasible angle  $\theta$  in the interval  $(\theta_0, \theta_1)$ , where  $\theta_1 = (2r+1)\pi/d$  and  $\theta_0 = \theta_1 - \pi/n$ .*

*Proof.* Let  $\theta$  be a feasible angle in  $\theta_0, \theta_1$ , with  $\theta_1 = (2r+1)\pi/d$  and  $\theta_0 = \theta_1 - \pi/n$ . By first, remark that  $\cos n\theta = 0$  if and only if  $\theta = \theta_c = [2n(2r+1)/d - 1]\pi/2n$ . Moreover, we have  $\cos n\theta > 0$  for  $\theta < \theta_c$  and  $\cos n\theta < 0$  for  $\theta > \theta_c$ . By Lemma 3.3, we have  $\sin k\theta > 0$  and  $\sin(n-k)\theta < 0$ . Then, we get  $R(\theta) > 0$  for any  $\theta$  in  $\theta_0, \theta_c$ . In the other case, that is when  $\theta$  belongs to  $\theta_c, \theta_1$ , remarking that  $R(\theta)$  can be expressed as  $R(\theta) = (n-k) \sin n\theta \cos(n-k)\theta - n \sin(n-k)\theta \cos n\theta$ , we will consider the function  $K(\theta) = R(\theta)/\cos n\theta \cos(n-k)\theta = (n-k) \tan n\theta - n \tan(n-k)\theta$ . In this case, we have  $\cos n\theta < 0$ . In addition, as  $(2r+1)\pi/d - \pi/2n < \theta < (2r+1)\pi/d$ , we get  $(n-k)(2r+1)\pi/d - (n-k)\pi/2n < (n-k)\theta < (n-k)(2r+1)\pi/d$ . Because  $(n-k)(2r+1)/d$  is even, there exists a nonnegative integer  $q$  such that  $2q = (n-k)(2r+1)/d$ . Then, this double inequality becomes  $2q\pi - (n-k)\pi/2n < (n-k)\theta < 2q\pi$ . Since  $(2q-1/2)\pi < 2q\pi - (n-k)\pi/2n$ , we obtain that  $\cos(n-k)\theta > 0$ . The sign of  $R(\theta)$  is so opposed to the sign of  $K(\theta)$ , which is derivable with  $K'(\theta) = n(n-k)[\tan^2 n\theta - \tan^2(n-k)\theta]$ . Since  $\tan n\theta < 0$  and  $\tan(n-k)\theta < 0$ , the zeros of  $K'(\theta)$  verify the equation  $\tan n\theta = \tan(n-k)\theta$ . Therefore, the unique solution of this equation is of the form  $\theta = l\pi/k$  where  $l$  is an integer. However,  $l\pi/k \in \theta_c, \theta_1$  if and only if  $k(2r+1)/d - k/2n < l < k(2r+1)/d$ . As  $k(2r+1)/d$  is odd, there exists a nonnegative integer  $p$  such that  $(2p+1) = k(2r+1)/d$ . Hence, we get  $(2p+1) - k/2n < l < (2p+1)$ . Because  $k < n$ , it follows that  $(2p+1/2) < l < (2p+1)$ , which is not possible since  $l$  is an integer. We conclude that  $K'(\theta)$  is never zero. Moreover,  $K(\theta)$  goes to  $-\infty$  as  $\theta$  tends on the right to  $\theta_c$  and  $K(\theta_1) = 0$ . This implies that  $K(\theta) < 0$  and that  $R(\theta) > 0$  for any  $\theta$  in  $\theta_c, \theta_1$ . Therefore,  $R(\theta) > 0$  for any  $\theta$  in the interval  $\theta_0, \theta_1$ .  $\square$

LEMMA 5.2. Let  $d = \gcd(k, n)$  and  $r \in \{0, 1, \dots, d-1\}$ , one assumes that  $0 < k < n$ ,  $k/d$  and  $n/d$  are odd. Then, one has  $U(\theta) = -[n \sin(n-k)\theta + (n-k) \sin n\theta] > 0$  for any feasible angle  $\theta$  in the interval  $(\theta_0, \theta_1)$ , where  $\theta_1 = (2r+1)\pi/d$  and  $\theta_0 = \theta_1 - \pi/n$ .

*Proof.* Let  $\theta$  be a feasible angle in  $\theta_0, \theta_1$ . The function  $U(\theta)$  is derivable, with  $U'(\theta) = -n(n-k)[\cos(n-k)\theta + \cos n\theta]$ . The zeros of  $U'(\theta)$  are of the form  $\theta = (2l-1)\pi/k$  or of the form  $\theta = (2l+1)\pi/(2n-k)$  where  $l$  is an integer. However,  $(2l-1)\pi/k \in ]\theta_0, \theta_1[$  if and only if  $k(2r+1)/d + 1 - k/n < 2l < k(2r+1)/d + 1$ . From the proof of the precedent lemma, we have  $(2p+1) = k(2r+1)/d$ , then, this double inequality becomes  $p+1 - k/2n < l < p+1$ . As  $k < n$ , one has  $(p+1/2) < p+1 - k/2n$ . Hence,  $(p+1/2) < l < (p+1)$ , which is impossible as  $l$  is an integer. On the other side,  $(2l+1)\pi/(2n-k) \in ]\theta_0, \theta_1[$  if and only if  $(2n-k)(2r+1)/2d - 3/2 + k/2n < l < (2n-k)(2r+1)/2d - 1/2$ . As  $n(2r+1)/d$  is odd, there exists a nonnegative integer  $s$  such that  $(2s+1) = n(2r+1)/d$ . Moreover, since  $(2p+1) = k(2r+1)/d$ , it follows that  $(2n-k)(2r+1)/2d - 1/2 = 2s - p$ . Therefore, the double inequality above becomes  $2s - p - 1 + k/2n < l < 2s - p$ , then  $2s - p - 1 < l < 2s - p$ , which is not possible. It follows that  $U'(\theta)$  is never zero. In addition, because  $U(\theta_0) > 0$  and  $U(\theta_1) = 0$ , we deduce that  $U(\theta) > 0$  for any angle  $\theta$  in  $\theta_0, \theta_1$ .  $\square$

Using the two lemmas above, we can prove the next main result for the trinomial arcs  $B(n, k, r)$ .

THEOREM 5.3. Let  $d = \gcd(k, n)$  and  $r \in \{0, 1, \dots, d-1\}$ , one assumes that  $0 < k < n$ ,  $k/d$  and  $n/d$  are odd. For every trinomial arc  $B(n, k, r)$ ,  $\rho(\theta)$  is a decreasing function on the interval  $(\theta_0, \theta_1)$ , where  $\theta_1 = (2r+1)\pi/d$  and  $\theta_1 - \theta_0 = \pi/n$ .

*Proof.* Let  $B(n, k, r)$  be a trinomial arc. Lemmas 5.1 and 5.2 imply, respectively, that  $R(\theta) > 0$  and  $U(\theta) > 0$  for any  $\theta$  in the interval  $\theta_0, \theta_1$ . Therefore, the relation  $R(\theta)[\rho^n - \rho^{n-k}] = U(\theta)[1 - \cos k\theta]$  given by (5.5) implies that  $\rho^n - \rho^{n-k} > 0$ , which is impossible as  $\rho < 1$ . Thus, we have proved that for each trinomial arc  $B(n, k, r)$ , we have  $d\rho/d\theta \neq 0$ , that is,  $\rho(\theta)$  is a monotonic function, for any angle  $\theta$  in  $\theta_0, \theta_1$ .

Moreover, putting  $\theta = \theta_0 = (2r+1)\pi/d - \pi/n$  in (2.3), we have  $\rho^{n-k}(\theta_0)\sin n\theta_0 - \rho^n(\theta_0)\sin(n-k)\theta_0 - \sin k\theta_0 = 0$ . Using the facts that  $(2s+1) = n(2r+1)/d$  and  $(2p+1) = k(2r+1)/d$ , we get  $\sin n\theta_0 = 0$  and  $\sin(n-k)\theta_0 = -\sin k\theta_0$ . So, it follows that  $[\rho^n(\theta_0) - 1]\sin k\theta_0 = 0$ . Hence, because  $2\pi p < k\theta_0 < (2p+1)\pi$ , it yields that  $\sin k\theta_0 \neq 0$  and that  $\rho(\theta_0) = 1$ . Since  $\rho(\theta)$  is less than or equal to 1 for any feasible angle  $\theta$ , we deduce that  $\rho(\theta)$  is a decreasing function on the interval  $(\theta_0, \theta_1)$ .  $\square$

## 6. Conclusion

In this work, we have studied the behavior of the family of trinomial arcs  $B(n, k, r)$ , composed of all solutions of (1.1) in the case  $0 < \alpha < 1$  with the feasible angles  $\theta$  in the interval  $[\arg(\gamma), \arg(\delta)]$ , where  $\gamma$  is an  $n$ th root of unity and  $\delta$  is both a  $k$ th root of  $-1$  and an  $(n-k)$ th root of unity. The problem of monotonicity of the trinomial arcs is completely solved in this case. During the description and definition of  $B(n, k, r)$ , we have evoked another type of trinomial arcs, defined as the solutions of (1.1) in the case  $0 < \alpha < 1$  with the feasible angles  $\theta$  in the interval  $[\arg(\delta), \arg(\gamma')]$ , where  $\delta$  is both a  $k$ th root of  $-1$  and

an  $(n - k)$ th root of unity and  $\gamma'$  is an  $n$ th root of unity. A later study of the behavior of this family of arcs would be interesting.

## References

- [1] H. Fell, “The geometry of zeros of trinomial equations,” *Rendiconti del Circolo Matematico di Palermo*, vol. 29, no. 2, pp. 303–336, 1980.
- [2] S. Dubuc and M. Zaoui, “Sur la quasi-convexité des arcs trinomiaux,” *Rendiconti del Circolo Matematico di Palermo*, vol. 45, no. 3, pp. 493–514, 1996.
- [3] S. Dubuc and A. Malik, “Convex hull of powers of a complex number, trinomial equations and the Farey sequence,” *Numerical Algorithms*, vol. 2, no. 1, pp. 1–32, 1992.

Kaoutar Lamrini Uahabi: Forces Armées Royales Boulevard 49, Apartment no. 9,  
Nador 62000, Morocco  
*Email address:* lamrinika@yahoo.fr

Mohammed Zaoui: Department of Mathematics, Faculty of Sciences, Mohamed First University,  
P.O. Box 524, Oujda 60000, Morocco  
*Email address:* zaouiimoh@menara.ma

## Special Issue on Intelligent Computational Methods for Financial Engineering

### Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods:** artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields:** asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects:** decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site <http://www.hindawi.com/journals/jamds/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

### Guest Editors

**Lean Yu**, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; [yulean@amss.ac.cn](mailto:yulean@amss.ac.cn)

**Shouyang Wang**, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; [sywang@amss.ac.cn](mailto:sywang@amss.ac.cn)

**K. K. Lai**, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; [mskklai@cityu.edu.hk](mailto:mskklai@cityu.edu.hk)